GEOMETRY OF BRAID GROUPS IN COMBINATORICS

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1. INTRODUCTION

This proposal is for a research program on the combinatorics of braid groups and fundamental groups of complexified arrangements, with strong applications to geometric group theory and classical combinatorics. This point of view yields **an interconnected library of concrete combinatorial problems especially suitable for graduate and undergraduate students** (see Section 5, as well as Section 8), while its broad perspective allows for consequential results and relevance to other fields.

Algebraic combinatorics interprets classical combinatorial identities or objects in the highly structured setting of algebra, with the expectation of removing unnecessary hypotheses and generalizing. A few successful examples, both recent and classical:

- generating functions become the Hilbert series of a graded algebra;
- symmetric functions are interpreted as a Weyl character formula or Schubert polynomials for the Grassmannian, while symmetric function theory is recalled to be a particular case of invariant theory; and
- q-analogues generalize enumerative formulas using the general linear group over a finite field, a Hecke algebra, or even a quantum group.

As experience with the above examples illustrates, there are difficulties associated with moving from a well-understood special case to a more general framework—but such difficulties can leave interesting combinatorics undiscovered.

The difficulty addressed in this proposal comes from replacing a real hyperplane arrangement with its complexification.

For example, moving from real to complex reflection groups further requires the sacrifice of many valuable tools rooted in the real geometry—simple generators and inversion sets are no longer accessible, which nullifies many proof techniques and geometric intuition. Indeed, classical objects like the braid group become alien when the real geometry is forgotten [BKL98, Bes03]. The prospect of leaving the real geometry behind has resulted in a general combinatorial neglect of the area; notable exceptions are David Bessis and his Ph.D. student Vivien Ripoll, as well as Vic Reiner and his recent Ph.D. student Theodosios Douvropoulos (whose talk at FPSAC this year solved a conjecture from my thesis) [Rip12, Bes06, Dou17, Dou18b, Dou18a, Wil13].

Recall that a topological space X is called a $K(\pi, 1)$ (or *Eilenberg-MacLane space*) for the group G if $\pi_1(X) = G$ and all higher homotopy groups vanish. A natural first question is to relate two different combinatorial $K(\pi, 1)$ models for complexified real hyperplane complements. These two models arise from different choices of basepoint—the Salvetti complex is constructed relative to a *real* basepoint, while Bessis's model comes from a *complex* one (I review these models in Sections 3.4 and 4.2). Since a $K(\pi, 1)$ is unique up to weak homotopy equivalence:

Problem 1. Give an explicit homotopy between Salvetti's and Bessis's $K(\pi, 1)$ models for the braid groups and pure braid groups of finite Coxeter groups.

Although a purely topological statement, this problem has combinatorial meaning. A certain subset of the vertices of each model represent different Coxeter-Catalan object— the sortable elements live as vertices of Salvetti's model, while the noncrossing partitions are vertices of Bessis's model. So besides providing a tangible connection between the two different presentations of the same group, such a homotopy would lend a sense of inevitability to the bijections between these Coxeter–Catalan objects [Rea07a]. The Loday– Reading pulling triangulation of the permutahedron and associahedron, which relates the associahedron to the order complex of the noncrossing partition lattice will serve as a useful guide in constructing this homotopy [Lod07, Rea11].

As further evidence that this area is underexplored, it seems remarkable that the following fundamental question is open:

Problem 2. Let W be a finite Coxeter group. Give explicit presentations for the *pure braid group* $P(W) := \pi_1(V_{\mathbb{C}}^{\mathrm{reg}})$.

The history of work on this problem is quickly outlined:

- In [Art25, Art47], Artin worked out a presentation of the pure braid group in type A using the usual model as strands (five families of relations);
- the dihedral groups are trivial (having only one family of relations);
- Classical work of Fadell and Neuwirth on the braid group extends to hyperplane arrangements of *fiber type* [FN62, FR85]; since type B is of fiber type, its pure braid group is an iterated semidirect product of free groups—yet even so
- Cohen worked out an explicit presentation in type B [Coh01, Theorem 1.4.3], pointing out an erroneous presentation for type B in [Lei93, Section 3.8]; and also
- Digne and Gomi derived a different explicit presentation for type B (but did not succeed in type D) [DG01]; while finally
- in their second survey article [FR00], Falk and Randell point out that the presentation for type D published in [Mar91] is incorrect.

One of the difficulties in writing down these presentations correctly comes from the sheer number of relations (for example, [Lei93] gives 23 families of relations in type B, while [Coh01] and [DG01] each have nine families) with fewer organizing principles than would be desired. But by combining the classical combinatorial $K(\pi, 1)$ models of Deligne

and Salvetti with Coxeter-Catalan combinatorics, I have recently developed a new, controlled method to compute presentations of these pure braid groups. **This approach has been overlooked due to a focus in the area on variants of the Zariski-van Kampen method[1] for computing presentations of fundamental groups of hyperplane arrangements:** although an arrangement may always be generically projected to a line arrangement in $\mathbb{C}^2$ while preserving the fundamental group, different projections give different presentations and the original geometry is obscured.

In [Rea11], Reading defined a delicate slicing procedure on simplicial hyperplane arrangements that cuts hyperplanes into several pieces called *shards*, geometrically modeling the lattice-theoretic properties of weak order. The fundamental observation that allows for control over the presentations is that although a reflection in a hyperplane has many lifts to the pure braid group—corresponding to $w$here a loop wraps around a hyperplane—the homotopy classes of such loops are actually indexed by these shards (see Theorem 9). My method exploits this observation, and I have written explicit presentations in types A and B. In type A, I recover and explain a compact rephrasing of Artin's presentation due to Margalit and McCammond [MM09] without drawing a single braid (see Theorem 5). In principal, there is no reason the method shouldn't work in general, but the computations become longer; part of this proposal is therefore to perform these computations.

The resulting presentations are explainable in the language of Coxeter-Catalan combinatorics, but not overwhelmingly elegant. Instead, they serve as a starting point towards writing down beautiful, simple presentations for pure braid groups (resembling the elegant presentations arising from Garside theory for braid groups):

Conjecture 3. Let W be a finite irreducible Coxeter group. Then $P(W) = \left\langle \mathbb{T} : [\mathbb{c}] \right\rangle$, where $\mathbb{c}$ is the full twist (the generator of the center of $P(W)$), $\mathbb{T}$ is a certain lift of the reflections of W to $P(W)$ coming from Bessis' dual braid monoid, and $[\mathbb{c}]$ is the relation equating all words in $\mathbb{T}$ for $\mathbb{c}$. Furthermore, these words for $\mathbb{c}$ are exactly those reflection orderings with respect to which the noncrossing partition lattice is EL-shellable.

By apply rewriting rules to the presentations given by my method above, I have made some partial progress towards proving Conjecture 3 by confirming it in types A and B. For the other types—because the computations become increasingly lengthy in the presence of larger parabolic dihedral subgroups—I propose to automate this rewriting process. I am confident this method will settle my conjecture for the remaining Coxeter groups.

This is not just an empty exercise in group presentations—while the method above will allow me to prove the presentations in Conjecture 3 for finite Coxeter groups, it strongly suggests that there ought to be a conceptual, geometric proof: the order complex of the noncrossing partition lattice has a strong connection to Bessis's $K(\pi, 1)$, which embeds

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[1]In [Zar37, Zar36], Zariski showed that for a choice of a projection of an algebraic hypersurface in complex projective space to a generic hyperplane preserves the fundamental group. As recounted in [VK33a], Zariski asked van Kampen to develop a topological method to compute the relations among generators of the fundamental group which had previously been determined by Enriques [Enr24]. Zariski's question directly led to van Kampen's famous theorem [VK33b, Gra92].

into the complexified hyperplane complement. A shelling of the complex should therefore somehow specify a homotopy class of loop representing the full twist. Such a geometric proof should address a natural generalization of Conjecture 3 to the class of complexified real central hyperplane arrangements, whose details I omit in the interest of space.

Problem 4. Prove the natural generalization of Conjecture 3 to give explicit presentations for any complexified real central hyperplane arrangement.

1.1. **Further Directions.** It will be worthwhile—though considerably more speculative— to see if this new perspective and new methods give any insight into classical unsolved problems in the area. In particular, I am interested in considering:

- the orderability of fundamental groups of complexified real hyperplane arrangements (thus proving torsion-freeness) [Deh95],
- the $K(\pi, 1)$ problem for braid groups of infinite Coxeter groups (my work on Garside shadows seems particularly relevant to this problem [HNW16]), and
- the CAT(0) conjectures for braid groups of finite type [CD95].

I would also like to interpret my work on zeta/sweep maps [TW18b] using Ion's result that double affine braid groups can be realized as the fundamental group of a certain complex hyperplane arrangement related to the usual affine Weyl arrangement [Ion03].

1.2. **Organization.** The remainder of this proposal is structured to explain the statements of Problem 1, Problem 2, and Conjecture 3, as well as my approach towards their resolution. In Section 2, I give a historical overview in the setting of the symmetric group, the braid group, and the pure braid group. In Section 3, I generalize to finite Coxeter groups and their braid groups, introducing shards and Bessis' $K(\pi, 1)$. I define the Salvetti complex in Section 4 and summarize my approach towards Problem 2,. In Section 5, I give an example of the sorts of problems I have given students by summarizing the result of an REU project I supervised this summer (and discuss related future work). Finally, Section 8 addresses the broader impacts of my work.

2. The Symmetric Group, Braid Group, and Dual Braid Group

Before generalizing to Coxeter groups in Section 3, I first give a brief introduction to the symmetric group, and its associated braid and pure braid groups. In Section 5, I highlight the sort of combinatorial question and research directions that arise; many problems are accessible to and suitable for undergraduate and graduate students.

2.1. **The Symmetric Group.** For $n \geq 1$, write $S = \{(1, 2), \ldots, (n-1, n)\}$ for a set of abstract generators. Then the *symmetric group* $\mathfrak{S}_n$ has the Coxeter presentation

$$\mathfrak{S}_n = \left\langle S : \begin{array}{ll} (i, i+1)(j, j+1) = (j, j+1)(i, i+1) & \text{if } |i - j| > 1 \\ (i, i+1)(j, j+1)(i, i+1) = (j, j+1)(i, i+1)(j, j+1) & \text{if } |i - j| = 1 \\ (i, i+1)(j, j+1) = e & \text{if } |i - j| = 0 \end{array} \right\rangle.$$

As an example of interpreting classical combinatorial objects in the framework of the symmetric group, the 231-avoiding permutations in the symmetric group are enumerated by the Catalan numbers

$$\mathsf{Cat}(n) := \frac{1}{n+1}\binom{2n}{n} \longrightarrow 1, 1, 2, 5, 14, 42, 132, 429, 1430, \ldots .$$

The *Tamari lattice* Tam_n first appeared in Tamari's 1951 thesis as a partial order on the parenthesizations of a product of $n{+}1$ variables (the connectedness of the underlying graph under the "flip" $a(bc) \to (ab)c$ is equivalent to associativity of the product). It may also be defined as the one-skeleton of the *associahedron*, which Stasheff discovered in 1961 in his thesis work on homotopy theory. As shown by Bjorner and Wachs [BW97], restricting weak order (an orientation of the permutahedron, or the Cayley graph of $\mathfrak{S}_n$ with respect to the generating set S) to the 231-avoiding permutations gives Tam_n. Starting with this classical object, combinatorial problems abound (see Section 5).

2.2. **The Braid Group.** A permutation can be drawn as a pairing of n upper points with n lower points (each numbered from 1 to n), with multiplication given by stacking two pairings. Replacing a pairing with non-intersecting strands leads to the classical definition of the braid group B_n—elements (now called *braids*) are equivalence classes of n non-intersecting strands (up to non-intersecting deformations).

Writing $\mathbf{S} = \{(\mathbf{1}, \mathbf{2}), \ldots, (\mathbf{n{-}1}, \mathbf{n})\}$ for a new set of generators, Artin proved in [Art25, Art47] that B_n has presentation

$$B_n = \left\langle \mathbf{S} : \begin{matrix} (\mathbf{i}, \mathbf{i{+}1})(\mathbf{j}, \mathbf{j{+}1}) = (\mathbf{j}, \mathbf{j{+}1})(\mathbf{i}, \mathbf{i{+}1}) & \text{if } |i - j| > 1 \\ (\mathbf{i}, \mathbf{i{+}1})(\mathbf{j}, \mathbf{j{+}1})(\mathbf{i}, \mathbf{i{+}1}) = (\mathbf{j}, \mathbf{j{+}1})(\mathbf{i}, \mathbf{i{+}1})(\mathbf{j}, \mathbf{j{+}1}) & \text{if } |i - j| = 1 \end{matrix} \right\rangle .$$

Remarkably, Artin's presentation for B_n is obtained from the Coxeter presentation of $\mathfrak{S}_n$ simply by forgetting that generators square to the identity (this presentation was obtained *before* Coxeter's work). In his single mathematical publication [Gar69], Garside laid the foundations for his eponymous *Garside theory*—roughly, the idea is to use the long permutation $w_\circ = n \cdots 21$ of $\mathfrak{S}_n$ (the Garside element) and the lattice property of weak order to construct a normal form for any braid, solving the word problem for B_n [Del72].

At the level of presentations, Garside's work can be interpreted as the beautifully compact presentation $B_n = \langle \mathbf{S} : [\mathbf{w}_\circ] \rangle$, where $\mathrm{Red}_S(w_\circ)$ is the set of all reduced words for $w_\circ$ in the generators S, and $[\mathbf{w}_\circ]$ for the relation that replaces the generators in S by the corresponding generator in $\mathbf{S}$ and setting all such words equal. For example, the relation in B_3 sets the two reduced words for 321 equal, while the relation in B_4 sets equal the 16 reduced words for 4321.

It is natural to wonder if there is an analogue of the 231-avoiding permutations in B_n, now counted by the *Fuss-Catalan numbers* $\frac{1}{mn+1}\binom{(m+1)n}{n}$. In [STW15], Stump, Thomas, and I describe exactly such a generalization using the restriction of the interval $[e, w_\circ^m]$ in the weak order on the positive braid monoid B_n^+ to certain *sortable elements* (these elements have a reduced word with a special form; see Definition-Theorem 6). The interval

$[e, w_\circ^2]$ for B_3^+ is illustrated on the left-hand side of Figure 1, with the sortable elements shaded in gray.
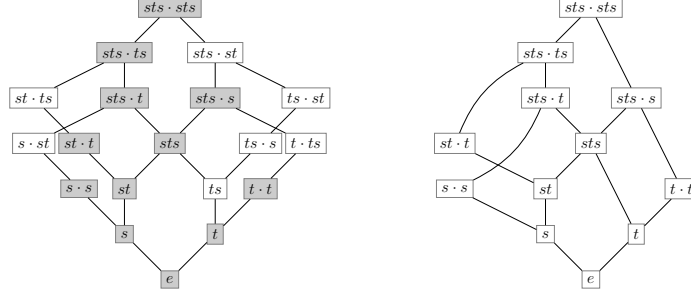

FIGURE 1. Left—the interval $[e, w_\circ^2]$ in B_3^+ with the sortable element highlighted in gray; center—the restriction to sortable elements gives $\mathrm{Camb}^2(\mathfrak{S}_3)$.

2.3. **The Pure Braid Group.** The kernel of the natural projection $B_n \to \mathfrak{S}_n$ obtained by remembering only the underlying pairing of the upper and lower points defines the *pure braid group* P_n (i.e. those braids that pair the upper point numbered i with the lower point numbered i), giving the short exact sequence $1 \to P_n \to B_n \to \mathfrak{S}_n \to 1$. For $1 \leq i < j \leq n$, define the braids

$$(\mathbf{ij}) := \Big((\mathbf{i,i+1})\cdots(\mathbf{j-1,j})\Big)(\mathbf{j,j+1})\Big((\mathbf{j-1,j})\cdots(\mathbf{i,i+1})\Big) \text{ and } (\mathring{\mathbf{i}}\mathring{\mathbf{j}}) = (\mathbf{ij})^2,$$

and write $T = \big\{(ij)\big\}_{1 \leq i < j \leq n}$, $\mathbf{T} := \big\{(\mathbf{ij})\big\}_{1 \leq i < j \leq n}$, and $\mathbb{T} := \big\{(\mathring{\mathbf{i}}\mathring{\mathbf{j}})\big\}_{1 \leq i < j \leq n+1}$ for the transpositions of $\mathfrak{S}_n$ and certain lifts to P_n and B_n. Artin proved that P_n was generated by $\mathbb{T}$, subject to a somewhat complicated set of five relations. Margalit and McCammond derived an easier equivalent presentation from Artin's original presentation [MM09]: say that two transpositions (ij) and (rs) are *noncrossing* if they are noncrossing when drawn as arcs in a circle connecting boundary vertices labelled $1, 2, \ldots, n$, and *crossing* otherwise. Say that three (ordered) transpositions $(ij), (ik), (jk)$ are *noncrossing* if $i < j < k$. Then the Margalit-McCammond presentation is:

Theorem 5 ([MM09, Theorem 2.3]).

$$P_n = \left\langle \ \mathbb{T} \ \middle| \ \begin{array}{ll} (\mathring{\mathbf{i}}\mathring{\mathbf{j}})(\mathbf{r}\mathbf{s}) = (\mathbf{r}\mathbf{s})(\mathring{\mathbf{i}}\mathring{\mathbf{j}}) & \textit{if } (ij) \textit{ and } (rs) \textit{ are noncrossing} \\ (\mathring{\mathbf{i}}\mathring{\mathbf{j}})(\mathring{\mathbf{i}}\mathring{\mathbf{k}})(\mathring{\mathbf{j}}\mathring{\mathbf{k}}) = (\mathring{\mathbf{i}}\mathring{\mathbf{k}})(\mathring{\mathbf{j}}\mathring{\mathbf{k}})(\mathring{\mathbf{i}}\mathring{\mathbf{j}}) = (\mathring{\mathbf{j}}\mathring{\mathbf{k}})(\mathring{\mathbf{i}}\mathring{\mathbf{j}})(\mathring{\mathbf{i}}\mathring{\mathbf{k}}) & \textit{if } (ij), (ik), (jk) \textit{ are noncrossing} \\ (\mathring{\mathbf{i}}\mathring{\mathbf{j}})(\mathring{\mathbf{j}}\mathbf{s})(\mathbf{r}\mathbf{s})(\mathring{\mathbf{j}}\mathbf{s})^{-1} = (\mathring{\mathbf{j}}\mathbf{s})(\mathbf{r}\mathbf{s})(\mathring{\mathbf{j}}\mathbf{s})^{-1}(\mathring{\mathbf{i}}\mathring{\mathbf{j}}) & \textit{if } (ij) \textit{ and } (rs) \textit{ are crossing} \end{array} \right\rangle.$$

As outlined in Section 4, I recently found a new proof of this presentation that does not rely on Artin's presentation (and my method extends to other types). The proof uses the combinatorics and geometry of the Tamari lattice inside of weak order.

Artin used his presentation to give a normal form for pure braids ("combing") and hence solve the word problem for P_n: any pure braid can^2 be written as a product $\mathtt{w} = \mathtt{w}_1\mathtt{w}_2\cdots\mathtt{w}_{n-1}$, where $\mathtt{w}_i$ belongs to the subgroup of P_n generated by $\{(\hat{\imath}\hat{\jmath}) : n \geq j > i\}$. In modern language, this algorithm uses the exact sequence (forgetting the last strand)

$$1 \to F_{n-1} \to P_n \to P_{n-1} \to 1, \text{ to show that } P_n = F_n \rtimes (F_{n-1} \rtimes (\cdots \rtimes F_1)\cdots)$$

is an iterated semidirect product of free groups, where F_{n-1} is the free group on $n-1$ generators. As B_n is a finite extension of P_n by $\mathfrak{S}_n$, this solves the word problem in B_n.

2.4. **Dual Presentations of the Braid Group.** I now return from the pure braid group back to the braid group. In 1998, Birman, Ko, and Lee had the remarkable insight to use **T** to give a new presentation of B_n, simultaneously giving a new solution to the word problem [BKL98]. Their presentation can be written as

$$B_n = \left\langle \mathbf{T} : \begin{array}{cc} (\mathbf{ij})(\mathbf{rs}) = (\mathbf{rs})(\mathbf{ij}) & \text{if } (\mathbf{ij}), (\mathbf{rs}) \text{ are noncrossing} \\ (\mathbf{ij})(\mathbf{jk}) = (\mathbf{ik})(\mathbf{ij}) = (\mathbf{jk})(\mathbf{ik}) & \text{if } 1 \leq i \leq j \leq k \end{array} \right\rangle.$$

Sergiescu somewhat anticipated these results in [Ser93], describing finite positive braid group presentations arising from a planar graph—Artin's presentation is read from a path graph (a Dynkin diagram of type A), while the Birman-Ko-Lee presentation comes from a complete graph [HK02]. Steinberg had previously considered presentations of the symmetric group (and, more generally, Coxeter groups) using all reflections, but the "obvious" suggested extension to a presentation for the braid group—obtained by dropping the relations that reflections square to the identity—is incorrect.

In the language of Garside theory, the Birman-Ko-Lee result replaces the long element $w_\circ$ of $\mathfrak{S}_n$ by the long cycle $c = (1, 2, \cdots, n)$ (a *Coxeter element*), which we can again interpret at the level of the presentations. Writing $\mathrm{Red}_T(c)$ for the set of all reduced words for $w_\circ$ in T and $[\mathbf{c}]$ for the relation replacing the generators in T by the corresponding generator in **T** and setting all such words equal, we again recover a simple presentation

$$B_n = \langle \mathbf{T} : [\mathbf{c}] \rangle.$$

By treating the blocks of a noncrossing partition as the cycles of a permutation, we obtain a special subset of elements of the symmetric group $\mathfrak{S}_n$ called the *noncrossing partitions*—those set partitions of $[n] := \{1, 2, \ldots, n\}$ whose blocks have disjoint convex hulls when drawn on a circle (explaining the nomenclature for noncrossing transpositions). Ordering the noncrossing partitions by refinement yields a lattice; and this lattice property turns out to be the reason why Garside theory applies. There are n^{n-2} maximal chains in the noncrossing partition lattice underlying the Birman-Ko-Lee presentation (in bijection with the reduced factorizations of the cycle c into transpositions), and so such chains are in bijection with trees, parking functions, etc.

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[2]Artin writes: "Although it has been proved that every braid can be deformed into a similar normal form the writer is convinced that any attempt to carry this out on a living person would only lead to violent protests and discrimination against mathematics. He would therefore discourage such an experiment."

Historically, the study of the chains in the noncrossing partition lattice go back to 1901 work of Hurwitz regarding the number of ramified covers of compact Riemann surfaces [Hur01] (the enormous amount of subsequent work on Hurwitz numbers, including connections to combinatorial map theory, is well beyond the scope of this section); they received further attention in a famous 1974 letter from Deligne to Looijenga [Del74]. The structure underlying the Birman-Ko-Lee presentation has been the subject of combinatorial attention since the work of Kreweras in 1972 [Kre72, Sim00, Rei97, Ede80, BW02, Bes03], and saw a surge of renewed interest with the appearance of [BKL98].

3. Coxeter Groups and their Braid Groups

I now generalize Section 2 from the symmetric group to finite Coxeter groups.

3.1. **Coxeter Groups.** A Coxeter system (W, S) is a group W with a given generating set of *simple reflections* S, subject to the Coxeter presentation:

$$(1) \qquad W = \left\langle s_1, \ldots, s_n : \underbrace{s_i s_j \cdots}_{m_{i,j} \text{ generators}} \overset{s_i^2 = e}{=} \underbrace{s_j s_i \cdots}_{m_{i,j} \text{ generators}} \right\rangle.$$

Denote the set of *all* reflections in the hyperplanes by $T := \{wsw^{-1} w \in W\}$, and let W act in the reflection representation on a real vector space V; associated to W is a real hyperplane arrangement $\mathscr{H} := \{H_t\}_{t \in T} \subset V$; the simple reflections S may be chosen to be the reflections in the walls of a particular chamber. Writing $V^{\text{reg}} := V \setminus \bigcup_{\alpha \in \Phi^+} H_\alpha$, we may identify elements of W with the connected regions of V^{reg}. The *inversion set* of an element $w \in W$ is the set of reflections whose corresponding hyperplanes separate w from the identity; ordering elements by inclusion of inversion sets gives the *weak order*, and for finite Coxeter groups the *long element* $w_\circ$ is the unique maximal element of weak order. A *cover reflection* is a $t \in T$ such that there is a cover $tw \lessdot w$ in weak order. A *Coxeter element* $c = s_{\pi_1} \cdots s_{\pi_{n-1}}$ of W is a product of the simple reflections of W in any order.

Each irreducible finite Coxeter group has an associated W-Catalan number, which comes in three levels of generality (stated, for simplicity, only for *crystallographic* Coxeter groups), corresponding to interpretations in the Coxeter group itself, the Artin group, and the rational Cherednik algebra (or affine Weyl group):

$$\mathsf{Cat}(W) := \prod_{i=1}^n \frac{h + d_i}{d_i}, \mathsf{Cat}^{(m)}(W) := \prod_{i=1}^n \frac{mh + d_i}{d_i}, \text{ and } \mathsf{Cat}^{[b]}(W) := \prod_{i=1}^n \frac{b + d_i - 1}{d_i},$$

where $d_1 < d_2 < \cdots < d_n$ are the degrees of W and $h := d_n$ is its Coxeter number. The parameters m and b are positive integers, and b is coprime to h. These are related by $\mathsf{Cat}(W) = \mathsf{Cat}^{(1)}(W)$ and $\mathsf{Cat}^{(m)}(W) = \mathsf{Cat}^{[mh+1]}(W)$. In particular, $\mathsf{Cat}(\mathfrak{S}_n) = \mathsf{Cat}(n)$ is associated to the symmetric group, and counts—in addition to the 231-avoiding permutations and noncrossing partitions—the number of triangulations of a convex $(n + 2)$-gon. Catalan numbers beautifully generalize to all other finite Coxeter groups: triangulations become finite-type clusters [FZ03] and 231-avoiding permutations become sortable elements [BW97, Rea07a, Rea07b, RS11]. Despite having uniform definitions, there are only
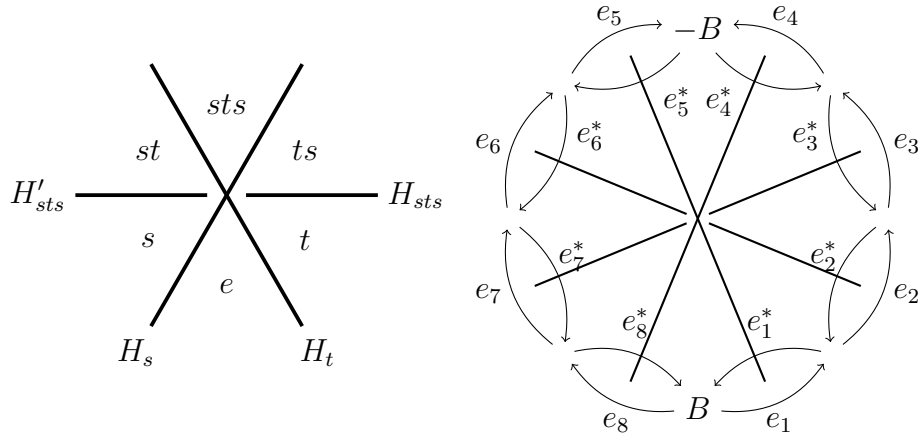
FIGURE 2. On the left is an illustration of the shards in type A_2—they are in bijection with the join-irreducible elements s, t, st, ts. On the right is the 1-skeleton of the Salvetti complex for the dihedral group $I_2(4)$ drawn over its hyperplane arrangement.

type-by-type proofs (using recursions or combinatorial models) that the noncrossing partitions, clusters, and sortable elements are counted by $\mathsf{Cat}(W)$. I have a **beautiful open conjecture** regarding a general bijection from clusters to nonnesting partitions (checked up to rank eight for all choices of Coxeter element) stated in [Wil14].

3.2. **Shards.** I now review a geometric construction of Reading [Rea11] that will return in Section 4.2. A hyperplane H in a subarrangement of the reflection arrangement is called *basic* if the connected region containing the fundamental chamber is bounded by H in the subarrangement. For any two hyperplanes H, H', define $\mathscr{A}(H, H')$ to be the subarrangement consisting of all hyperplanes containing $H \cap H'$. One says that H' *cuts* H if H' is a basic hyperplane of $\mathscr{A}(H, H')$ while H is not. In this way, all hyperplanes are cut into *shards*, defined as the closures of the connected pieces $H \setminus \bigcup_{H' \text{ cuts } H} H'$ of hyperplanes. The left of Figure 2 illustrates how the reflection arrangement of type A_2 is sliced into shards. A *lower shard* for an element $w \in W$ is a shard Σ belonging to a hyperplane H_t such that t is a cover reflections of w and $tw \lessdot w$ crosses Σ. Reading proved that shards are in bijection with join-irreducible elements in weak order [Rea11, Proposition 4.7].

3.3. **Braid Groups.** The theory of the braid group has a beautiful generalization to all finite Coxeter groups:

$$(2) \qquad B(W) := \pi_1 \left(V_{\mathbb{C}}^{\mathrm{reg}} / W \right) \overset{\mathrm{thm}}{=} \left\langle \mathbf{S} : \underbrace{\mathbf{s}_i \mathbf{s}_j \cdots}_{m_{i,j} \text{ generators}} = \underbrace{\mathbf{s}_j \mathbf{s}_i \cdots}_{m_{i,j} \text{ generators}} \right\rangle = \langle \mathbf{S} : [\mathbf{w}_\circ] \rangle,$$

where the first explicit presentation follows from work of Deligne and Brieskorn-Saito [Del72, BS72]; van der Lek proved that one still has this presentation for *any* Coxeter group, when $V_{\mathbb{C}}^{\mathrm{reg}}$ is replaced by $(V + iI)_{\mathbb{C}}^{\mathrm{reg}}$, where I is the Tits cone [VdL83]. The second presentation

$\langle \mathbf{S} : [\mathbf{w}_\circ] \rangle$ should be read as follows: for $w \in W$, let $\mathbf{w} \in B(W)$ be obtained by writing any reduced word for w in S and replacing all generators by their equivalents in $\mathbf{S}$; write $[\mathbf{w}_\circ]$ for the relation setting all reduced words in $\mathbf{S}$ for the long element $\mathbf{w}_\circ$ equal.

The most important idea Stump, Thomas, and I advance in [STW15] is that the correct setting for the Fuss–Catalan numbers is provided by the *positive braid monoid* $B^+(W)$. Our construction allows us to not only give a uniform treatment of previous work, but also supply a missing definition of sortable elements at the Fuss-Catalan level of generality.

Definition-Theorem 6 ([STW15])**.** Let W be a Coxeter group and $B^+(W)$ the corresponding positive braid monoid. The *c-sorting word* $\mathsf{w}(\mathsf{c})$ of an element $w \in B^+(W)$ is the lexicographically first subword of c^∞ that is a reduced word for w. An element $w \in B^+(W)$ is *c-sortable* if $\mathsf{w}(\mathsf{c})$ yields a decreasing sequence of subsets of positions in c.

Then for W finite, there are $\mathsf{Cat}^{(m)}(W)$ many c-sortable elements in the interval $[e, \mathbf{w}_\circ{}^m]$.

3.4. **Dual Presentations and $K(\pi, 1)$.** In the early 2000s Bessis and—independently— Brady-Watt generalized the work of Birman-Ko-Lee and showed that the *dual braid group* $B_c(W) \simeq B(W)$ admitted a presentation using *all* reflections, now depending on the choice of a Coxeter element c:

$$(3) \qquad B(W) := \pi_1(V_{\mathbb{C}}^{\mathrm{reg}}/W) \overset{\mathrm{thm}}{=} \left\langle \mathbf{T} : \mathbf{rt} = \mathbf{t^r r} \text{ if } \mathbf{rt} \leq_T \mathbf{c} \right\rangle = \langle \mathbf{T} : [\mathbf{c}] \rangle,$$

built from the *noncrossing partition lattice* $\mathsf{NC}(W) := [e, c]_T$ with Garside element the image of the Coxeter element c. Note that this is *not* the group obtained by taking all relations satisfied by the reflections in W. As Bessis points out, the difference between the presentations in Equation (2) and Equation (3) is the choice of real versus complex (an eigenvector of Coxeter element) basepoint.

The noncrossing partitions are enumerated by $\mathsf{Cat}(W)$, and through work of Armstrong (generalizing a construction of Edelman) [Arm09, Ede80], they admit *Fuss–Catalan* extensions as m-multichains in $\mathsf{NC}(W)$ enumerated by $\mathsf{Cat}^{(m)}(W)$. It is an **open problem** to give uniform generalizations of noncrossing partitions to objects counted by $\mathsf{Cat}^{[b]}(W)$. Recently, Michel found a uniform proof for the number of factorizations of a Coxeter element for Weyl groups using Deligne-Lusztig theory [Mic14].

In proving the $K(\pi, 1)$ conjecture for finite complex reflection groups, Bessis constructed simplicial complexes homotopy equivalent to $V_{\mathbb{C}}^{\mathrm{reg}}/W$ and $V_{\mathbb{C}}^{\mathrm{reg}}$: in the first case, the i-cells are indexed by i-chains in the noncrossing partition lattice; in the second case, the i-cells are indexed by an element of W and an i-chain.

4. Presentations of Pure Braid Groups: Problem 2

4.1. **Pure Braid Groups.** As for the symmetric group, the *pure braid group* $P(W) := \pi_1(V_{\mathbb{C}}^{\mathrm{reg}})$ is the kernel of the natural map $B(W) \to W$. As before, the pure braid is generated by the squares $\mathbb{T}$ of the elements of $\mathbf{T}$ (depending on the choice of Coxeter element)—for $\mathbf{t} \in \mathbf{T}$, I will abbreviate its square by $\mathbb{t} = \mathbf{t}^2$. **It is remarkable that the problem of giving a presentation of $P(W)$ has not been solved.**

One would like to apply Garside theory to $P(W)$ to derive a presentation, with the role of Garside element played by the *full twist* $\mathbb{c} = \mathbf{w}_\circ^2 = \mathbf{c}^h$. Unfortunately, this naive hope cannot work because the interval $[e, \mathbb{c}]$ in $B^+(W, c)$ is not generally a lattice. I still optimistically conjecture the analogues of the right-most presentations in Equations (2) and (3) given in Conjecture 3—that $P(W) = \langle \mathbb{T} : [\mathbb{c}] \rangle$, where $[\mathbb{c}]$ is the relation equating all reduced words in $\mathbb{T}$ for the full twist $\mathbb{c}$.

What makes such a presentation even more enticing is that I have found an elegant conjectural description of the reduced words for the full twist, inspired by Bessis's proof that the complements of complexified arrangements are $K(\pi, 1)$. To state this description, recall that any reduced word for the full twist defines a total ordering on the reflections of W. On the other hand—as a portion of the Cayley graph of W with respect to T—the noncrossing partition lattice comes equipped with a labeling by reflections and is EL-shellable with respect to a natural ordering on this labeling coming from $\mathbf{w}_\circ(\mathbf{c})$. Define EL_c as the set of total orderings of T for which $\mathrm{NC}_c(W)$ is EL-shellable.

Conjecture 7. The orderings of $\mathbb{T}$ defined by the reduced words for the full twist in $\mathrm{Red}_c(\mathbb{c})$ are exactly the orderings of T in EL_c.

The method of proof should be geometric: an element of EL_c coincides with a shelling of the order complex of the noncrossing partition lattice; but this complex embeds in $\mathbb{C}^n$, and the statement reads that a shelling coincides with a homotopy class of loop representing the full twist. Thomas has recently given a short proof that EL_c are exactly those orders of T that cyclically respect the rank two orderings on the reflections given by $\mathbf{w}_\circ(\mathbf{c})$.

4.2. **The Salvetti Complex and Shards.** For W a Coxeter group and $A \subseteq S$, write W_A for the parabolic subgroup of W generated by A. The *Salvetti complex* $\mathrm{Sal}(W)$ is the order complex of the poset $W \times \{A : A \subseteq S, W_A \text{ is finite}\}$, with relations $(w, A) \leq (w', A')$ if $A \subset A'$, $(w')^{-1}w \in W_{A'}$, and $(w')^{-1}w$ is a minimal coset representative of $(w')^{-1}wW_A$. **The vertices of $\mathrm{Sal}(W)$ are indexed by group elements, and its 1-skeleton coincides with the weak order on W with each edge doubled** (see Figure 2). There is an action of W on $\mathrm{Sal}(\mathscr{H})$ defined by $u \cdot (w, A) = (uw, A)$.

Theorem 8 ([Sal87, Theorem 1],[Del72]). *For W a finite Coxeter group, the Salvetti complex $\mathrm{Sal}(W)$ embeds in $V_{\mathbb{C}}^{\mathrm{reg}}$, and is a deformation retract of $V_{\mathbb{C}}^{\mathrm{reg}}$. The W-action induces a homotopy equivalence between $\mathrm{Sal}(\mathscr{H}/W)$ and $V_{\mathbb{C}}^{\mathrm{reg}}/W$. Furthermore, both $\mathrm{Sal}(W)$ and $\mathrm{Sal}(\mathscr{H}/W)$ are $K(\pi, 1)$ spaces.*

By definition, the group $P(W)$ is generated by certain loops indexed by the edges of the Hasse diagram of weak order. We may write these generators as the set

$$\mathbb{T}_{\mathrm{edge}} := \{\mathbb{t}_{w,i} := \mathbf{w} \cdot \mathbf{s}_i^2 \cdot \mathbf{w}^{-1} : w \in W, s_i \text{ a descent of } w\}.$$

If Σ is a shard corresponding to the join-irreducible element w_Σ with unique descent s_i, we define the loop $\mathbb{t}_\Sigma := \mathbf{w}_\Sigma \cdot \mathbf{s}_i^2 \cdot \mathbf{w}_\Sigma^{-1}$. Write $\mathbb{T}_{\mathrm{shard}} := \{\mathbb{t}_\Sigma : \Sigma \text{ a shard}\}$.

The main technical tool in finding presentations is the following observation:

Theorem 9 (W. 2018). *Each $\mathbb{t}_{w,i} \in \mathbb{T}_{\mathrm{edge}}$ is homotopic to a unique $\mathbb{t}_\Sigma \in \mathbb{T}_{\mathrm{shard}}$.*

This generating set $\mathbb{T}_{\text{edge}}$ can be reduced in size—in fact, it turns out that $P(W)$ is generated by a set indexed by the hyperplanes in $\mathcal{H}$ in the following way: any reduced word for the long element $\mathsf{w}_\circ := s_1 s_2 \cdots s_N$ specifies a set of pure braids

$$\mathbb{T}_{\mathsf{w}_\circ} = \left\{ \left(\mathbf{s}_1 \mathbf{s}_2 \cdots \mathbf{s}_{i-1}\right) \mathbf{s}_i^2 \left(\mathbf{s}_{i-1}^{-1} \cdots \mathbf{s}_2^{-1} \mathbf{s}_1^{-1}\right) \right\}_{i=1}^N.$$

It turns out that $\mathbb{T}_{\mathsf{w}_\circ}$ is a generating set for $P(W)$—any element of $\mathbb{T}_{\text{edge}}$ can (explicitly) be written as a product of elements of $\mathbb{T}_{\mathsf{w}_\circ}$ and their inverses. A first step towards a proof of Conjecture 3 is then a complete list of relations. In fact, by a theorem of Salvetti, we only need one family of relations $[\mathscr{A}]_{\mathbb{T}_{\mathsf{w}_\circ}}$ for each full rank-two subarrangement $\mathscr{A}$ of $\mathcal{H}$, encoding homotopic loops for the small full-twist $\mathbb{c}_{\mathscr{A}}$ of $\mathscr{A}$. When an element $\mathbb{t}_e$ used to express the relations for $\mathbb{c}_{\mathscr{A}}$ is not a generator from our chosen $\mathbb{T}_{\mathbb{T}_{\mathsf{w}_\circ}}$, we must rewrite $\mathbb{t}_e$ in terms of the elements $\mathbb{T}_{\mathbb{T}_{\mathsf{w}_\circ}}$. **The difficulty in writing down simple presentations for $\pi_1(V^{\text{reg}})$ is concentrated in this step** [Ran82, Ran85].

Coxeter-Catalan combinatorics now comes to the rescue: having chosen a set of shards through a choice of Coxeter element c, giving the c-sorting word $\mathsf{w}_\circ(c)$ and generators $\mathbb{T}_{\mathsf{w}_\circ(c)}$, the relations from rank-two c-noncrossing subspaces can be easily expressed using Theorem 9. On the other hand, most rank-two subspaces can be realized as a non-crossing subspace for some Coxeter element c. The theory now allows control over the presentations, and rewriting gives a method of attack for proving Conjecture 3.

5. REU Project: Hurwitz Presentations

In the interest of space, I will not give a long list of the problems I have for students and instead summarize an REU project I proposed and supervised this past summer.

In [Kra08], Krammer came up with a family of presentations of the braid group, a special case of which used triangulations of an n-gon. His construction can be placed in the setting of cluster algebras as follows: the Dynkin diagram of a simply-laced Coxeter group W encodes a presentation for the braid group where vertices correspond to generators, edges to braid relations, and missing edges to commutations, as in the top line of Figure 3. Orienting the edges of a Dynkin diagram gives a *quiver*, for which Fomin and Zelevinsky have defined a notion of *mutation*. The input is a quiver $\mathbb{Q}$ and any vertex v of $\mathbb{Q}$, and the output is a new quiver $\mu_v(\mathbb{Q})$. Building on work of Barot and Marsh [BM15], Grant and Marsh [GM17] showed that each quiver $\mathbb{Q}$ in the same mutation class as a Dynkin quiver encoded a presentation $B_{\mathbb{Q}}(W) \simeq B(W)$. For example, Figure 3 illustrates a mutation of a quiver and the corresponding presentations of the braid group B_3.

By associating the simple reflections to the vertices of a Dynkin diagram, a Coxeter element specifies an orientation of the edges—giving a quiver. Similarly, there is a sort of mutation available. Given a factorization of c into reflections, we can perform a *Hurwitz move* to obtain a new factorization:

$$(4) \qquad \mu_k(t_1 t_2 \cdots t_k t_{k+1} \cdots t_{n-1}) = t_1 t_2 \cdots t_{k+1}^{t_k} t_k \cdots t_{n-1}, \text{ where } t^s = sts^{-1}.$$

It turns out that clusters and the cluster exchange graph can be modeled as certain special "two-part" factorizations of a Coxeter element into reflections [ST13, STW15]—and
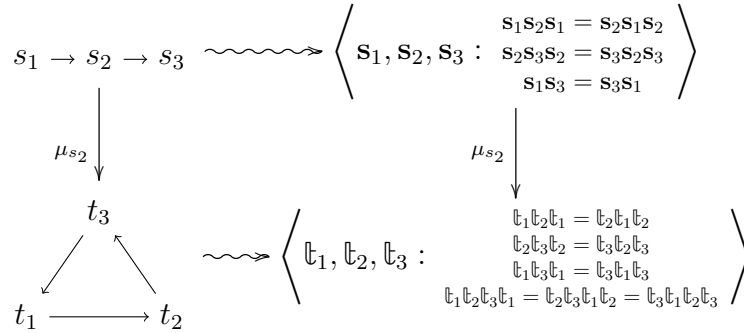
$$s_1 \to s_2 \to s_3 \quad \rightsquigarrow \quad \left\langle \mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3 : \begin{array}{c} \mathbf{s}_1\mathbf{s}_2\mathbf{s}_1 = \mathbf{s}_2\mathbf{s}_1\mathbf{s}_2 \\ \mathbf{s}_2\mathbf{s}_3\mathbf{s}_2 = \mathbf{s}_3\mathbf{s}_2\mathbf{s}_3 \\ \mathbf{s}_1\mathbf{s}_3 = \mathbf{s}_3\mathbf{s}_1 \end{array} \right\rangle$$

$$\mu_{s_2} \downarrow \qquad\qquad \mu_{s_2} \downarrow$$

$$\left\langle \mathbb{t}_1, \mathbb{t}_2, \mathbb{t}_3 : \begin{array}{c} \mathbb{t}_1\mathbb{t}_2\mathbb{t}_1 = \mathbb{t}_2\mathbb{t}_1\mathbb{t}_2 \\ \mathbb{t}_2\mathbb{t}_3\mathbb{t}_2 = \mathbb{t}_3\mathbb{t}_2\mathbb{t}_3 \\ \mathbb{t}_1\mathbb{t}_3\mathbb{t}_1 = \mathbb{t}_3\mathbb{t}_1\mathbb{t}_3 \\ \mathbb{t}_1\mathbb{t}_2\mathbb{t}_3\mathbb{t}_1 = \mathbb{t}_2\mathbb{t}_3\mathbb{t}_1\mathbb{t}_2 = \mathbb{t}_3\mathbb{t}_1\mathbb{t}_2\mathbb{t}_3 \end{array} \right\rangle$$

FIGURE 3. The top line illustrates how the Dynkin diagram of type A_3 encodes a presentation of the braid group B_3. The left column gives an example of mutation between two quivers; the right column gives the corresponding group presentations for B_3 derived from these quivers.

mutation factors as a composition of a sequence of Hurwitz moves on the factorization. It is therefore natural to wonder if any factorization of the Coxeter element encodes a presentation of the braid group. This summer, my REU student Reed Hubbard and I proved that this was the case, generalizing the Grant-Marsh result to a much larger family of presentations [Del74, Rea08, Bes03].

Theorem 10 (Hubbard, W. 2018). *Factorizations of Coxeter elements in a finite simply-laced Coxeter group encode presentations of $B(W)$.*

In generalizing, our theorem explains results in [BM15, GM17, HHLP17] in the context of Bessis's dual braid monoid [Bes03], using the beautiful combinatorics on the factorizations. Our proof is based upon the theorem that any factorization can be transformed to any other by Hurwitz moves—one must then check that certain obvious maps between two groups whose underlying factorization differ by a single Hurwitz move are homomorphisms. A natural extension is to tackle other reflection groups.

Problem 11. Extend Theorem 10 to all finite Coxeter groups, to finite complex reflection groups, and even to infinite Coxeter groups.

The same proof technique should continue to work for the Hurwitz orbit of a factorization (for complex reflection groups, use the presentations of Michel-Malle-Rouqier, and Bessis-Michel as a starting point [BMR97, BM04]), but the subtlety will lie in identifying the correct definition of the presentations.

6. Prior Support: Not Applicable

I have not held an NSF grant before.

7. Intellectual Merit

My research is in algebraic combinatorics, with a broad interest in motivation from other areas of mathematics such as Lie theory, geometric group theory and Artin/braid groups, and reflection groups. The proposed research project draws on connections between geometric group theory and Coxeter-Catalan combinatorics to produce elegant and useful combinatorics.

I believe that my research has had a positive effect on the combinatorics community, and many results have applied to research problems outside of the context in which they originally arose. I have a record of producing problems and research areas accessible to beginning researchers, and I have been selected to give talks on my research at FPSAC in 2012, 2015, 2016, 2017, and 2018.

My work with Striker in [SW12] has served as a catalyst for the involvement of undergraduate and young graduate students in cutting-edge research at REUs and doctoral programs—there were many developments motivated by the appearance of our paper [SW12]: [CHHM15, EP13, EFG⁺15, Had14, Hop16, GR14, GR15, GR16, PR15, Rob16, RS13, RW15, Rus16, DPS17, Str15, Str16]. In 2015, Striker, Propp, Roby and I organized an AIM workshop that launched a new field of combinatorics that J. Propp has termed "Dynamical Algebraic Combinatorics", and many papers have resulted from and been inspired by our workshop, including [DPS17, EFG⁺15, JR17, STWW17, HMP16, GHMP17b, GHMP17a, GP17]. We organized a successful session at the Joint Mathematics Meetings in 2018, and J. Striker and M. Arnold and I are organizing an AMS special session in Hawaii in spring 2019. I revisited this area with Thomas this past year in two papers [TW17, TW18a].

My work with Hamaker, Patrias, and Pechenik [HPPW16]—using K-theoretic Schubert calculus to resolve a long-standing open bijective problem involving plane partitions—led to two separate REU projects over the past two years: one at Morrow's REU at the University of Washington mentored by Hamaker and Griffith, and one supervised by Pechenik [BHK16, BHK17]. One of these REU projects resulted in a FSPAC poster.

My work with Thomas inverting sweep and zeta maps [TW18b] solved a long-standing problem in the field of diagonal coinvariants, and has already found applications outside of the field [HV17, Proposition 4.4]. Our follow-up project extending this work to resolving conjectures from [GMV16] has led to further interesting problems related to the Littleman path model and random walks on weight lattices of simple Lie algebras [LLP12].

8. Broader Impacts

8.1. **Education and Service.** Although I have only been in my current position at the University of Texas at Dallas for one year, I have already supervised independent coursework with Austin Marstaller (masters student), the undergraduate honors thesis of Kevin Zimmer (undergraduate), and an REU student Reed Hubbard last summer (undergraduate). I now have a Ph.D. student Amit Kaushal who is currently pursuing research with me—funding will allow me to support Amit's research past the initial two

years of coursework. As a service to the dynamical systems community at UTDallas (and to my own students), I began a representation theory seminar this fall, for which I provide written notes; we typically have four graduate students and four professors in attendance.

Other activities over the last year include organizing the Graduate Student Combinatorics Conference at UT Dallas in 2018 (which hosted over 75 graduate students from around the country) and appearing as a mathematical consultant in a televised report (WFAA) regarding the NCAA basketball bracket, which since aired in over 15 cities nationwide. I am interested in continuing to increase the visibility and participation of women in mathematics at UT Dallas by establishing an AWM chapter. As the only combinatorialist at UT Dallas, I have also designed new undergraduate and graduate courses in combinatorics. I have already had one combinatorics course approved to be part of the curriculum for our new bachelor's program in Data Science.

8.2. **Mentoring and REUs.** Because of its many elementary problems, combinatorics is a discipline in which undergraduate and graduate students can immediately become involved in research-level mathematics. I have formulated a large interconnected library of concrete combinatorial problems especially suitable for graduate and undergraduate students, and I have substantial past experience in involving students and underrepresented students in research. I will continue to seek out such opportunities with the goal to eventually build a strong combinatorics program at UT Dallas; for example, our current Pioneer REU program only allows for four students.

- Beginning in 2018, I am now the Ph.D. advisor of Amit Kaushal.
- In 2018, I mentored a summer Pioneer REU student Reed Hubbard, who produced some high-quality research on braid groups that we will submit for publication.
- In 2018, I mentored Kevin Zimmer's undergraduate honors on fixed point theorems in algebraic combinatorics (he recieved distinction).
- In 2017 and 2018, I pursued an independent study course with Austin Marstaller on the classification of root systems
- In 2016, I co-mentored Florence Maas-Gariepy on a research/study project involving finite reflection groups, which led to her project report (in French) being featured on the funding agency's website [MG16].
- In 2014, I mentored Stephanie Schanack, Fatiha Djermane, and Sarah Ouahib on an original research problem involving the characterization of the fixed points of a certain combinatorial set under a cyclic group action. I guided them through a case-by-case analyses which the three wrote up (in French) [SSD14].

I was also involved with the very successful combinatorics Research Experience for Undergraduates (REU) at the University of Minnesota:

- At the 2011 REU, I provided support to David B Rush and XiaoLin Shi [RS13], who found a generalization of my work in [SW12].
- For the 2010 Minnesota REU, I helped direct Gaku Liu's research in partition identities [Liu] and helped a second group formulate and computationally test conjectures on a combinatorial reformulation of the four-color theorem [CSS14].

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