PROJECT DESCRIPTION

INTRODUCTION

Dynamical algebraic combinatorics (DAC) is a relatively new field pioneered by the PI. DAC extends classical enumerative combinatorics to accommodate group actions—well-behaved actions hint at deep connections between combinatorial objects and other, more algebraic, constructions (such as integrable systems, bases in quantum groups, or cluster variables in cluster algebras). Conceptual proofs often exploit these connections and have led to fruitful interchanges between combinatorics, representation theory, and algebraic geometry.

Rowmotion is a ubiquitous action that has been discovered and generalized in many disparate settings—it serves as a defining property of semidistributive lattices, it is related to the Auslander-Reiten translate of heritary algebras, it recovers the Kreweras complement on noncrossing partitions, it recovers K-theoretic promotion on increasing tableaux and its piecewise-linear generalization has the same orbit structure as promotion on semistandard tableaux, etc. This proposal builds on the PI's work of *independence posets*, *independence polytopes*, and *semidistrim lattices* to unify and generalize these settings in several directions, including piecewise-linear and birational directions, as well as a lattice-theoretic directions.
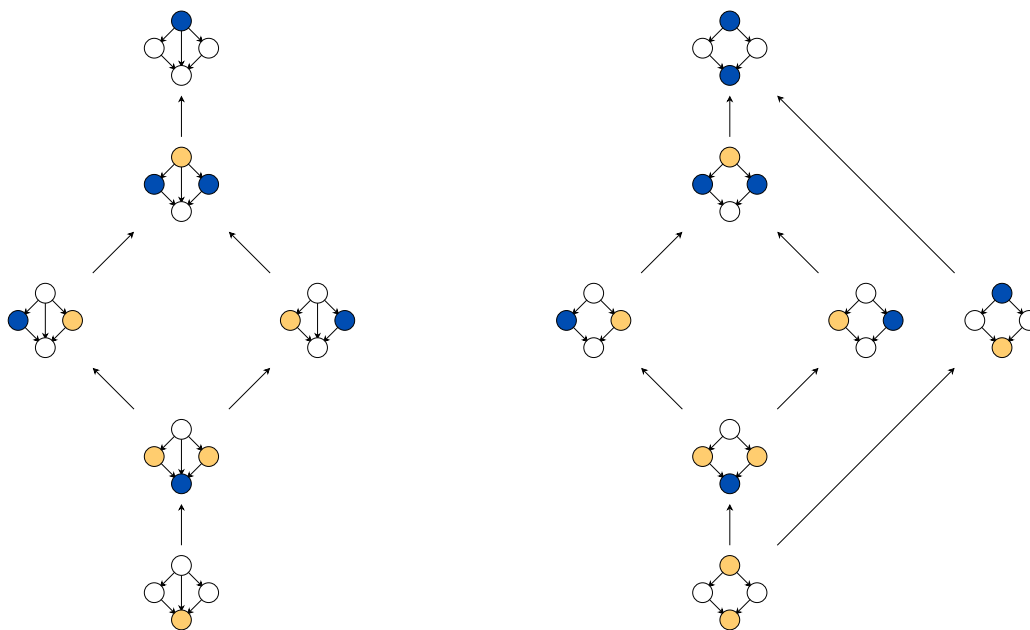

FIGURE 1. *Left:* independent sets in the comparability graph of the product of two chains arranged as an independence poset. *Right:* the independence poset obtained by removing a single edge from this comparability graph.

The PI has already laid some of the theoretical groundwork underpinning this proposal in the two recent publications with Hugh Thomas [TW19b, TW19a], as well as very recent work with Colin Defant [DW21a, DW21b]. Over the course of this research,

the PI has developed an original toolkit and perspective that has yielded substantial new progress in related fields. Based on this new perspective, the PI has created an interconnected library of concrete combinatorial problems especially suitable for early-stage students.

The specific goals of the proposed research include:

- Relate independence posets with integral points in dilations of polytopes constructed by Chvátal [Chv75], generalizing P-partitions from distributive lattices to independence posets [Sta86]; generalize independence posets to *any* convex lattice polytopes;
- Research a new lattice family that simultaneously generalizes semidistributive and trim lattices, proving that it preserves many properties common to both;
- Exploit the monotone structure on independent sets provided by independence posets to extend the applicability of the Propp-Wilson coupling from the past algorithm to randomly sample independent sets [PW98] (Section 5);
- Find new examples of relevance to Dynamical Algebraic Combinatorics; i.e., combinatorial, piecewise-linear, and birational rowmotion periodicity in the generalized setting of independence posets [Rob16, Str17] (Section 3).

1. Rowmotion

Rowmotion was introduced by Duchet in [Duc74]; studied for the Boolean lattice (and the product of two chains) by Brouwer and Schrijver [BS74, Bro75]; and (still for the Boolean lattice) related to matroid theory by Deza and Fukuda [DF90]. Cameron and Fon-der-Flaass considered rowmotion on the product of two and then three chains [FDF93, CFDF95]. Its study then apparently lay dormant for over a decade until Panyushev resurrected it in the form of a series of conjectures coming from Lie theory [Pan09]. The focus then shifted to finding equivariant bijections to natural combinatorial objects, and Stanley and Thomas completely characterized the orbit structure of rowmotion on the product of two chains combinatorially (using the Stanley-Thomas word) [Sta09]. Striker and the PI unified and extended various results by relating rowmotion to *jeu-de-taquin* and made terminological innovations to the theory [SW12]. This popularization of rowmotion led to a swell of related work falling under Propp's heading of *dynamical algebraic combinatorics*.

Work of Dilks, Pechenik, Striker, and later Vorland connected rowmotion to Thomas and Yong's K-theoretic *jeu-de-taquin*, developed to compute structure coefficients in K-theoretic Schubert calculus [DPS17, DSV19a]. The quasi-periodicity (under the name *resonance*) of K-theoretic promotion of rectangular tableau was studied by Dilks, Pechenik, and Stiker using the relationship to rowmotion on plane partitions [DPS17]; this relationship was exploited in the other direction by Patrias and Pechenik to resolve a long-standing conjecture of Cameron and Fon-Der-Flaass [PP20]. The relationship to K-theoretic slides was later picked up by Dao, Wellman, Yost-Wolff, and Zhang [DWYWZ20] via a bijection of Hamaker, Patrias, Pechenik, and the PI between plane partitions of trapezoidal and rectangular posets [HPPW20].

Motivated by Berenstein and Kirillov's piecewise-linear (PL) Bender-Knuth involutions on Gelfand-Tsetlin patterns [KB96], Einstein and Propp considered a PL-lifting of rowmotion to the order polytope of a poset [EP13, EP14]. Einstein and

Propp [EP14] (and Hopkins [H+20, Appendix A]) elucidated the connection between PL-rowmotion on rectangular plane partitions and promotion of rectangular semistandard Young tableaux, further solidifying the representation-theoretic connections. Thus, while PL-rowmotion on plane partitions recovers promotion of semistandard tableaux, rowmotion recovers K-theoretic promotion on increasing tableaux.

The most general combinatorial approach to date that encompasses the varied settings of the study of rowmotion is a novel notion of *independence polytope*, developed independently by the PI, which extends the earlier notion of *independence posets* introduced jointly by the PI and Hugh Thomas in [TW19a]. A separate generalization in a lattice-theoretic direction has been recently considered by the PI with Colin Defant.

2. Independence Posets

The definitions found below can be explored using the PI's FPSAC 2020 online interactive poster [TW20b]; Figure 1 illustrates two simple examples. Let G be a finite acyclic directed graph (without oriented cycles, loops, or multiple edges) so that the transitive closure of G admits a partial order on its vertices called *G-order*. An *independent set $\mathcal{I} \subseteq G$* is a set of pairwise non-adjacent vertices of G. *Independence posets*, introduced by the PI in [TW19a], are a certain partial ordering on the independent sets of G—depending on the orientation of G—whose cover relations are given by a novel definition of (non-local) *flips*.

As a more involved example, an independence poset structure is given in Figure 2 on the set partitions of $\{1, 2, 3, 4\}$. This poset is not a lattice; adding a single directed edge to the underlying graph from the vertex (13) to the vertex (24) recovers the well-known Tamari lattice on the 14 noncrossing partitions on $\{1, 2, 3, 4\}$.
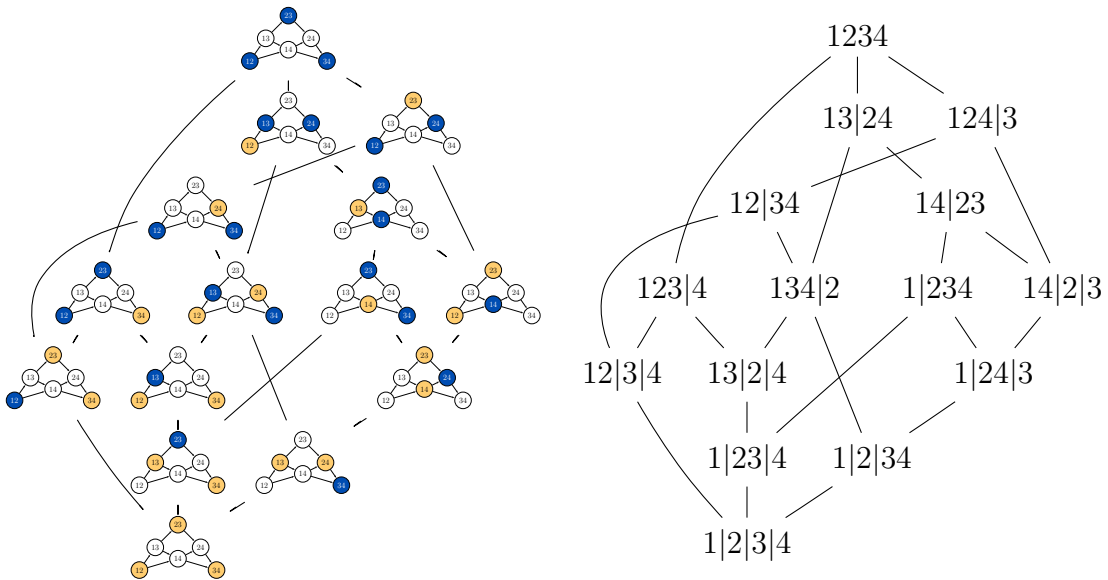

FIGURE 2. *Left:* The 15 set partitions of $\{1, 2, 3, 4\}$ as tight orthogonal pairs of an acyclic digraph G. *Right:* the corresponding independence poset on set partitions. Note that this poset is *not* a lattice (the elements 1|23|4 and 1|2|34 do not have a unique join).

The notion of "independence poset" is a natural generalization of that of "distributive lattice," but where the lattice requirement is eliminated. An independence poset that is a graded lattice is a distributive lattice. Many other well-known posets (such as Tamari and Cambrian lattices) turn out to be special cases of independence posets.

Definition 1. A pair $(\mathscr{D}, \mathscr{U})$ of disjoint independent sets of G is called *orthogonal* if there is no edge in G from an element of $\mathscr{D}$ to an element of $\mathscr{U}$. An orthogonal pair of independent sets $(\mathscr{D}, \mathscr{U})$ is called *tight* if whenever any element of $\mathscr{D}$ is increased (that is, removed and replaced by a larger element with respect to G-order) or any element of $\mathscr{U}$ is decreased, or a new element is added to either $\mathscr{D}$ or $\mathscr{U}$, then the result is no longer an orthogonal pair of independent sets.

We write $\mathrm{top}(G)$ for the set of all **t**ight **o**rthogonal **p**airs of G. One can show that for any independent set $\mathscr{I}$, there is a unique $(\mathscr{I}, \mathscr{U}) \in \mathrm{top}(G)$ and a unique $(\mathscr{D}, \mathscr{I}) \in \mathrm{top}(G)$. *Rowmotion* is defined as the map that sends an independent set $\mathscr{D}$ to $\mathscr{U}$, where $(\mathscr{D}, \mathscr{U}) \in \mathrm{top}(G)$.

Tight orthogonal pairs allow us to define a non-local *flip* operation, which generate the cover relations of a partial order which we call the *independence poset*.

Definition 2. The *flip* of $(\mathscr{D}, \mathscr{U}) \in \mathrm{top}(G)$ at an element $g \in G$ is the tight orthogonal pair $\mathrm{flip}_g(\mathscr{D}, \mathscr{U})$ defined as follows: if $g \notin \mathscr{D}$ and $g \notin \mathscr{U}$, the flip does nothing. Otherwise, preserve all elements of $\mathscr{D}$ that are not less than g and all elements of $\mathscr{U}$ that are not greater than g (and delete all other elements); after switching the set to which g belongs, then greedily add elements to $\mathscr{D}$ and $\mathscr{U}$ (respecting the conditions to form an orthogonal pair) in reverse G-order and G-order, respectively.

Figure 3 illustrates a flip on a top in an orientation of $[7] \times [7]$. The *independence relations* on $\mathrm{top}(G)$ are the reflexive and transitive closure of the relations $(\mathscr{D}, \mathscr{U}) < (\mathscr{D}', \mathscr{U}')$ if there is some $g \in \mathscr{U}$ such that $\mathrm{flip}_g(\mathscr{D}, \mathscr{U}) = (\mathscr{D}', \mathscr{U}')$.

Theorem 3. *Independence relations are antisymmetric, and hence define an* independence poset, *denoted* $\mathrm{top}(G)$. *Flips and cover relations of* $\mathrm{top}(G)$ *coincide.*

Birkhoff's *fundamental theorem of finite distributive lattices* proves that finite distributive lattices are parametrized by finite posets P (as the lattice $J(P)$ of order ideals under inclusion). Independence posets generalize Birkhoff's theorem: they are parametrized by acyclic directed graphs and their elements are independent sets.

Problem 1. Systematically investigate independence posets and rowmotion from the point of view of dynamical algebraic combinatorics (simple order, homomesy, etc.).

In general, we expect that directed graphs coming from representation theory ought to have interesting behavior under these actions, including possible connections with cluster algebras, R-systems, and various generalizations of periodicity. Recently Sam Hopkins—with the help of Ira Gessel—found a new class of poset with connections to certain colored A_3 webs.

Since distributive lattices $J(P)$ are recovered by independence posets when $G = \mathrm{Comp}(P)$ is the comparability graph of the poset P (antichains in P become independent sets of $\mathrm{Comp}(P)$), many classical combinatorial objects (to name a few: integer partitions in a box, various classes of plane partitions, domino tilings, stable marriages, alternating sign matrices, and minuscule lattices) can all be represented using
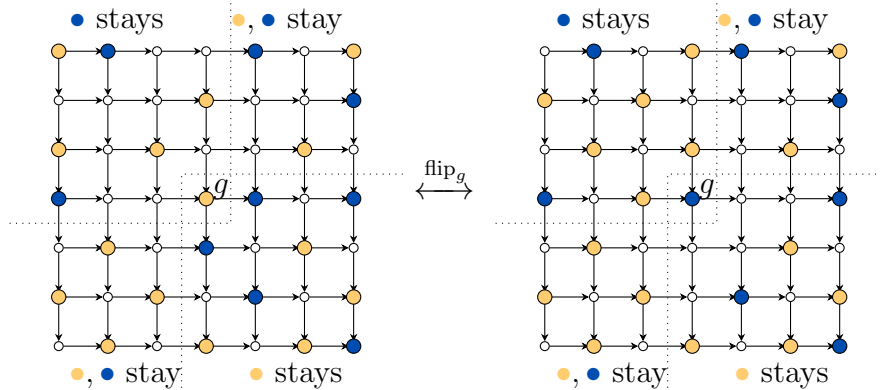
FIGURE 3. A flip on a top $(\mathscr{D}, \mathscr{U})$ in the 7×7 grid oriented from top left to bottom right. Flipping at the vertex g changes its color, and divides the grid into 5 connected regions (delineated by the dotted lines): the blue vertices not less than g (i.e., not in the bottom right) and the orange vertices not greater than g (i.e., not in the top left) are preserved by the flip. The orange vertices in the top left are filled in greedily from bottom right to top left; the blue vertices in the bottom right are filled in greedily from top left to bottom right.

independence posets. Placing different acyclic orientations on the comparability graph gives new partial orders—different from the one obtained from their distributive lattice structure—on these classical objects.

The real interest is that many objects in combinatorics can be encoded as independent sets of particular graphs. Using the framework of independence posets, these objects can now be endowed with a wide variety of new partial orders. For example, we can now obtain new orientations on Coxeter–Catalan objects (such as trees or triangulations), including both noncrossing and nonnesting objects (generalizing Cambrian lattices), as well as on the intersection lattice of a real central hyperplane arrangement (including, for example, set partitions as the intersections of the type A braid arrangement, as in Figure 2 for the the 15 set partitions of $\{1, 2, 3, 4\}$ with 126 possible independence posets depending on the acyclic orientation chosen).

Problem 2. Find a collection of "local moves" on digraphs (preserving the number of independent sets) to give a framework for producing useful new bijections between combinatorial objects.

For example, the graphs ◇ and ⟨→⟩ have the same number of independent sets (one graph encodes the seven 3×3 alternating sign matrices, the other the corresponding set of totally symmetric self-complementary plane partitions). Useful places to search for such generalized moves from Auslander-Reiten quivers, derived categories, and the theory of heaps.

3. INDEPENDENCE POLYTOPES

In this section we discuss a generalization of independence posets that allows for the efficient generation of all lattice points inside convex lattice polytopes. In particular,

we propose a piecewise-linear generalizations of independence posets, thereby providing a generalization of the construction of P-partitions. Parts of this work are currently in progress with the PI's graduate student, Amit Kaushal.

For G an acyclic directed graph, we conjecture in Problem 3 that a certain PL-generalization of flips given in Definition 5 defines the cover relations in a partial order $\text{top}^{(m)}(G)$ on the integer points in the m-fold dilation of Chvátal's *independence polytope* $C(G)$. Order ideals of a poset have a natural generalization to the theory of P-partitions [GHL$^+$16], which (after a piecewise-linear transfer map) can be interpreted as the lattice points inside of a certain polytope called the *chain polytope*. Given a set $X = \{x_1, \ldots, x_n\}$, we write $\mathbb{R}^X$ for the set of functions $f : X \to \mathbb{R}$. For a poset P, the *chain polytope* in $\mathbb{R}^P$ is defined as the set of points $f \in \mathbb{R}^P$ satisfying the inequalities $0 \le f(p)$ for all $p \in P$ and $\sum_{i=1}^k f(p_i) \le 1$ for any chain $p_1 < \cdots < p_k$ in P. Stanley proved that the chain polytope is the convex hull of the characteristic functions of antichains of P [Sta86]. In fact (as Stanley remarks), this is a special case of a beautiful construction of Chvátal [Chv75]. Replacing order ideals of P by antichains in the comparability graph $G = \text{Comp}(P)$ leads to the definition of the polytope $\mathscr{C}(G)$ as the set of points $f \in \mathbb{R}^G$ satisfying the inequalities

$$(1) \qquad 0 \le f(g) \text{ for all } g \in G \text{ and } \sum_{g \in C} f(g) \le 1 \text{ for any clique } C \subseteq G.$$

We call this the *independence polytope* of G. On the combinatorial side, the number of lattice points inside the m-fold dilation of the chain polytope is given by the number of multichains of order ideals $\emptyset = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_{m+1} = P$ in $J(P)$, or equivalently by $J(P \times [m])$. Since $\text{top}(\text{Comp}(P)) \simeq J(P)$, it is natural to search for a definition of the poset "$\text{top}(G \times [m])$"—a partial order on the integer points in $m\mathscr{C}(G)$ that recovers $J(P \times [m])$ for $G = \text{Comp}(P)$.

The PI has recently defined what appears to be the correct generalization—the PI has written Sage code to confirm this, and the task remains to prove these results. Rather than first defining the tight orthogonal pairs $\text{top}(G)$ and then using these to define rowmotion, we *first* define rowmotion and then use rowmotion to give the correct generalization of $\text{top}(G)$.

Definition 4. Given $\mathscr{D} \in m\mathscr{C}(G)$, define *PL-toggle* operators $\text{tog}_g^{(m)} : m\mathscr{C}(G) \to m\mathscr{C}(G)$ by

$$(2) \qquad \text{tog}_g^{(m)}(\mathscr{D}(x)) = \begin{cases} \mathscr{D}(x) & \text{if } x \neq g \\ m - \max_{\substack{C \text{ a clique} \\ g \in C}} \sum_{h \in C} \mathscr{D}(h) & \text{otherwise} \end{cases} \cdot$$

Rowmotion is the operator $\text{row}^{(m)} : m\mathscr{C}(G) \to m\mathscr{C}(G)$

$$\text{row}^{(m)}(\mathscr{D}) = \prod_{g \in G} \text{tog}_g^{(m)}(\mathscr{D}),$$

where the product is in G-order.

Definition 4 matches the combinatorial and piecewise-linear definitions of rowmotion for $G = \text{Comp}(P)$ [SW12, EP13, Jos19, JR20]).

Definition 5. The directed graph $\text{top}^{(m)}(G)$ has vertices that are pairs $(\mathscr{D}, \text{row}^{(m)}(\mathscr{D}))$ for an integer point $\mathscr{D} \in m\mathscr{C}(G) \cap \mathbb{Z}^G$. Its directed edges are defined using a PL-generalization of *flips*: for a vertex $g \in G$, subtract 1 from $\mathscr{U}(g)$ and add one to $\mathscr{D}(g)$ (if possible), and fill in the remainder of $\mathscr{U}$ and $\mathscr{D}$ above and below g in G-order using PL-toggles (exactly as in Definition 2).

An example is given in Figure 4; this definition has been coded in Sage although no theoretical properties have yet been proven.
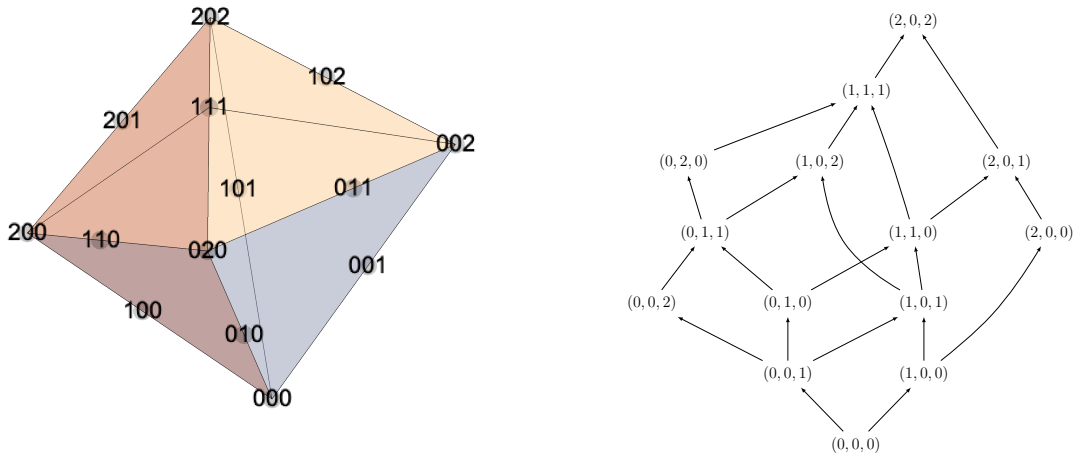

FIGURE 4. *Left:* the two-fold dilation of the independence polytope for $G = 1 \rightarrow 2 \rightarrow 3$, with its 14 integer points labeled. *Right:* the same 14 lattice points in the generalized independence poset $\text{top}^{(2)}(G)$.

Problem 3.
- Prove that $\text{top}^{(m)}(G)$ defines a partial order on the integer points in the polytope $\mathscr{C}(G)$ with cover relations given by PL-flips with unique minimal element $(0, \text{row}^{(m)}(0))$.
- Prove that $\text{top}^{(m)}(\text{comp}(P))$ recovers the distributive lattice structure on the integer points on Stanley's chain polytope.
- Generalize central properties of P-partitions and Stanley's chain polytopes to $\text{top}^{(m)}(G)$ and $\mathscr{C}(G)$. For example, a triangulation of $\mathscr{C}(G)$ should suggest a notion of "linear extensions" for independence posets, and then ought to give a formula for the Ehrhart generating function of $\mathscr{C}(G)$.
- Show that the polytopal interpretation of $\text{top}^{(m)}(G)$ gives an efficient algorithm to generate the lattice points in $m\mathscr{C}(G)$.
- Extend existing problems (such as Problem 1) and theorems from $\text{top}(G)$ to $\text{top}^{(m)}(G)$.

4. Semidistrim Lattices

In this section we explain some preliminary work with Colin Defant. Birkhoff's well-known *fundamental theorem of finite distributive lattices* proves that finite distributive lattices are parametrized by finite posets P (as the lattice $J(P)$ of order

ideals under inclusion). Markowsky (a student of Birkhoff) generalized Birkhoff's theorem to a lesser-known representation theorem for finite *extremal lattices*—that is, lattices whose longest chain is equal to both the number of join irreducible elements and meet irreducible elements—showing that finite extremal lattices are parametrized by (finite) acyclic graphs G (as the lattice of maximal orthogonal pairs of G under inclusion) [Mar92b].[1]

Independence posets are a different generalization of Birkhoff's theorem: although independence posets (like extremal lattices) are still parametrized by acyclic directed graphs, their elements are independent sets, rather than the more technical "maximal orthogonal pairs". When an independence set happens to be a lattice, then it is a special kind of extremal lattice called a *trim lattice* (which then admit a canonical labeling of cover relations by join and meet irreducibles [TW19b]).

Theorem 6. *If* $\mathrm{top}(G)$ *is a lattice, then it is a trim lattice. Every trim lattice can be realized as* $\mathrm{top}(G)$ *for a unique (up to isomorphism) acyclic directed graph* G.

Recall that a *semidistributive lattice* is a lattice such that if when $x, y, z \in L$ satisfy $x \vee y = x \vee z$ then $x \vee (y \wedge z) = x \vee y$ and also if when $x, y, z \in L$ satisfy $x \wedge y = x \wedge z$ then $x \wedge (y \vee z) = x \wedge y$. Although trim lattices and semidistributive lattices are distinct families of lattices (a semidistributive lattice that is not trim and a trim lattice that is not semidistributive are illustrated on the left and in the middle of Figure 5), they share many common properties:

- intervals are again trim/semidistributive;
- there is a canonical bijection between join- and meet-irreducible elements;
- cover relations are canonically labeled by a join-irreducible element;
- elements are specified by the set of labels of their down- and up-covers; and
- the set of downward labels equals the set of upward labels, and these are independent sets in a certain graph.

As trim lattices and semidistributive lattices share so many properties, it is reasonable to suspect there is a simultaneous generalization of both. In ongoing work with Colin Defant, we propose *semidistrim lattices* as a common generalization, a kind of "least upper bound" of trim and semidistributive lattices. Our semidistrim lattices are analogous to interval-dismantlable lattices that additionally require a certain compatibility of join- and meet-irreducible elements. An example of a semidistrim lattice that is neither semidistributive nor trim is illustrated on the right of Figure 5. Cover relations is a semidistrim lattice are labeled by join/meet-irreducible elements, and elements x in a semidistrim lattice are the join of their downward labels, the meet of their upward labels, and a subset of their upward (or downward) labels are independent sets in a related graph.

For $j \in \mathcal{J}$, write j^* for its unique covered element and define $\mathcal{M}(j)$ to be the set of maximal elements from the set $\{x : j \wedge x = j^*\}$. For $m \in \mathcal{M}$, write m^* for its unique covering element and define $\mathcal{J}(m)$ to be the set of minimal elements from the set $\{x : m \vee x = m^*\}$. $\mathcal{M}(j)$ has a unique element for all $j \in \mathcal{J}$ iff L is

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[1]In fact, Markowksy's work included a representation theorem for *any* finite lattice, previously anticipated by Barbut [Bar65]; these ideas were later developed by Wille in the guise of *Formal Concept Analysis*. [Wil82].
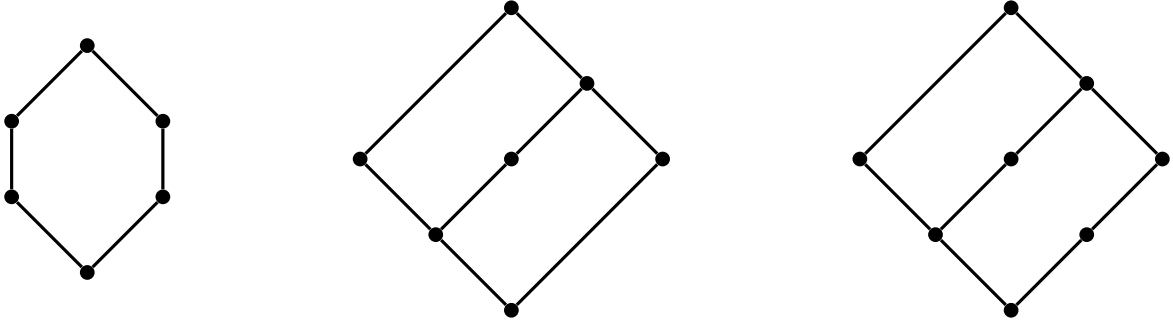
FIGURE 5. *Left*: A semidistributive lattice that isn't trim. *Middle*: A trim lattice that isn't semidistributive. *Right*: A semidistrim lattice that is neither trim nor semidistributive.

meet-semidistributive, while $\mathcal{J}(m)$ has a unique element for all $m \in \mathcal{M}$ iff L is join-semidistributive. It turns out that $\mathcal{J}(m)$ only contains join-irreducible elements, while $\mathcal{M}(j)$ only contains meet-irreducible elements.

Definition 7. Say that L is *paired* if there is a unique bijection row $: \mathcal{J} \to \mathcal{M}$ so that $\mathrm{row}(j) \in \mathcal{M}(j)$ and $\mathrm{row}^{-1}(m) \in \mathcal{J}(m)$.

Note that in a paired lattice, $j \not\leq \mathrm{row}(j)$—for if $j \leq m$, then $j \wedge m = j \neq j^*$ and $m \vee j = m \neq m^*$ so that $j \notin \mathcal{J}(m)$ and $m \notin \mathcal{M}(j)$, contradicting that j and m were paired. In fact, if $m = \mathrm{row}(j)$ and j are comparable, then it is in the following very specific way: if $m \leq j$, then $j \wedge m = m$ and $m \vee j = j$ so that $m = j^*$ and $j = m^*$.

Proposition 8. *Semidistributive and extremal lattices are paired.*

Even though $\mathcal{J}(m)$ and $\mathcal{M}(j)$ can contain multiple elements in extremal lattices, such lattices are still paired. An element $j \in L$ is called *join-prime* if for all $x, y \in L$, if $x \vee y \geq j$ then $x \geq j$ or $y \geq j$. Similarly, an element $m \in L$ is called *meet-prime* if for all $x, y \in L$, if $x \wedge y \leq m$ then $x \leq m$ or $y \leq m$. Join-prime elements are join-irreducible, and meet-prime elements are meet-irreducible, and each join-prime element j_0 has a corresponding meet-prime element m_0 such that $L = [\hat{0}, m_0] \sqcup [j_0, \hat{1}]$. We call the pair of join-prime and meet-prime elements (j_0, m_0) with $L = [\hat{0}, m_0] \sqcup [j_0, \hat{1}]$ a *prime pair*. By [Mar92a, Theorem 15], extremal lattices have a join-prime atom, while all atoms of semidistributive lattices are join-prime by [GN81, Lemma 1].

Proposition 9. *If L is a paired lattice with a prime pair (j_0, m_0), then $\mathrm{row}(j_0) = m_0$.*

The following definition is an analogue of interval-dismantlability for paired lattices that additionally requires a compatibility of join- and meet-irreducible elements.

Definition 10. A paired lattice L is *semidistrim* if it contains a prime pair (j_0, m_0) such that

- $[j_0, \hat{1}]$ is semidistrim and the map $j_0(j') = j_0 \vee j'$ is a bijection from $\{j' \in \mathcal{J} : j_0 \leq \mathrm{row}(j')\}$ to $\mathcal{J}_{[j_0, 1]}$ with $\mathrm{row}_{[j_0, 1]}(j_0 \vee j') = m'$ for every $m' \in [j_0, 1]$ with $\mathrm{row}^{-1}(m') = j'$, and
- $[\hat{0}, m_0]$ is semidistrim and the map $m_0(m') = m_0 \wedge m'$ is a bijection from $\{m' \in \mathcal{M} : \mathrm{row}^{-1}(m') \leq m_0\}$ to $\mathcal{M}_{[0, m_0]}$ with $\mathrm{row}_{[0, m_0]}(j') = m_0 \wedge m'$ for every $j' \in [0, m_0]$ with $\mathrm{row}(j') = m'$.

Theorem 11. *Semidistributive and trim lattices are semidistrim.*

Definition 12. The *Galois graph* G_L of a paired lattice L is the directed graph with vertices $\mathcal{J}$ and edges $j \to j'$ when $j \not\leq \operatorname{row}(j')$ and $j \neq j'$.

We write $j \to j'$ to mean that there is a directed edge from j to j' in G_L, and we define $\operatorname{Out}(j) = \{j' \in \mathcal{J} : j \to j'\}$ and $\operatorname{In}(j) = \{j' \in \mathcal{J} : j' \to j\}$. When L is paired, for $x \in L$ we write $J(x) = \{j \in \mathcal{J} : j \leq x\}$ and $M(x) = \{\operatorname{row}^{-1}(m) : m \in \mathcal{M}, m \geq x\}$.

Definition 13. A paired lattice L is *overlapping* if for every cover $x \lessdot y$ in L, $M(x) \cap J(y)$ contains a single element, which we denote j_{xy}. If L is overlapping, for $x \in L$ we define its set of *downward labels* $\mathcal{D}(x) = \{j_{yx} : y \lessdot x\}$ and its set of *upward labels* $\mathcal{U}(x) = \{j_{xy} : x \lessdot y\}$.

Note that in a paired lattice, we must have that $M(x) \cap J(x) = \emptyset$—for if $j \in M(x) \cap J(x)$ with $\operatorname{row}(j) = m$, then $j \leq x \leq m$, contradicting that the pair j and m must be incomparable. By [RST19, Lemma 4.4] and [TW19b, Theorem 3.4], both semidistributive and trim lattices are overlapping.

Theorem 14. *A semidistrim lattice is overlapping.*

Definition 15. Let L be overlapping. For $x \in L$ we define its set of *reduced downward labels*
$$\overline{\mathcal{D}}(x) = \{j \in \mathcal{D}(x) : \operatorname{In}(j) \cap \mathcal{D}(x) = \emptyset\}$$
and its set of *reduced upward labels*
$$\overline{\mathcal{U}}(x) = \{j \in \mathcal{U}(x) : \operatorname{Out}(j) \cap \mathcal{U}(x) = \emptyset\}.$$

Let $\mathcal{I}(G)$ denote the collection of independent sets of a directed graph G. If L is a semiditrim lattice and $x \in L$, then by definition $\overline{\mathcal{D}}(x)$ and $\overline{\mathcal{U}}(x)$ are independent sets of G_L. For semidistributive lattices, we have $\mathcal{D}(x) = \overline{\mathcal{D}}(x)$ because $\mathcal{D}(x)$ is a canonical join representation and is already an independent set in G_L [Bar19] (and similarly for $\mathcal{U}(x), \overline{\mathcal{U}}(x)$ and canoncial meet representations). Similarly, for trim lattices we have $\mathcal{D}(x) = \overline{\mathcal{D}}(x)$ because again $\mathcal{D}(x)$ is an independent set in G_L [TW19b, Corollary 5.6] (and similarly for $\mathcal{U}(x)$ and $\overline{\mathcal{U}}(x)$). On the other hand, for semidistrim lattices it can happen that $\mathcal{D}(x) \neq \overline{\mathcal{D}}(x)$ and $\mathcal{U}(x) \neq \overline{\mathcal{U}}(x)$.

Theorem 16. *Let L be semidistrim. Then any element $x \in L$ is uniquely determined by its reduced downward labels $\overline{\mathcal{D}}(x)$, and also by its reduced upward labels $\overline{\mathcal{U}}(x)$:*
$$x = \bigvee_{j \in \overline{\mathcal{D}}(x)} j = \bigwedge_{j \in \overline{\mathcal{U}}(x)} \operatorname{row}(j).$$

Theorem 17. *Let L be semidistrim. The maps $\overline{\mathcal{D}}_L \colon L \to \mathcal{I}(G_L)$ and $\overline{\mathcal{U}}_L \colon L \to \mathcal{I}(G_L)$ are bijections.*

The ability to associate elements in a semidistrim lattice with pairs of independent sets allows for a *rowmotion* to be defined: $\operatorname{row}(x)$ is defined to be the unique y with $\overline{\mathcal{D}}(y) = \overline{\mathcal{U}}(x)$.

Problem 4.   • If L is a semidistrim lattice with $\overline{\mathcal{D}}(x) = \mathcal{D}(x)$ and $\overline{\mathcal{U}}(x) = \mathcal{U}(x)$ for all $x \in L$, do other common properties of semidistributive and trim lattices also hold (for example, are intervals again semidistrim)?

- In the presence of cycles, is it still possible to define a local flip operation on the pair $(\overline{\mathscr{D}}(x), \overline{\mathscr{U}}(x))$?
- Does rowmotion on semidistrim lattices exhibit good dynamical algebraic combinatorial properties?

5. Random Sampling

In this section, we propose using independence posets to extend Propp and Wilson's *coupling from the past algorithm* from distributive lattice theory to independence sets of any graph.

The random sampling of independent sets (weighted by the number of vertices in the set) is termed the *hard-core model* in statistical mechanics. Given a graph G with independent sets $I(G)$ and a *fugacity* $\lambda > 0$, define the partition function

$$P_G(\lambda) = \sum_{a \in I(G)} \lambda^{|a|}.$$

Efficiently sampling independent sets according to this measure is a well-known problem that has only been solved in certain special cases (certain types of graphs, ex. comparability graphs of posets; graphs with low maximum vertex degree; etc.)—note that it is known to be NP-complete to determine the maximum size of an independent set of a graph G, which forces this problem to be intractable in general (take $\lambda \to \infty$).

Let δ denote the maximum degree of a graph G. We briefly summarize some of the known results. It is already #P-complete to count independent sets in graphs with $\delta = 3$; Glauber dynamics has a mixing time $O(n \ln n)$ when $\lambda < 2\delta - 2$; for $\lambda = 1$ and all $\Delta \geq 6$, Dyer, Frieze and Jerrum proved there exists a bipartite graph for which the mixing time of any Markov chain making only "local moves" is exponential (but recall that our flips from Definition 2 are highly *non-local*, so that this result does not apply!); Luby and Vigoda describe a Markov chain that approximately counts independent sets in graphs with $\delta \leq 4$ in polynomial time[LV97, LV99, HN98].

Propp and Wilson's *coupling from the past* (CFTP) algorithm allows for uniform sampling without knowing the mixing time of the underlying Markov chain. In general, CFTP requires as many instances to be run as states; in practice, an additional monotonicity assumption reduces the number of concurrent running instances to just two. Applications are numerous, but the most well-known example is to distributive lattices (which allows, for example, random sampling of domino tilings of an Aztec diamond). It is known that CFTP cannot be fast in general—the simple example of a complete bipartite graph already produces a bottleneck that forces the sampling to take exponential time—but the theoretical guarantees provided by CFTP make it an attractive method for sampling.

A trick due to Shor and Winkler encodes independent sets of *bipartite* graphs as order ideals in a corresponding distributive lattice, but no such trick is known in general: "For general (non-bipartite) graphs G there is no monotone structure which would allow one to use monotone CFTP"[LP17, 22.4]. But independence posets would seem to now provide such a structure, and it is worth at least running experiments to see if this structure provides an improvement over traditional Glauber/heat-bath dynamics.

Problem 5. Use independence posets to extend CFTP and its theoretical implications to the independent sets of any graph. A first step is to consider semidistributive lattices, and then semidistrim lattices. Produce computational and experimental evidence for the efficacy of this method.

6. Prior Support

The PI applied for and received the NSF conference award number 1801331 with title "Graduate Student Combinatorics Conference 2018," for an amount of $20,000 and period of support 3/1/18–2/28/19. **Intellectual Merit:** The GSCC has been an annual conference for graduate students in combinatorics since 2005. The 2018 GSCC focused on graduate student research presentations and included keynote addresses by four leading researchers in the field of combinatorics. **Broader Impacts:** UTD hosted over 70 outside graduate student participants at the conference. The 2018 GSCC provided a unique and invaluable opportunity for graduate students whose research focuses on combinatorics to experience the benefits of taking part in a research conference. No publications were produced under this award.

7. Intellectual Merit

The PI's research is in algebraic combinatorics, with a broad interest in motivation from other areas of mathematics such as Lie theory, geometric group theory, and reflection groups. The PI has a strong record of solving long-standing problems using an original toolkit and perspective: he has been selected to give six talks (only around 5% of submissions are accepted for talks) at the international conference Formal Power Series and Algebraic Combinatorics (FPSAC) and was an invited speaker at the 2020 Triangle Lectures in Combinatorics as well as Open Problems in Algebraic Combinatorics 2021 at the University of Minnesota. There have been many developments motivated by the appearance of the PI's paper [SW12]—to name a few: [CHHM15, EP13, EFG+15, Had14, Hop16, GR14, GR15, GR16, PR15, Rob16, RS13, RW15, Rus16, DPS17, Str15, Str16, JR18, MR19, DSV19b, Jos19, JR20, Hop20, JR20]. In 2015, the PI, Striker, Propp, and Roby organized an AIM workshop that launched a new field of combinatorics now termed "Dynamical Algebraic Combinatorics." This same group organized a follow-up BIRS online workshop in the Fall 2020 and the PI has additionally organized several successful AMS and JMM special sessions in this field. An integral part of this proposal is to continue supporting the PI's ongoing and future efforts to involve students in cutting-edge research in algebraic combinatorics and related areas.

The PI has already laid some of the theoretical groundwork underpinning this proposal in the two recent publications [TW19b, TW19a, DW21a, DW21b].

8. Broader Impacts

The PI has substantial past experience in involving students and underrepresented students in research: he has mentored undergraduate research over six different summers (at UTD, LaCIM, and UMN), supervised three honors theses at UTD, and he currently has two Ph.D. students pursuing thesis research in areas related to this proposal. The PI's related work in [SW12] has served as a catalyst for the involvement of

undergraduate and beginning graduate students in cutting-edge research at REUs and doctoral programs, and there have been many papers motivated by the appearance of [SW12]. The PI has a record of producing problems and research areas accessible to beginning researchers, including the now-active area of dynamical algebraic combinatorics. At least four of the PI's papers have independently led to Research Experience for Undergraduates (REU) projects at four different institutions. In 2015, the PI, Striker, Propp, and Roby organized an AIM workshop that launched a new field of combinatorics now termed "Dynamical Algebraic Combinatorics." This same group organized follow-up BIRS workshops in Fall 2020 (originally accepted in-person, but held online due to COVID-19; the PI took advantage of this to arrange for his undergraduate honors reading class to attend the workshop) and 2021. The PI has additionally organized several successful AMS and JMM special sessions in this field.

An integral part of this proposal is to continue supporting the PI's ongoing and future efforts to involve students in cutting-edge research in algebraic combinatorics and related areas. As the only combinatorialist at UTD, the PI has designed new undergraduate and graduate courses in combinatorics; due to the success of his undergraduate Discrete Math and Combinatorics class, the PI was asked by the honors college to teach honors reading courses in Fall 2019, 2020, and 2021. The PI has a history of service to the combinatorial community: he has refereed for over twenty journals, became an editor for Annals of Combinatorics in 2019, served on the program committee of FPSAC in 2019, serves currently on the organizing committee as the US funding coordinator, and has organized over ten conferences, workshops, and special sessions. He has represented the larger mathematical community to the public by appearing as a mathematical consultant in a 2018 nationally televised report (WFAA) regarding the NCAA basketball bracket, and hosting mathematical events at UTD (freshman orientations, MATHCOUNTS competitions, π-day events, etc.).

8.1. **Interactive JavaScript textbook.** During the COVID-19 pandemic, the PI experimented with novel methods to disseminate his research. The 2020 summer international conference Formal Power Series and Algebraic Combinatorics (FPSAC) was held online, and the PI used the opportunity of the remote poster session to develop a JavaScript browser-based interactive poster (see Figure 6 and [TW20b]). This poster was a highly successful experiment: the conference organizers selected it as an example for other presenters of how the online format could be harnessed to be even more engaging than a static in-person poster, and asked the PI for advice on how other presenters could develop a similar poster. The PI is currently developing a new interactive poster for his FPSAC 2021 submission.

The PI would like to build on this success by extending such interactive materials from his research to his undergraduate teaching by creating a **browser-based interactive discrete math textbook**. The PI designed a discrete math and combinatorics course as part of the new data science program at UTD. He has currently taught the course five times, and he would like to use the expertise he developed while creating interactive posters to render his notes of course content and classroom activities more engaging by using JavaScript to both animate concepts and allow students to interact with new definitions and proofs. Materials include introduction to proof, naive set theory, relations, introduction to algorithms, modular arithmetic, basic combinatorial
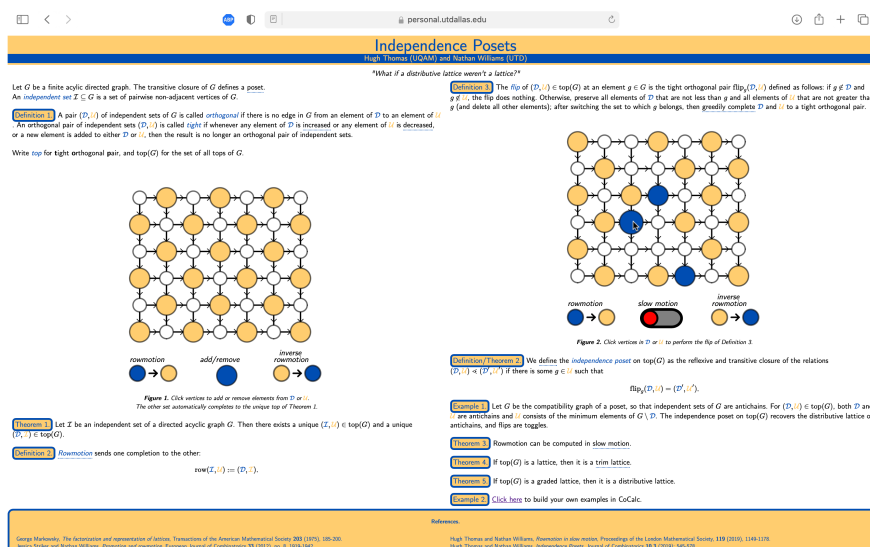
FIGURE 6. A screen shot of the PI's interactive poster presented at FPSAC 2020. Each of the grids is a JavaScript applet that allows the participant to experiment with various definitions, including Figure 3.

objects (combinations and permutations), recurrences, inclusion-exclusion, the cycle lemma, and trees. The PI has notes in TeX for this course, as well as lecture recordings and handwritten notes from the past two online semesters. An example of how classroom content could be made interactive is the following two-player game introducing inductive reasoning, typically clumsily played on paper: "There are nine coins. The players take turns, each of which consists of taking either one or two coins. A player loses if they can't take a coin. Do you want to go first or second, and why?" As a virtual exercise, students will be able to more easily play and experiment and more productively engage with the problem.

8.2. **Conferences and Workshops.** The PI has been very active in **organizing conferences and workshops**: • 2015 - week-long workshop at the American Institute of Mathematics • 2018 - Graduate Student Combinatorics Conference at UTD, with over 75 attendees (also obtaining $20,000 of NSF funding); • 2019 - FPSAC program committee • 2017–2021 - Organized four AMS special sessions on interactions between dynamical systems and combinatorics • 2018 - two-week "research-in-pairs" program at Oberwolfach • 2019 - two minisymposia on "Coinvariant Spaces and Parking Functions" at the SIAM Texas Louisiana Section at Southern Methodist University under the meta-organization of Sottile • 2020 and 2021 - BIRS workshops with Propp, Roby, and Striker on "Dynamical Algebraic Combinatorics" and • 2021–2023 - Member of the FPSAC organizing committee as US funding coordinator.

The PI intends to use his past experience in conference organization and research mentoring to set up a **yearly online workshop** with the goal of bringing together **early graduate and undergraduate students** (including the honors students in his honors reading courses, as well as Ph.D. students of the PI's collaborators).

8.3. **Mentoring.** The PI has substantial past experience in involving students and underrepresented students in research: this past Spring 2020, the PI supervised two undergraduate honors theses (both summitted for publication, one already accepted), and this past Summer 2021, the PI supervised two graduate students on a research project. He currently has two Ph.D. students (A. Kaushal and P. Palit) pursuing their thesis research in areas related to this proposal. The PI will continue to seek out such opportunities with the goal to eventually build a strong combinatorics program at UTD; part of this proposal includes summer funding for the PI to perform REU-like activities with graduate and undergraduates at UTD. While at UTD the PI has worked with **graduate students** in the following ways: • currently the **thesis advisor** of of P. Palit and A. Kaushal • organized the 2018 **Graduate Student Combinatorics Conference** • supervised several **independent study/research** courses with graduate students (Fall 2017, Spring 2019, Summer 2020, Summer 2021). While at UTD the PI has worked with **undergraduates** in the following ways: • Spring 2018 - Supervised K. Zimmer's **senior honors thesis** • Summer 2018 - Mentored rising senior R. Hubbard for eight weeks as part of the **Pioneer REU program** (now pursuing his Ph.D. at UNC Chapell Hill) • Spring 2019 - Supervised **independent research** with junior J. Marsh • Spring/Summer 2019 - Supervised **independent research** with undergraduates C. Kondor and M. Patten • Due to the success of the Discrete Math and Combinatorics course the PI designed for the new BS in Data Science program, the PI was asked by the honors college to teach an **honors reading course** in Fall 2019, 2020, and 2021. In Fall 2020, this reading class took part in the BIRS Dynamical Algebraic Combinatorics conference (held online due to COVID-19)• Spring 2020 - Supervised J. Marsh's **senior honors thesis** (now pursuing his Ph.D. studies at GA tech; submitted for publication) • Spring 2020 - supervised B. Cotton's **senior honors thesis** (accepted for publication). Further past experience involving undergraduate students in research includes two summers as an REU mentor at the University of Minnesota and two summers mentoring undergraduate students at LaCIM.

8.4. **Digital database of independence posets.** Since distributive lattices $J(P)$ are recovered by independence posets when $G = \text{Comp}(P)$ is the comparability graph of the poset P (antichains in P become independent sets of $\text{Comp}(P)$), many classical combinatorial objects (to name a few: integer partitions in a box, various classes of plane partitions, domino tilings, stable marriages, alternating sign matrices, and minuscule lattices) can be encoded as independent sets of particular graphs, and these objects can now be endowed with a wide variety of new partial orders using the framework of independence posets. For example, the graphs ⬦ and ⬦ each have 7 independent sets (one graph encodes the seven 3×3 alternating sign matrices, the other the corresponding set of totally symmetric self-complementary plane partitions). A broader impact of this proposal is to **compile a library of combinatorially-relevant graphs and their independent sets, and integrate them into Sage for widespread use**. The PI has written and made publicly available code for independence posets [TW20a].

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