PROJECT DESCRIPTION

The PI proposes a research program in algebraic combinatorics based on a diverse circle of ideas: braid Richardson varieties over finite fields, Hecke algebra traces, Lusztig's exotic Fourier transform, rational Cherednik algebras, and a new relationship between Deodhar decompositions and noncrossing combinatorics. This program is based on a general framework for producing new and interesting combinatorial results, leveraging braid varieties as a unifying tool. This framework has already proven successful in producing substantial new results: the PI's recent joint work (with Galashin, Lam, and Trinh) resolved two decades-long open problems in Coxeter–Catalan combinatorics [GLTW22]. The framework is built from three objects giving the same q-polynomial:

- the number of points in a particular variety over a finite field $\mathbb{F}_q$ (Definition 1.1);
- a trace of certain elements in a suitable Hecke algebra (Theorem 1.8); and
- a recursively defined R-polynomial (Section 4).

The first item produces combinatorial objects via the Deodhar decomposition and specializing $q \mapsto 1$ (Section 1.5); the second item allows the use of representation-theoretic techniques for proving enumerative formulas (Section 1.6); and the third item allows for computational experimentation. At different levels of generality, different techniques become available.

For **finite** Weyl and Coxeter groups (Section 2), it is possible to compute everything in a case-by-case manner using an explicit decomposition of the Hecke algebra (Equation (2.1)), and there are many interesting combinatorial and representation-theoretic problems that are open for immediate attack and resolution using these case-by-case methods by the PI and his students. Special classes of elements in finite type have favorable representation-theoretic properties that allow for uniform approaches using Lusztig's exotic Fourier transform, Springer theory, and graded modules of rational Cherednik algebras (Section 2.2). A special case of this has already led to the solution to two long-standing open problems in Coxeter–Catalan combinatorics (Theorem 2.5). We also propose an exciting new extension to complex reflection groups (Section 2.4).

For **affine** Weyl groups (Section 3), the main tool is a trace formula for translation elements due to Opdam (Theorem 3.2). In this setting, the proposed framework recovers some Tessler matrix identities due to Haglund [Hag11] (Theorem 3.4)—and the PI has formulated many other interesting conjectures of intermediate difficulty (Section 3), some of which can be attacked with Opdam's trace formula.

For **general** Weyl groups (Section 4), we are reduced to generic recursive and cluster-theoretic methods. These methods also apply in both the finite and affine cases, but require software implementation before further exploration is possible (Section 4)—and yet we are cautiously optimistic that we can make new progress in this case also.

These more general techniques have the advantage that they also provide a tool for studying braid varieties over $\mathbb{C}$, with the eventual aim of computing **mixed Hodge decompositions**. This direction is much more speculative (Section 5), with probable links to q, t-polynomials arising in the study of diagonal harmonics. The PI has previous experience with these q, t-polynomials through his work inverting the sweep map and the zeta map on parking functions [TW18, MTW17], and hopes to lift this experience to the noncrossing context of braid varieties, at least in type A.

1. Braid Varieties and the Deodhar Decomposition

In this section we describe our proposed framework for producing new and interesting combinatorial results. We first define our **central object of study**.

1.1. Braid Varieties.
Let $\mathbb{F}$ be a field. Fix a split, connected reductive algebraic group G over $\mathbb{F}$ with Weyl group W (later, we will take G to be a Kac–Moody group). Let $\mathcal{B}$ be the *flag variety* of G, i.e., the variety of all Borel subgroups of G. The group G acts on $\mathcal{B}$ by conjugation—if $g \in G$ and $B \in \mathcal{B}$, then $g \cdot B := gBg^{-1}$. Fix a pair of opposed $\mathbb{F}$-split Borel subgroups $B_+, B_- \in \mathcal{B}$, and set $H := B_+ \cap B_-$ a split maximal torus of G. We can identify W with $N_G(H)/H$. We write $w \cdot B_+ := \dot{w} \cdot B_+$, where $\dot{w} \in G$ is any lift of $w \in W$ to $N_G(H)$. For any two Borels $B_1, B_2 \in \mathcal{B}$, there is a unique w such that $(B_1, B_2) = (g \cdot B_+, gw \cdot B_+)$ for some $g \in G$. In this case, we write $B_1 \xrightarrow{w} B_2$ and say that (B_1, B_2) are in *relative position w*. For a fixed Borel B, the set $\{B' \in \mathcal{B} \mid B \xrightarrow{w} B'\}$ can be seen to be isomorphic as an algebraic variety to an affine space of dimension $\ell(w)$.

Definition 1.1. Let W be a Weyl group with simple reflections S, $\mathsf{w} = (s_1, s_2, \ldots, s_m)$ be a word in S of length m, and fix $u \in W$. Define the *braid Richardson variety* by

$$R_{u,\mathsf{w}}^{(v)} = \left\{ (vB_+ = B_0 \xrightarrow{s_1} B_1 \xrightarrow{s_2} \cdots \xrightarrow{s_m} B_m \xleftarrow{vuw_\circ} B_-) \mid B_i \in \mathcal{B} \text{ for all } i \right\}.$$

For $v = e$, we write $R_{u,\mathsf{w}} := R_{u,\mathsf{w}}^{(v)}$. When w is the reduced word of an element $w \in W$, $R_{u,\mathsf{w}}$ recovers the usual Richardson variety $R_{u,w}$. It is natural to view w as an element of the braid group, as one obtains isomorphisms between braid varieties whose words w are related by braid moves.

Example 1.2. Let $G = \mathrm{SL}_2$ with $W = \{e, s\}$. Then $|\mathcal{B}(\mathbb{F}_q)| = q + 1$ and by analyzing which Borel subgroups are equal to B_-, one can compute by hand that $\left|R_{e,(s,s,s)}\right| = (q-1)^3 + 2q(q-1) = (q-1)(q^2+1)$.

To systematically decompose $R_{u,\mathsf{w}}^{(v)}$ into understandable pieces, we require the technology of distinguished subwords.

1.2. Coxeter Groups.
Let W be a Coxeter group—that is, a group for which we can find a subset $S \subset W$ (the *simple reflections*) and a group presentation

$$W = \left\langle s \in S \mid (st)^{m(s,t)} = 1 \right\rangle$$

in which $m(s,t) \geq 1$ and $m(s,s) = 1$ for all $s, t \in S$. The *rank* of W is the integer $r := |S|$. The *reflections* T of W are the conjugates of the simple reflections. For an arbitrary element $w \in W$, the *length* (resp. *absolute length*) $\ell(w)$ (resp. $\ell_T(w)$) of w is the smallest integer $m \geq 0$ such that w can be expressed as a product of m simple reflections (resp. reflections), possibly with repetition. There is a unique element of maximal length called the *longest element*, which we denote by $w_\circ \in W$. For $w \in W$ and $s \in S$, we write $ws < w$ if $\ell(ws) < \ell(w)$ and $ws > w$ if $\ell(ws) > \ell(w)$. The *weak order* on W is the partial order formed by the transitive closure of these relations.

1.3. Distinguished Subwords.
A *word* is any finite sequence $\mathsf{w} = (s_1, s_2, \ldots, s_m)$ of elements of S, possibly with repetition. If $w = s_1 s_2 \cdots s_m$, then we refer to w as a *w-word*, and if $m = \ell(w)$, then we say it is *reduced*. A *subword* of $\mathbf{w}$ is a sequence $\mathbf{u} = (u_1, u_2, \ldots, u_m)$ in which $u_i \in \{s_i, e\}$ for all i. For any such sequence, we set $u_{(i)} = u_1 u_2 \cdots u_i \in W$. If $u_{(m)} = u$, then we refer to $\mathbf{u}$ as a *u-subword* of $\mathbf{w}$.

Definition 1.3 ([Deo85, MR04]). Let $u \in W$. We say that a u-subword $\mathbf{u}$ of $\mathbf{w}$ is v-*distinguished* if $vu_{(i)} \leq vu_{(i-1)}s_i$ for all i. We write $\mathcal{D}_{u,\mathbf{w}}^{(v)}$ for the set of v-distinguished u-subwords of $\mathbf{w}$ (and set $\mathcal{D}_{u,\mathbf{w}} = \mathcal{D}_{u,\mathbf{w}}^{(e)}$). We write $\mathrm{e}_{\mathbf{u}} = |\{i \in [m] \mid vu_i = e\}|$, $\mathrm{d}_{\mathbf{u}} = |\{i \in [m] \mid vu_{(i)} < vu_{(i-1)}\}|$, $k = \min_{\mathbf{u} \in \mathcal{D}_{u,\mathbf{w}}^{(v)}} \mathrm{e}_{\mathbf{u}}$, $\mathcal{M}_{u,\mathbf{w}}^{(v)} = \{\mathbf{u} \in \mathcal{D}_{u,\mathbf{w}}^{(v)} \mid \mathrm{e}_{\mathbf{u}} = k\}$ for the set of *maximal v-distinguished u-subwords*, and $\mathcal{M}_{u,\mathbf{w}} = \mathcal{M}_{u,\mathbf{w}}^{(e)}$.

When $u = e$, this minimal value k is given by $\ell_T(w)$ [GLTW22].

1.4. **Deodhar Decomposition.** The following result appears in [Deo85] (see also [MR04, WY07]) for reduced words $\mathbf{w}$, but the argument in [Deo85] extends to the case where $\mathbf{w}$ is arbitrary.

Theorem 1.4 ([Deo85, MR04, WY07]). *Let W be a Weyl group. For a u-subword $\mathbf{u}$ of $\mathbf{w} = (s_1, s_2, \ldots, s_m)$, let*

$$R_{\mathbf{u},\mathbf{w}}^{(v)} = \left\{ (vB_+ = B_0 \xrightarrow{s_1} B_1 \xrightarrow{s_2} \cdots \xrightarrow{s_m} B_m \xleftarrow{vuw_\circ} B_-) \mid B_- \xrightarrow{vu_{(i)}w_\circ} B_i \right\}.$$

Then

$$(1.1) \qquad R_{u,\mathbf{w}}^{(v)} = \bigsqcup_{\mathbf{u} \in \mathcal{D}_{u,\mathbf{w}}^{(v)}} R_{\mathbf{u},\mathbf{w}}^{(v)} \quad \text{with} \quad R_{\mathbf{u},\mathbf{w}}^{(v)}(\mathbb{F}) \simeq (\mathbb{F}^*)^{\mathrm{e}_{\mathbf{u}}} \times \mathbb{F}^{\mathrm{d}_{\mathbf{u}}}.$$

Example 1.5. Continuing Example 1.2, let $W = \mathfrak{S}_2 = \{e, s\}$. Then $\mathcal{D}_{e,(s,s,s)} = \{(e,e,e), (e,s,s), (s,s,e)\}$ and we find that

$$\sum_{\mathbf{u} \in \mathcal{D}_{e,(s,s,s)}} (q-1)^{\mathrm{e}_{\mathbf{u}}} q^{\mathrm{d}_{\mathbf{u}}} = (q-1)^3 + 2(q-1)q = (q-1)(q^2+1).$$

1.5. **Combinatorial Objects: Distinguished Subwords.** Using the Deodhar decomposition of braid Richardson varieties over finite fields, we identify maximal distinguished subwords as the **combinatorial objects** in our framework. Suppose that $\mathbb{F} = \mathbb{F}_q$ is a finite field with q elements, where q is a prime power. Then we have

$$\left| R_{u,\mathbf{w}}^{(v)}(\mathbb{F}_q) \right| = \sum_{\mathbf{u} \in \mathcal{D}_{u,\mathbf{w}}^{(v)}} (q-1)^{\mathrm{e}_{\mathbf{u}}} q^{\mathrm{d}_{\mathbf{u}}}.$$

By definition of $\mathcal{M}_{u,\mathbf{w}}^{(v)}$, we conclude that

$$(1.2) \qquad \lim_{q \to 1} \frac{1}{(q-1)^k} \left| R_{u,\mathbf{w}}^{(v)}(\mathbb{F}_q) \right| = \left| \mathcal{M}_{u,\mathbf{w}}^{(v)} \right|.$$

Thus, any technique for computing $\left| R_{u,\mathbf{w}}^{(v)}(\mathbb{F}_q) \right|$ gives a formula for the combinatorial set $\mathcal{M}_{u,\mathbf{w}}^{(v)}$ of maximal distinguished subwords—in certain settings, we will even be able to identify $\mathcal{M}_{u,\mathbf{w}}^{(v)}$ with existing combinatorial objects.

Problem 1.6. *If the polynomial $\frac{1}{(q-1)^k} \left| R_{u,\mathbf{w}}^{(v)}(\mathbb{F}_q) \right|$ has nonnegative coefficients, find a combinatorial statistic* stat *on the maximal distinguished subwords $\mathcal{M}_{u,\mathbf{w}}^{(v)}$ so that*

$$\frac{1}{(q-1)^k} \left| R_{u,\mathbf{w}}^{(v)}(\mathbb{F}_q) \right| = \sum_{\mathbf{u} \in \mathcal{M}_{u,\mathbf{w}}^{(v)}} q^{\mathrm{stat}(\mathbf{u})}.$$

Example 1.7. Continuing Example 1.5, we have $\mathcal{M}_{e,(s,s,s)} = \{(e,s,s),(s,s,e)\}$. Problem 1.6 asks for a way to assign a statistic so that

$$\frac{1}{(q-1)^k}\left|R_{u,\mathsf{w}}^{(v)}(\mathbb{F}_q)\right| = 1 + q^2 = q^{\mathrm{stat}(e,s,s)} + q^{\mathrm{stat}(s,s,e)} = \sum_{\mathbf{u}\in\mathcal{M}_{e,(s,s,s)}} q^{\mathrm{stat}(\mathbf{u})}.$$

1.6. Formulas: Hecke Algebras. We find **formulas** for the number of points of a braid Richardson variety over a finite field using traces in the corresponding Hecke algebra $\mathcal{H}_W$. Recall that the group ring $\mathbb{Z}[W]$ can be deformed to a $\mathbb{Z}[q^{\pm 1}]$-algebra called the *Hecke algebra* $\mathcal{H}_W$. Every word w in the simple reflections of W gives rise to a corresponding element T_{w} of the Hecke algebra. We will show that $R_{u,\mathsf{w}}^{(v)}(q)$ can be expressed in terms of the value of T_{w} under a certain $\mathbb{Z}[q^{\pm 1}]$-linear trace. Let $A = \mathbb{Z}[q^{\pm 1}]$. The *Hecke algebra* of (W,S) is the A-algebra $\mathcal{H}_W$ freely generated by symbols T_w for $w \in W$, modulo the relations

$$T_w T_s = \begin{cases} qT_{ws} + (q-1)T_w & \text{if } ws < w, \\ T_{ws} & \text{if } ws > w, \end{cases}$$

for all $w \in W$ and $s \in S$. For any word $\mathsf{w} = (s_1, s_2, \ldots, s_m)$, we set $T_{\mathsf{w}} := T_{s_1} T_{s_2} \cdots T_{s_m}$. Note that if w is a reduced w-word, then $T_{\mathsf{w}} = T_w$. Let $\tau : \mathcal{H}_W \to A$ be the *trace* defined linearly by:

$$\tau(T_w^{-1}) := \begin{cases} 1 & w = e, \\ 0 & w \neq e \end{cases} \qquad \text{for } w \in W.$$

Theorem 1.8 ([GLTW22, Corollary 5.3]). *For any word w and $u, v \in W$, we have*

$$\left|R_{u,\mathsf{w}}^{(v)}(\mathbb{F}_q)\right| = q^{\ell(v)}\tau(T_{v^{-1}}^{-1} T_{\mathsf{w}} T_{vu}^{-1}).$$

2. Techniques and Problems in Finite Coxeter Groups

In this section we work with **finite** Weyl groups W. We will describe techniques and problems that apply, and extensions to other finite (complex) reflection groups.

2.1. Techniques. When the Coxeter group W is finite, then the Hecke algebra $K\mathcal{H}_W$ decomposes in the same way as the group algebra $K[W]$, with certain weights called *Schur elements*. Let $\mathrm{Irr}(W)$ be the set of characters of simple $K[W]$-modules up to isomorphism. Each $\chi \in \mathrm{Irr}(W)$ restricts to a class function $\chi : W \to \mathbb{Q}_W$. At the same time, via the isomorphism $K\mathcal{H}_W \xrightarrow{\sim} K[W]$, we can pull back χ to the character of a simple $K\mathcal{H}_W$-module. We denote the resulting character by $\chi_q : K\mathcal{H}_W \to K$. Schur orthogonality for $K\mathcal{H}_W$ says that for $\tau : K\mathcal{H}_W \to K$, we have a decomposition

$$(2.1) \qquad\qquad \tau = \sum_{\chi \in \mathrm{Irr}(W)} \frac{1}{\mathbf{s}_\tau(\chi_q)} \chi_q,$$

where $\mathbf{s}_\tau(\chi_q) \in K$ is a scalar characterized by the property that $\bar{\Sigma}(\tau)$ acts by $\chi(e)\mathbf{s}_\tau(\chi_q)$ on any $K\mathcal{H}_W$-module with character χ_q. The expression $\mathbf{s}_\tau(\chi_q)$ is the *Schur element* for χ_q; this decomposition of $K\mathcal{H}_W$, along with explicit identifications of $\mathbf{s}_\tau(\chi_q)$ gives a general technique for computing $\left|R_{u,\mathsf{w}}^{(v)}(\mathbb{F}_q)\right|$ in a case-by-case manner for any finite Coxeter group. As the bases $(T_v^{-1})_{v\in W}$ and $(q^{\ell(v)}T_{v^{-1}}^{-1})_{v\in W}$ are dual to each other with respect to τ, we have the following dramatic simplification of Theorem 1.8 when taking a union of braid varieties $R_{e,\mathsf{w}}^{(v)}$ over all elements $v \in W$.

Theorem 2.1 ([GLTW22, Section 6.6]). *For W finite,*

$$\sum_{v \in W} \left| R^{(v)}_{e,\mathsf{w}}(\mathbb{F}_q) \right| = \sum_{\chi \in \mathrm{Irr}(W)} \dim(\chi) \cdot \chi_q(T_\mathsf{w}).$$

The PI proposes to systematically search for interesting braid Richardson varieties in finite Coxeter groups using code he has developed in Sage [SCc13]. This is an excellent open-ended project for an undergraduate or beginning graduate student; once a candidate variety $R^{(v)}_{u,\mathsf{w}}$ has been identified as "interesting", there are sufficient representation-theoretic tools available for finite Coxeter groups that it is relatively straightforward to count $\left| R^{(v)}_{u,\mathsf{w}}(\mathbb{F}_q) \right|$. The PI has already identified many interesting examples, as we now discuss.

2.2. **Periodic Elements.** While the above decomposition enables brute-force computations of any traces for finite Coxeter groups, there are also more powerful specific tools available for sufficiently nice words w. We say that a braid $\mathsf{w} = \mathsf{s}_1 \cdots \mathsf{s}_m \in B^+(W)$ is *periodic* if $\mathsf{w}^m = \mathsf{w}^{2p}_\circ$ for some p, m with $m \neq 0$; we say that $\frac{p}{m}$ is the *slope* of w. For example, if $\mathbf{c}$ is a Coxeter word, then $\sigma^p_\mathbf{c}$ is periodic of slope $\frac{p}{h}$ for any integer p. There a classification of periodic braids up to conjugacy using Springer theory as the dth roots of the full twist, where d is a regular number of W [Bes06, LL11, Gar22]: in type A we have only (conjugates of) $\mathbf{c} = \mathsf{s}_1 \cdots \mathsf{s}_n$ and $(\mathsf{s}_1 \cdots \mathsf{s}_n)\mathsf{s}_1$; in type D we have $\mathbf{c} = \mathsf{s}_1 \cdots \mathsf{s}_n$ and $(\mathsf{s}_1 \cdots \mathsf{s}_{n-2})\mathsf{s}_{n-1}(\mathsf{s}_1 \cdots \mathsf{s}_{n-2})\mathsf{s}_n$.

Problem 2.2. *Find formulas for $|R_{e,\mathsf{w}^p}(\mathbb{F}_q)|$ for powers of periodic elements w^p. Extend to formulas for $\sum_{v \in W} \left| R^{(v)}_{e,\mathsf{w}^p}(\mathbb{F}_q) \right|$.*

Small computations suggest that there should be a uniform formula, generalizing the usual Coxeter–Catalan numbers, using the regular number d, the eigenvalues of the periodic element, and a subset of the degrees—such a formula should immediately follow after tracing through the method outlined below, taking advantage of formulas for character values of periodic elements. The extension to the sum should be straightforward using Theorem 2.1.

Example 2.3. For type D_4 with $d = 4$ and $\mathsf{w} = \mathsf{s}_1\mathsf{s}_2\mathsf{s}_3\mathsf{s}_1\mathsf{s}_2\mathsf{s}_4$, we have that

$$|R_{e,\mathsf{w}^3}(\mathbb{F}_q)| = q^{-18}(q-1)^4(1 + q^2 + 3q^4 + 4q^6 + 4q^8 + 3q^{10} + q^{12} + q^{14}), \text{ and}$$

$$\sum_{v \in W} \left| R^{(v)}_{e,\mathsf{w}^3}(\mathbb{F}_q) \right| = q^{-18}(q-1)^4(1 + q + q^2)^4(1 + 4q^3 + q^6).$$

At $q = 1$ and for p odd, we appear to have $\lim_{q \to 1}(q-1)^{-4}|R_{e,\mathsf{w}^p}(\mathbb{F}_q)| = \frac{((p+1)(p+3))^2}{32}$; note that the order d of w is 4, and that the eigenvalues of $w = s_1s_2s_3s_1s_2s_4$ are i^1 and i^3 (each with multiplicity 2).

There is a concrete, systematic, and uniform approach to Problem 2.2, building on the PI's work in [GLTW22]. For all $\chi \in \mathrm{Irr}(W)$, the *fake* and *generic degrees* of χ are

$$\mathrm{Feg}_\chi(q) := \frac{(\chi, [\mathfrak{S}]_q)_W}{(1, [\mathfrak{S}]_q)_W} \text{ and } \mathrm{Deg}_\chi(q) := \frac{\mathsf{s}^+(1_q)}{\mathsf{s}^+(\chi_q)}.$$

It turns out that $\mathrm{Feg}_\chi(q) \in \mathbb{Z}[q]$ and $\mathrm{Deg}_\chi \in \mathbb{Q}_W[q]$; at $q = 1$, both polynomials specialize to the *degree* of χ. Then we have the following result on traces of periodic

elements. If $\chi \in \mathrm{Irr}(W)$ and w is a periodic braid of slope $\nu \in \mathbb{Q}$ write $\sigma_{\mathsf{w}} = q^{-\ell(\mathsf{w})/2}T_{\mathsf{w}}$. Then $\chi_q(\sigma_{\mathsf{w}}) = q^{\nu\,\mathsf{c}(\chi)}\mathrm{Feg}_\chi(e^{2\pi i\nu})$ so that

$$\tau(\sigma_{\mathsf{w}}) = \frac{\varepsilon(w)}{\mathsf{s}^+(1_q)} \sum_{\chi \in \mathrm{Irr}(W)} q^{-\nu\,\mathsf{c}(\chi)}\mathrm{Feg}_\chi(e^{2\pi i\nu})\mathrm{Deg}_\chi(q)$$

$$= \frac{\varepsilon(w)}{\mathsf{s}^+(1_q)} \sum_{\chi \in \mathrm{Irr}(W)} q^{-\nu\,\mathsf{c}(\chi)}\mathrm{Feg}_\chi(q)\mathrm{Deg}_\chi(e^{2\pi i\nu}),$$

where the second equality follows from Lusztig's exotic Fourier transform. This last formula gives a dramatic simplification of the trace—even though periodic elements often have many vanishing characters, transferring the root of unity from the generic degree to the fake degree typically gives a dramatic reduction in the number of terms vanishing in the sum.

Problem 2.4. *Develop graphical models for the maximal distinguished subwords $\mathcal{M}_{e,\mathsf{w}^p}$ (generalizing the usual depictions of noncrossing partitions in classical types).*

2.3. **Rational Noncrossing Coxeter–Catalan Combinatorics.** The proposal of Section 2.2 has been successfully implemented in [GLTW22] for the special case that the periodic element is a Coxeter element, thereby solving two long-standing open problems in Coxeter–Catalan combinatorics. (There are still many open combinatorial problems stemming from this work.) For any positive integer p coprime to h, we set

$$(2.2) \qquad \mathsf{Cat}_p(W;q) := \prod_{i=1}^{r} \frac{[p + (pe_i \bmod h)]}{[d_i]},$$

where $d_1 \leq d_2 \leq \cdots \leq d_r$ are the *degrees* of W, $e_i = d_i - 1$ are the exponents, h is the Coxeter number, and where $0 \leq (pe_i \bmod h) < h$ is the integer in that range congruent to pe_i modulo h. For well-generated finite complex reflection groups, $\mathsf{Cat}_p(W;q)$ is the graded character of the finite-dimensional irreducible representation $eL_{p/h}(\mathsf{triv})$ of the rational Cherednik algebra at the parameter p/h; we write $\mathrm{Cat}(W) := \mathrm{Cat}_{h+1}(W;1)$, which is known case-by-case to enumerate the noncrossing partitions. Two long-standing open problems in Coxeter–Catalan combinatorics were:

- uniformly prove that noncrossing objects are enumerated by $\mathrm{Cat}(W)$.
- uniformly construct rational noncrossing objects enumerated by $\mathrm{Cat}_p(W;1)$.

Using the framework of this proposal, the PI recently solved both of these problems in [GLTW22], along with their parking analogues.

Theorem 2.5 ([GLTW22]). *For any (finite) Weyl group W of rank r and Coxeter number h, Coxeter word $\mathbf{c}$, and integer p coprime to h, we have*

$$|R_{e,\mathbf{c}^p}(\mathbb{F}_q)| = (q-1)^r\mathsf{Cat}_p(W;q) \ \text{and} \ \sum_{v \in W}\left|R_{e,\mathbf{c}^p}^{(v)}(\mathbb{F}_q)\right| = (q-1)^r[p]^r.$$

Sending $q \to 1$, we conclude that for any (irreducible, finite) Coxeter group W of rank r and Coxeter number h, Coxeter word $\mathbf{c}$, and (positive) integer p coprime to h, we have $|\mathcal{M}_{e,\mathbf{c}^p}(W)| = \mathsf{Cat}_p(W)$ and $\sum_{v \in W}\left|\mathcal{M}_{e,\mathbf{c}^p}^{(v)}\right| = p^r$. There are already interesting combinatorial problems when $W = \mathfrak{S}_n$ is the symmetric group. It seems likely several of these could be resolved by a suitable application of the Edelman-Greene bijection, as this map has been applied with success to the subword complex and related crystal models [SS12, MS14].

Problem 2.6. *Let p, n be two coprime positive integers.*

- *Find a bijection between $\mathcal{M}_{e,\mathbf{c}^p}(\mathfrak{S}_n)$ and $p \times n$ rational Dyck paths.*
- *Find a bijection between $\mathcal{M}_{e,\mathbf{c}^p}(\mathfrak{S}_n)$ and $\mathcal{M}_{e,(\mathbf{c}')^n}(\mathfrak{S}_p)$.*
- *Find a bijection between $\mathcal{P}_{e,\mathbf{c}^p}(\mathfrak{S}_n) := \bigcup_{v \in W} \mathcal{M}_{e,\mathbf{c}^p}^{(v)}$ and rational parking functions.*
- *Find a statistic stat on $\mathcal{M}_{e,\mathbf{c}^p}(\mathfrak{S}_n)$ and on $\mathcal{P}_{e,\mathbf{c}^p}(\mathfrak{S}_n)$ such that*

$$\mathsf{Cat}_p(\mathfrak{S}_n; q) = \sum_{\mathbf{u} \in \mathcal{M}_{e,\mathbf{c}^p}(\mathfrak{S}_n)} q^{\mathrm{stat}(\mathbf{u})} \quad and \quad [p]^{n-1} = \sum_{\mathbf{u} \in \mathcal{P}_{e,\mathbf{c}^p}(\mathfrak{S}_n)} q^{\mathrm{stat}(\mathbf{u})}.$$

We showed that the objects in $\mathcal{M}_{e,\mathbf{c}^p}(W)$ are truly *noncrossing* by giving a natural uniform bijection with noncrossing partitions $p = mh + 1$ [Arm09, STW15].

Problem 2.7. *Develop graphical models for rational Catalan objects (generalizing the usual depictions of noncrossing partitions in classical types).*

One possible approach in the symmetric group would be to attempt to match $\mathcal{M}_{e,\mathbf{c}^p}(W)$ up to the PI's work with Rhoades and Armstrong on rational noncrossing partitions [ARW13], relating to Problem 2.6. A second approach would be to relate these constructions to Reading's work on noncrossing partitions and the Coxeter plane (this last is particularly attractive, as one can view distinguished subwords as walks in the weak order that stay relatively close to the Coxeter plane) [Rea10].

Even in the Catalan case $p = h + 1$, all previous results in the literature on the enumeration of W-noncrossing objects relied on the classification of Coxeter groups. The PI's work therefore provides the first uniform proof that each of these W-noncrossing families is counted by the W-*Catalan numbers* $\mathsf{Cat}(W) := \mathsf{Cat}_{h+1}(W)$. In particular, our results give the first uniform proof that the number of clusters in a finite-type cluster algebra is counted by $\mathsf{Cat}(W)$ [FZ02, Theorem 1.9].

Beyond the maximal distinguished subwords, enumerative results are lacking. Preliminary calculations in type A suggest that these lower order terms have a deeper, perhaps not unexpected, connection to maps—just as there are the same number of maximal distinguished subwords and rooted bicolored unicellular maps of genus 0 on n edges, there appear to be the same number of distinguished subwords with $r + 2$ skips as rooted bicolored unicellular maps of genus 1 on $n + 2$ edges (given by $\frac{(2n+3)!}{6n!(n+1)!}$).

Problem 2.8. *Relate distinguished subwords in $\mathcal{D}_{e,\mathbf{c}^p}$ to the theory of maps to obtain enumerations.*

2.4. **Complex reflection groups.** In this section we propose an exciting extension of the PI's work in [GLTW22] to well-generated complex reflection groups. Well-generated complex reflection groups still have a preferred set of "simple reflections", Coxeter elements, and a well-defined rational Catalan number [GG12]. Moreover, the notion of periodic elements naturally generalizes to well-generated complex reflection groups, and such elements have been classified. We may therefore consider Problem 2.2 in this context.

The easy first step in this direction would be to compute the trace $\tau(T_{\mathbf{c}}^p)$ in the Hecke algebra of the well-generated complex reflection group [BMR98, Bro10] (whose braid groups have Artin-like presentations, by Bessis); many of the favorable representation-theoretic properties of Coxeter elements that we used in Section 2.3 still carry over to the complex setting [BMR98]. While we no longer have Lusztig's exotic Fourier transformation, there are partial results due to Lasy and Lacabanne [Las12, Lac21], and a reciprocity result due to Malle that has been used by Douvropoulos in a related

setting [Dou18]. In any event, the computation can be carried out case-by-case because we still have access to the Schur elements and the decomposition of the Hecke algebra.

Conjecture 2.9. *Let W be a well-generated complex reflection group. Then we have*

$$\tau(T_{\mathbf{c}}^p) = (q-1)^r \prod_{i=1}^{r} \frac{[p + e_i(V^p)]}{[d_i]},$$

where the $e_i(V^p)$ are the fake degrees of the p-th Galois twist of the reflection representation and the trace is taken in the Hecke algebra $\mathcal{H}_W$.

Example 2.10. The complex reflection group G_4 has rank $r = 2$, Coxeter number $h = 6$. Its reflection representation has fake degrees 3 and 5. We computed using GAP3 that

$$\tau(T_{\mathbf{c}}^7) = (q-1)^2(q^{12} + q^8 + q^6 + q^4 + 1) = (q-1)^2 \frac{[7+3][7+5]}{[4][6]}.$$

The Deodhar decomposition allows us to build combinatorial models of braid Richardson varieties for general Coxeter groups. Besides the representation-theoretic computation above, the PI also has a proposed combinatorial definition of distinguished subwords for complex reflection groups using his interpretation of the Deodhar condition in terms of colored inversions—rather than being forced to use a simple reflection when the color of the corresponding colored inversion is odd, one should instead be forced to use a simple reflection s when the color is not 0 modulo $\mathsf{ord}(s)$. The harder second step would be to relate this trace to the combinatorial definition above, but this likely follows from a similar computation as [GLTW22, Proposition 5.1].

2.5. **Rational Nonnesting Catalan Combinatorics.** Catalan numbers naturally appear in a markedly different context—in the study of affine Weyl groups and affine Springer fibers. The state of the art has now changed with the PI's recent new definition of rational noncrossing objects—both noncrossing and nonnesting objects are finally defined at almost the same level of generality: both are defined for Weyl groups and for any parameter p coprime to h. Specializing to crystallographic Coxeter groups, $\mathsf{Cat}_p(W)$ (uniformly) counts the number of coroot points inside a b-fold dilation of the fundamental alcove in the corresponding affine Weyl group [Hai94, Sut98]. For $p = h + 1$, these coroot points are called *nonnesting partitions*, and are in bijection with order ideals in the root poset (or, equivalently, ad-nilpotent ideals in a Borel subalgebra of the corresponding complex simple Lie algebra). Although nonnesting and noncrossing partitions have many similarities, finding a uniform bijection between the two sets has been an active and motivating area of research since the late 1990s [Rei97, Ath98].

In [Wil13], we conjectured exactly such a bijection between nonnesting and noncrossing objects for any Coxeter element and any finite Weyl group, suggesting that the root poset encodes a remarkable amount of information related to the corresponding Weyl group (compare with the duality between the heights of roots and the degrees). Our conjectural bijection between noncrossing and nonnesting objects comes from mimicking walks on the W-associahedron—drawing inspiration from [Pan09, BR11, AST13], our methods produce remarkable conjectural (compatible) bijections from nonnesting partitions to clusters and noncrossing partitions which have been exhaustively checked up to rank eight [Wil13, Wil14, STW17].

Problem 2.11 ([Wil13, Wil14, STW17]). *Show that these maps are bijections. Use the PI's new definition of rational noncrossing objects to extend them to the Fuss and rational levels of generality.*

The PI and coauthors at LaCIM have recently made substantial progress, proving the conjecture for $p = h + 1$ by purely combinatorial means in type A, for all Coxeter elements, by finding an element that realizes the Cambrian recurrence on nonnesting partitions. A similar approach might work for the other classical types.

2.6. **Other traces in finite Coxeter groups.** For W a finite Coxeter group and $\chi \in \mathrm{Irr}(W)$ an irreducible character, write $\mathrm{cont}(\chi) = \frac{1}{\dim(\chi)} \sum_{t \in T} \chi(t)$ for the *content* of χ [Tri21]. When $W = \mathfrak{S}_{n+1}$ is the symmetric group and the irreducible characters are indexed by integer partitions, this agrees with the usual definition of content.

Conjecture 2.12. *Let W be a finite Coxeter group, and write $\mathsf{w}_\circ^2$ for the full twist in the braid group $B(W)$.*

$$\sum_{v \in W} \left| R_{e,\mathsf{w}_\circ^2}^{(v)}(\mathbb{F}_q) \right| = \sum_{\chi \in \mathrm{Irr}(W)} \dim(\chi)^2 q^{\mathrm{cont}(\chi)}.$$

More generally, for fixed $x \in W$, the expression $\sum_{v \in W} \left| R_{e,\mathbf{x}^{\mathrm{ord}(x)}}^{(v)}(\mathbb{F}_q) \right|$ appears to have positive coefficients. The PI has many other open problems of this kind in finite Weyl groups, but space prevents listing them here.

3. Techniques and Problems in Affine Weyl Groups

In this section we work with **affine** Weyl groups $\widetilde{W}$.

3.1. **Techniques.** Write Φ^+ for the positive roots of a simple Lie group, $Q = \bigoplus_{i=1}^r \mathbb{Z}\alpha_i$ for the *root lattice*, $Q^+ \subset Q$ for the positive span of the simple roots, and Λ for the weight lattice. Given $\lambda \in Q^+$, we express λ in the basis of fundamental weights as $\lambda = \sum_{i=1}^{n-1} a_i \lambda_i$ and define $\lambda_+ = \sum_{i:a_i>0} a_i \lambda_i$ and $\lambda_- = -\sum_{i:a_i<0} a_i \lambda_i$. For $x \in \Lambda$, we write t_x for the translation in the extended affine Weyl group $\widehat{W}$.

Definition 3.1. A *Kostant partition* $(a_\alpha)_{\alpha \in \Phi^+}$ for $\lambda \in Q^+$ is a sequence of nonnegative integers indexed by positive roots such that $\lambda = \sum_{\alpha \in \Phi^+} a_\alpha \alpha$. We denote the set of all Kostant partitions for λ by $K(\lambda)$.

Opdam proved the following trace formula, which—when combined with Theorem 1.8—is our main technique in this setting.

Theorem 3.2 ([Opd03, Corollary 1.18]). *Let $[k]_q = \frac{(q-1)^2}{q} \frac{q^k - q^{-k}}{q - q^{-1}}$. For $\lambda \in Q^+$,*

$$\tau(T_{t_{\lambda_-}} T_{t_{\lambda_+}}^{-1}) = q^{(\ell(t_{\lambda_-}) - \ell(t_{\lambda_+}))/2} \sum_{(a_\alpha) \in K(\lambda)} \prod_{\substack{\alpha \in \Phi^+ \\ a_\alpha > 0}} [a_\alpha]_q.$$

3.2. **Haglund and Tesler Matrices.** Let S_n act diagonally on the polynomial ring $\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$. The *quotient ring of diagonal coinvariants* DH_n is the quotient of this polynomial ring by the ideal generated by the invariants with no constant term; there is a more general S_n module DH_n^m depending on an integral parameter m. In [Hag11], Haglund proved a remarkable formula for the bigraded (in x- and y-degree) Hilbert series of DH_n^m. Haglund stated the formula in terms of *Tesler matrices*, which are a simple combinatorial rephrasing of Kostant partitions.

Theorem 3.3 ([Hag11, Corollary 1 and Theorem 3]). *Write* $[k]_{q,t} = (q-1)(1-t)\frac{q^k - t^k}{q-t}$ *and let* $\lambda = (m(n-1)+1)\lambda_{n-1} - (m-1)\lambda_1 \in Q^+$. *Then*

$$\mathrm{Hilb}(\mathrm{DH}_{n-1}^m; q, t) = \left(\frac{1}{(q-1)(t-1)}\right)^{n-1} \sum_{(a_\alpha) \in K(\lambda)} \prod_{\substack{\alpha \in \Phi^+ \\ a_\alpha > 0}} [a_\alpha]_{q,t}.$$

Since $[k]_q = [k]_{q,q^{-1}}$, we can specialize Theorem 3.3 using a result of Haiman to conclude the following.

Theorem 3.4 (W. 2022+). *Fix the extended affine Weyl group* $\widehat{W} = \widehat{S}_n$, *and let* $v = t_{(m-1)\lambda_1}$ *and* $w = t_{(m(n-1)+1)\lambda_{n-1}}$. *Then the number of* $\mathbb{F}_q$-*points in the open Richardson variety* $R_{v,w}^\circ$ *is given by*

$$|R_{v,w}(\mathbb{F}_q)| = (q-1)^{2(n-1)}\mathrm{Hilb}(\mathrm{DH}_{n-1}^m; q, q^{-1}).$$

The following combinatorial problem is easily stated, and the PI expects it should have a beautiful solution.

Problem 3.5. *Find a bijection between* $\mathcal{M}_{e, t_{n\lambda_{n-1}}}$ *in* $\widetilde{S}_n$ *and parking functions of length* $n-1$. *Extend this bijection to* $\mathcal{M}_{t_{(m-1)\lambda_1}, t_{(m(n-1)+1)\lambda_{n-1}}}$ *and Fuss parking functions.*

It actually appears that nice enumerations extend to other fundamental weights beyong λ_{n-1}. It should be possible evaluate the sum over the Kostant partitions using inductive methods similar to those Haglund used to establish Theorem 3.3.

3.3. Problems in the Affine Symmetric Group.
We propose an extension of Theorem 2.1 to rational parking functions.

Conjecture 3.6. *Fix* $\widetilde{W}$ *to be the affine symmetric group* $\widetilde{S}_m$, *let* n *and* m *be relatively prime, and define the element* $c_{nm} = (s_{m-n+1}s_{m-n+2}\cdots s_{m-1}s_0 s_{m-n}s_{m-n-1}\cdots s_1)^n \in \widetilde{\mathfrak{S}}_m$. *Then*

$$|R_{e,c_{nm}}(\mathbb{F}_q)| = (q-1)^{n+m-1}[m]^{n-1}.$$

This agrees with the computatations in Section 3.2 for $(n, m) = (n, n+1)$, since it is easily seen that $w_{n,n+1}$ has the same R-polynomial as $t_{n\lambda_{n-1}}$. We have checked Conjecture 3.6 by computer for $(n, m) \in \{(3, 5), (3, 8), (4, 7), (5, 7)\}$; note that Opdam's formula does not apply because we don't get cancellation of the non-translation elements in this more general case. One can again ask for bijections between $\mathcal{M}_{e,c_{nm}}$ and rational parking functions.

The following conjecture is a restatement of a conjecture of Armstrong, Garsia, Haglund, Rhoades, and Sagan in the language of Kostant partitions and our framework, and would generalize Theorem 2.1.

Conjecture 3.7 ([AGH+11, Conjecture 7.1]). *Fix* $\widetilde{W}$ *to be the affine symmetric group* $\widetilde{S}_n$. *Let* $\lambda \in Q^+$ *satisfy* $\lambda = \sum_{i=1}^{n-1} a_i \alpha_i = t_{\lambda_+} - t_{\lambda_-}$ *with* $a_1 > a_2 > \cdots > a_{n-1} \geq a_n = 0$. *Then*

$$\left|R_{t_{\lambda_-}, t_{\lambda_+}}(\mathbb{F}_q)\right| = q^{(\ell(t_{\lambda_+}) - \ell(t_{\lambda_-}))/2}(q-1)^{2n} \prod_{i=1}^{n-1}[(i+1)a_i - ia_{i+1}].$$

While we can express the left-hand side of the conjecture as a sum of Kostant partitions, the form of the right-hand side strongly suggests that there is another decomposition of the variety available. It would be interesting to try to extend the previous conjecture to other affine Weyl groups, and other affine Weyl group elements

other than translations. The PI has many other open problems of this kind in affine Weyl groups, but space prevents listing them here.

4. Techniques and Problems in Kac-Moody groups

In this section, we fix a general Kac–Moody Lie group G and describe general techniques that apply. We then propose software implementation as the first step towards applying our framework in this generality.

4.1. R-polynomials.

Let W be the Weyl group of a Kac-Moody Lie group. There are two general technique that persist at this level of generality. The first exploits a recursive structure on the set of distinguished subwords to construct polynomials that count $\left|R_{u,\mathbf{w}}^{(v)}(\mathbb{F}_q)\right|$: setting $R_{u,\varnothing}^{(v)}(q) := R_{u,\varnothing}(q)$, for any word $\mathbf{w}$ and $s \in S$ we have

$$(4.1) \qquad R_{u,\mathbf{ws}}^{(v)}(q) = \begin{cases} R_{us,\mathbf{w}}^{(v)}(q) & \text{if } vus < vu \\ qR_{us,\mathbf{w}}^{(v)}(q) + (q-1)R_{u,\mathbf{w}}^{(v)}(q) & \text{if } vus > vu. \end{cases}$$

Theorem 4.1. *For arbitrary Weyl groups W, we have $R_{u,\mathbf{w}}^{(v)}(q) = \left|R_{u,\mathbf{w}}^{(v)}(\mathbb{F}_q)\right|$.*

The first step would be for the PI to develop code to work with Hecke algebras (or at least R-polynomials) of general Coxeter groups. At the moment, the implementation of reflection groups within Sage is limited to finite Coxeter groups, affine and extended affine Weyl groups, and complex reflection groups—as such, it is not yet possible in Sage to work with the simple recurrence given in Equation (4.1) for general W. The PI has previous experience writing code (in Mathematica) to deal with general Coxeter groups from his work with Hohlweg and Nadeau on small roots and low elements [HNW16]; setting up the basic framework to be able to run the recurrence in Equation (4.1) should be a relatively easy task. This will enable interesting varieties to be identified. Braid varieties arising from powers of Coxeter elements are likely varieties of interest for general W, and serve as analogues of rational noncrossing Catalan objects for general Coxeter groups.

4.2. Cluster Varieties.

Having identified interesting varieties, the second technique is a cluster-theoretic approach for computing the mixed Hodge decomposition. For particular choices of u and $\mathbf{w}$, it is possible to put a (locally acyclic) cluster structure on $R_{u,\mathbf{w}}^{(v)}(\mathbb{C})$. With this cluster structure, we may invoke technology of Lam and Speyer [LS22, Mul14] to recursively compute $|R_{u,\mathbf{w}}^{(v)}(\mathbb{F}_q)|$. This requires finding an artful sequence of mutations to isolate a separating edge, thereby enabling recursive arguments. This technique succeeds in small examples, but the PI will need to develop a more systematic approach using the specific properties of the quivers corresponding to the braid varieties of interest to handle large cases.

5. Cohomology and Mixed Hodge Structure

This section is speculative, as general techniques for computing mixed Hodge structures are limited. Working over $\mathbb{C}$ now (rather than $\mathbb{F}_q$, as in the previous parts of this proposal), we have the *Deligne splitting* of cohomology

$$H^k(R_{u,\mathbf{w}}^{(v)}(\mathbb{C})) = \bigoplus_{p=0}^{k} \bigoplus_{q=0}^{k} H^{k,(p,q)}(R_{u,\mathbf{w}}^{(v)}(\mathbb{C})).$$

Other than the long exact sequence for relative cohomology, we have only one tool. For particular choices of u and w—even, for example, in affine type—it is reasonably easy to put a (locally acyclic) cluster structure on $R_{u,\mathsf{w}}^{(v)}(\mathbb{C})$, from which we may conclude that $H^{k,(p,q)}(R_{u,\mathsf{w}}^{(v)}(\mathbb{C})) = 0$ for $p \neq q$. With this cluster structure, we may invoke technology of Lam and Speyer [ST13], building on work of Muller [Mul14]. As in the previous section, the approach is to find an artful sequence of mutations to isolate a separating edge. Following [GL20], when $R_{u,\mathsf{w}}^{(v)}(\mathbb{C})$ has dimension d we define the *mixed Hodge polynomial*

$$\mathcal{P}(R_{u,\mathsf{w}}^{(v)}(\mathbb{C}); q, t) := \sum_{k,p\in\mathbb{Z}} q^{p-k/2} t^{(d-k)/2} \dim(H^{k,(p,p)}(R_{u,\mathsf{w}}^{(v)}(\mathbb{C}))).$$

Problem 5.1. *Compute the mixed Hodge polynomial $\mathcal{P}(R_{u,\mathsf{w}}^{(v)}(\mathbb{C}))$ in all cases where the point count $|R_{u,\mathsf{w}}^{(v)}(\mathbb{F}_q)|$ has been established.*

This is already fascinating in the case of the Coxeter–Catalan varieties $R_{e,\mathbf{c}^p}$ and the parking varieties $\bigcup_{v\in W} R_{e,\mathbf{c}^p}^{(v)}$—here, the mixed Hodge polynomials should compute the rational (W, q, t)-analogues of Catalan numbers and parking functions [GG12].

Example 5.2. For $W = \mathfrak{S}_3$ and $\mathsf{w} = (s_1, s_2, s_1, s_2, s_1, s_2, s_1, s_2)$, we have six varieties in $\bigsqcup_{v\in W} R_{e,\mathbf{c}^p}^{(v)}$ (one for each element v of W): as cluster varieties, they are of type E_6, A_4 (twice), A_2 (twice), and A_0. Using the tables from [LS22], we obtain the sum

$$(q^3 + q^2 t + qt + tq^2 + t^3) + 2(q^2 + qt + t^2) + 2(q + t) + 1,$$

which is the usual parking q, t-analogue of 4^2.

There are some subtleties to this decomposition that appear to be closely linked to Foata's inv-to-maj bijection. The PI has experience with these q, t-analogues through his work inverting sweep maps [TW18, MTW17].

Problem 5.3. *Find combinatorial statistics $\mathrm{stat}_q, \mathrm{stat}_t$ on all distinguished subwords $\mathcal{D}_{u,\mathsf{w}}^{(v)}$ so that*

$$\mathcal{P}(R_{u,\mathsf{w}}^{(v)}(\mathbb{C})) = \sum_{\mathbf{u}\in\mathcal{D}_{u,\mathsf{w}}^{(v)}} [(q-1)(t-1)]^{e_{\mathbf{u}}/2} q^{\mathrm{stat}_q(\mathbf{u})} t^{\mathrm{stat}_t(\mathbf{u})}.$$

When $\mathcal{P}(R_{u,\mathsf{w}}^{(v)}(\mathbb{C}))$ is a positive q, t-polynomial, find combinatorial statistics $\mathrm{stat}'_q, \mathrm{stat}'_t$ on all maximal distinguished subwords $\mathcal{M}_{u,\mathsf{w}}^{(v)}$ so that

$$\mathcal{P}(R_{u,\mathsf{w}}^{(v)}(\mathbb{C})) = \sum_{\mathbf{u}\in\mathcal{M}_{u,\mathsf{w}}^{(v)}} q^{\mathrm{stat}'_q(\mathbf{u})} t^{\mathrm{stat}'_t(\mathbf{u})}.$$

These are both is highly speculative; since stat_q on $\mathcal{M}_{e,\mathbf{c}^p}$ in $\mathfrak{S}_n$ appears to be just the usual area of a rational Dyck path, one strategy would be to try to lift the PI's work on inverting the sweep map to the noncrossing context [TW18, MTW17].

Prior Support

Impacts: UTD hosted over 70 outside graduate student participants at the conference. The 2018 GSCC provided a unique and invaluable opportunity for graduate students whose research focuses on combinatorics to experience the benefits of taking part in a research conference. No publications were produced under this award.

Intellectual Merit

The PI's research is in algebraic combinatorics, with a broad interest in motivation from other areas of mathematics such as Lie theory, geometric group theory, and reflection groups. The PI has a strong record of solving long-standing problems using an original toolkit and perspective: he has been selected to give six talks in nine years (only around 5% of submissions are accepted for talks) at the international conference Formal Power Series and Algebraic Combinatorics (FPSAC) and was an invited speaker at the 2020 Triangle Lectures in Combinatorics as well as Open Problems in Algebraic Combinatorics 2022 at the University of Minnesota. There have been many developments motivated by the appearance of the PI's paper [SW12]—to name a few: [CHHM15, EP13, EFG$^+$16, Had21, Hop17, GR14, GR15, GR16, PR15, Rob16, RS13, RW15, Rus16, DPS17, Str15, Str16, JR18, MR19, DSV19, Jos19, RJ21, Hop20, RJ21]. In 2015, the PI, Striker, Propp, and Roby organized an AIM workshop that launched a new field of combinatorics now termed "Dynamical Algebraic Combinatorics." This same group organized a follow-up BIRS online workshop in the Fall 2020 and the PI has additionally organized several successful AMS and JMM special sessions in this field. An integral part of this proposal is to continue supporting the PI's ongoing and future efforts to involve students in cutting-edge research in algebraic combinatorics and related areas. The PI has already laid some of the theoretical groundwork underpinning this proposal in his recent work with Galashin, Thomas, and Trinh [GLTW22].

Broader Impacts

An integral part of this proposal is to support the PI's ongoing and future efforts to **involve students and faculty** in cutting-edge research in algebraic combinatorics and related areas, thereby strengthening the combinatorics program at the University of Texas at Dallas (UTD). The PI has a record of producing problems and research areas accessible to beginning researchers—many of the PI's papers have independently led to Research Experience for Undergraduates (REU) projects at different institutions, most recently leading to four different projects over 2021–22 at Gallian's Minnesota Duluth REU (based on the PI's multiple collaborations with Defant).

The PI graduated his first Ph.D. student in the summer of 2022 (P. Palit, who has started a tenure-track position at Spring Hill College). This year, the PI also recruited a Masters student from Simon Fraser University (Giftson Santhosh) to come pursue his Ph.D. at UTD. The PI has also been successful in attracting fellow faculty members with no prior experience in combinatorics into the combinatorics community; his colleague Arreche who gave a talk on their joint work at FPSAC in 2020 [AW21], and he organized a AMS special session with his colleague Arnold.

The PI intends to use his past experience in conference organization and research mentoring to set up a **yearly online workshop** with the goal of bringing together **early graduate and undergraduate students** (including the honors students in his honors reading courses, as well as Ph.D. students of the PI's collaborators).

As the only combinatorialist at UTD, the PI has designed new undergraduate and graduate courses in combinatorics; due to the success of his undergraduate Discrete

Math and Combinatorics class, the PI was asked by the honors college to teach honors reading courses in Fall 2019, 2020, 2021, and 2022. The PI would like to build on his success with undergraduate education and his experience with designing, coding, and presenting JavaScript browser-based interactive research posters by extending such interactive materials from his research to his teaching by creating a **browser-based interactive discrete math textbook**.

The PI has a history of **service** to the combinatorial community: he has refereed for over twenty journals, became an editor for Annals of Combinatorics in 2019, served on the program committee of FPSAC in 2019, serves currently on the organizing committee as the US funding coordinator, and has organized over ten conferences, workshops, and special sessions. He has represented the larger mathematical community to the public by appearing as a mathematical consultant in a 2018 nationally televised report (WFAA) regarding the NCAA basketball bracket, in two 2022 televised interviews on the recent Mega Millions lottery jackpot, and has hosted many mathematical events at UTD (freshman orientations, MATHCOUNTS competitions, Putnam supervisor, π-day events, algebra and combinatorics seminar organizer, etc.).

5.1. **Mentoring.** The PI graduated one Ph.D. student in the summer of 2022 (P. Palit, who has started a tenure-track position at Spring Hill College). The PI has substantial past experience in involving students and underrepresented students in research: he has mentored undergraduate research over six different summers (at UTD, LaCIM, and UMN), supervised five honors theses at UTD and one senior capstone poject, and he currently has one Ph.D. student pursuing thesis research in areas related to this proposal. The PI will continue to seek out such opportunities with the goal to eventually build a strong combinatorics program at UTD. While at UTD the PI has worked with **students** in the following ways: • organized the 2018 **Graduate Student Combinatorics Conference** • supervised several **independent study/research** courses with graduate students (Fall 2017, Spring 2019, Summer 2020, Summer 2021) While at UTD the PI has worked with **undergraduates** in the following ways: • Spring 2018 - Supervised K. Zimmer's **senior honors thesis** • Summer 2018 - Mentored rising senior R. Hubbard for eight weeks as part of the **Pioneer REU program** (now pursuing his Ph.D. at UNC Chapell Hill) • Spring 2019 - Supervised **independent research** with junior J. Marsh • Spring/Summer 2019 - Supervised **independent research** with undergraduates C. Kondor and M. Patten • Due to the success of the Discrete Math and Combinatorics course the PI designed for the new BS in Data Science program, the PI was asked by the honors college to teach an **honors reading course** in Fall 2019, 2020, 2021, and 2022. In Fall 2020, this reading class took part in the BIRS Dynamical Algebraic Combinatorics conference (held online due to COVID-19)• Spring 2020 - Supervised J. Marsh's **senior honors thesis** (now pursuing his Ph.D. studies at GA tech; submitted for publication) • Spring 2020 - Supervised B. Cotton's **senior honors thesis** (accepted for publication) • Spring 2022 - Supervised a **senior capstone project** with team members D. Desai, A. Tahsin, R. Lofton, S. Venkatesh, and R. Kanakala (Software Implementation of Independence Posets; recieved honorable mention for creativity at the UTDesign Capstone poster presentation) • Spring 2022 - Supervised M. Sferrazza's **senior honors thesis** (EL-shellability of noncrossing partition lattices) • Spring 2022 - Supervised Y. Ahsanullah's **senior honors thesis** (cluster fans in quantum groups) • Further past experience involving undergraduate students in research includes two summers as an REU mentor at the University of Minnesota and two summers mentoring undergraduate students at LaCIM.

5.2. **Conferences and Workshops.** The PI has been very active in **organizing conferences and workshops**. In 2015, the PI, Striker, Propp, and Roby organized an AIM workshop that launched a new field of combinatorics now termed "Dynamical Algebraic Combinatorics." This same group organized follow-up BIRS workshops in Fall 2020 (originally accepted in-person, but held online due to COVID-19; the PI took advantage of this to arrange for his undergraduate honors reading class to attend the workshop) and 2021. The PI has additionally organized several successful AMS and JMM special sessions in this field: ● 2015 - week-long workshop at the American Institute of Mathematics ● 2018 - Graduate Student Combinatorics Conference at UTD, with over 75 attendees (also obtaining $20,000 of NSF funding); ● 2019 - FPSAC program committee ● 2017–2021 - Organized four AMS special sessions on interactions between dynamical systems and combinatorics ● 2018 - two-week "research-in-pairs" program at Oberwolfach ● 2019 - two minisymposia on "Coinvariant Spaces and Parking Functions" at the SIAM Texas Louisiana Section at Southern Methodist University under the meta-organization of Sottile ● 2020 and 2021 - BIRS workshops with Propp, Roby, and Striker on "Dynamical Algebraic Combinatorics" and ● 2021–2023 - Member of the FPSAC organizing committee as US funding coordinator.
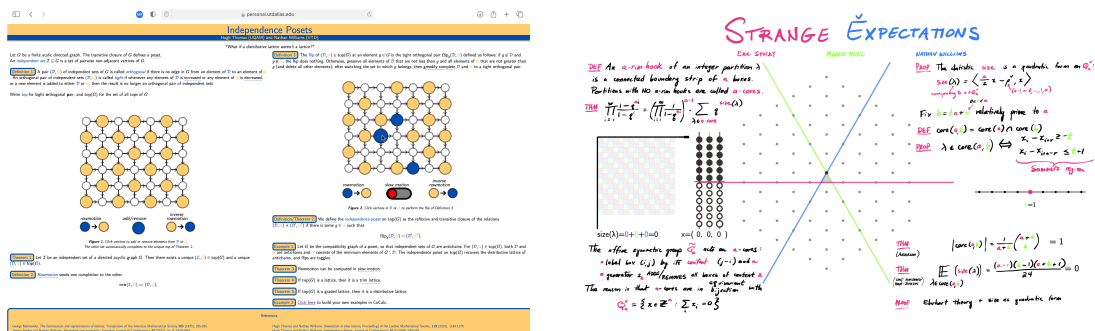

FIGURE 1. Screen shots of the PI's interactive posters presented at FPSAC 2020 and 2021 [TW20, STW21]. Each page is animated using JavaScript code that allows the participant to experiment with various objects, definitions, and theorems.

5.3. **Interactive JavaScript textbook.** During the COVID-19 pandemic, the PI experimented with novel methods to disseminate his research. The 2020 and 2021 FPSAC conferences had remote poster sessions, and the PI developed two JavaScript browser-based interactive posters (see Figure 1 and [TW20, STW21]). These posters were highly successful: the conference organizers selected them as an example for other presenters of how the online format could be harnessed to be even more engaging than a static in-person poster. The PI has designed a discrete math and combinatorics course as part of the new data science program at UTD—he has currently taught the course five times, and he would like to use the expertise he developed while creating interactive posters to render his notes of course content and classroom activities more engaging by using JavaScript to both animate concepts and allow students to interact with new definitions and proofs. Materials include introduction to proof, naive set theory, relations, introduction to algorithms, modular arithmetic, basic combinatorial objects (combinations and permutations), recurrences, inclusion-exclusion, the cycle lemma, and trees. The PI already has LaTeX notes for this course.