# BACHELOR'S THESIS: WALKS IN THE QUANTUM GROUP AND THE POSITIVE CLUSTER FAN

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ABSTRACT. Let $\mathfrak{g}$ be a simple Lie algebra. In this paper we use a particular PBW basis connect a lifting of the positive root lattice of $\mathfrak{g}$ into its quantum group, and connect the leading term of the lifting to the positive cluster fan.

## 1. MOTIVATION

In linear algebra, we often learn a great deal about a vector space and its constituent vectors by endowing it with a particular basis, and studying the properties of the basis. For example, choosing an orthogonal basis of an inner product space allows us to easily calculate coefficients of an expansion of a given vector by projecting onto each basis vector. The content of the expansion holds information which sometimes has an intuitive interpretation (like the "harmonic content" of a function).

While we will not be working in an inner product space, we will aim to accomplish a similar thing, finding the expansion of a given monomial in terms of a chosen basis in the quantum group associated to a simple Lie algebra $\mathfrak{g}$. We study the terms that show up in the expansion, and distinguishing characteristics of their coefficients. On a more combinatorial note, we will also wish to track these coefficients in their "straightening" to the basis, and to see the significance of the "leading term" in a combinatorial setting. Informally, we will be lifting elements of the quantum group up to the Weyl group (more accurately, to the positive cluster fan) in a specific manner, and tracking what happens in the latter to make conclusions about the former. Before discussing this, we will need some background knowledge to formulate the quantum group associated to a simple Lie algebra, PBW basis, and the positive cluster fan, which we address in the following section.

## 2. SEMISIMPLE LIE ALGEBRAS, ROOT SYSTEMS, AND COXETER GROUPS

The quantum group is a deformation of the universal enveloping algebra of a Lie algebra $\mathfrak{g}$. We will use root systems or Dynkin diagrams in its formulations, and to this end we give some background on the subject to have the tools needed. As most of the subject is a little removed from the result we wish to prove and the area we will concern ourselves with, provided here are only definitions and relevant theorems from the study of Lie algebras which are necessary to understand where everything is coming from.

**Definition 2.1.** A *Lie algebra* is a vector space over a field $F$ (usually of characteristic 0) combined with an operation called the Lie bracket $[\cdot, \cdot]$, which satisfies:
(a) $[ax + by, z] = a[x, z] + b[y, z]$ (Bilinearity)
(b) $[x, x] = 0$ (Alternativity)
(c) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ (Jacobi Identity),
where $x$, $y$, $z$ are vectors, and $a$, $b$ are scalars. Endowing $\mathbb{R}^3$ with the cross product gives an example

of a familiar Lie Algebra most have performed calculations in before. We will be restricting ourselves to Lie algebras where $F$ will stand for the complex numbers unless stated otherwise.

**Definition 2.2.** A *subalgebra* $H$ of a given Lie algebra $L$ is a subspace closed under the Lie bracket. An *ideal* of $L$ is a subspace $I$ such that $[L, I] \subset I$, where $[L, I]$ is the set $\{[l, i]\colon l \in L, i \in I\}$.

**Definition 2.3.** A Lie algebra is said to be *simple* if it has no proper nontrivial ideals. It is said to be *semisimple* if it may be written as a direct sum of simple Lie algebras.

The main tool for classifying simple Lie algebras is the notion of a root system.

**Definition 2.4** (Root System)**.** A finite set of non-zero vectors $\Phi \subset \mathbb{R}^n$ is said to be a *root system* of a subspace $E$ if it satisfies the following conditions.
(a) $Span\Phi = E$

(b) $\forall \alpha \in \Phi$, the only vectors parallel to $\alpha$ in $\Phi$ are $\pm\alpha$
(c) $\forall \alpha \in \Phi$, $\Phi$ is closed under reflection by the hyperplane perpendicular to $\alpha$
(d) if $\alpha$, $\beta \in \Phi$, the projection of $\beta$ on $\alpha$ is an integer or half integer multiple of $\alpha$

Reflecting $\alpha$ through the hyperplane perpendicular $\beta$, for example, to corresponds to projecting $\alpha$ onto $\beta$, flipping this portion, and then adding the remaining part of $\alpha$ to this. For $\Phi$ to be closed under this, we need that $\alpha$ - $2\frac{(\beta,\alpha)}{(\beta,\beta)}\beta \in \Phi$ for $\alpha$, $\beta \in \Phi$, where $(\cdot,\cdot)$ is the normal dot product. Similarly, condition (d) just says $\forall \alpha, \beta \in \Phi$, $2\frac{(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$. As it will be useful later in writing the Cartan matrix, we will write this asymmetrical product as $\langle a, b\rangle = 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}$

The collection of these reflections forms a group under composition, as function composition is always associative, and each reflection is its own inverse.

**Definition 2.5.** The *Weyl group* of a root system $\Phi$ is the collection of reflections $\{\sigma_\alpha\}_{\alpha\in\Phi}$. This group is a subgroup of $GL(E)$, as reflections are invertible linear transformations. It is also viewable very clearly as a subgroup of the symmetric group, $S_\Phi$, since each reflection just permutes around elements of $\Phi$.

It is notable that the Weyl group is finite as a subgroup of the permutations on the roots.
We get to the main theorem motivating these definitions .

**Theorem 2.6.** *Every semisimple Lie Algebra is uniquely determined by an associated root system.*

The origin of root systems at least vaguely explained, we now wish to establish some key theorems and definitions relating to root systems and the Weyl group. Most of these results are contained in [Hum12], and we follow the structure of a survey of root systems done by Joshua Ruiter in 2019 [Rui19].

**Definition 2.7.** A root system is said to be *reducible* if it is writable as a disjoint union of orthogonal sets $\Phi = \Phi_1 \sqcup \Phi_2$ where orthogonality means that any vector from $\Phi_1$ is orthogonal to any vector from $\Phi_2$

**Theorem 2.8.** *Every root system decomposes into a direct sum of irreducible root systems.*

From here it becomes clear that classification of irreducible root systems gives us classification of all root systems. Fortunately, this famous problem is already done for us, and we explore the irreducibles explicitly later. Another question related to root systems is similar to that of finding a basis for a vector space, and thus gets the following name.

**Definition 2.9.** A *base* of root system $\Phi$ spanning vector space $E$ is a basis $\Delta$ of E, $\Delta \subset \Phi$, with the property that every element of $\Phi$ is expressible as an integer combination of of elements of $\Delta$, where the sign of all the integer coefficients is the same. We have, for $\beta \in \Phi$

$$\beta = \sum_{\alpha\in\Delta} k_\alpha \alpha$$

with $k_\alpha \in \mathbb{Z}$ all having the same sign.

This condition of sign consistency allows us the notion of positive and negative roots with respect to a given base, the elements of which are called simple roots.

**Definition 2.10.** With $\Delta$ as before, we say $\alpha \in \Delta$ is a *simple root*, and say $\Delta$ composes the *simple roots*. Denote by $\Phi^+$, the set of *positive roots*, that is, the set $\Phi^+ = \{\sum_{\alpha\in\Delta} k_\alpha\alpha : k_\alpha \geq 0\}$, and the *negative roots* by $\Phi^- = \{\sum_{\alpha\in\Delta} k_\alpha\alpha : k_\alpha \leq 0\}$. The *height* of a positive root $\beta$ is the sum of the coefficients of the simple roots required to write it.

As every vector space has a basis, so too will every root system have a base. We state it as a theorem.

**Theorem 2.11.** *Every root system has a base.*

Just as bases for a vector space are not unique, bases for a root system are not either. To formulate Dynkin diagrams and furthermore the quantum group, We will want to capture some information about the root system itself through a base without worrying about the specifics of which base we selected and whether or not it impacted that information, and so it is useful to have the following fact as well. I particularly like Ruiter's ordering in presenting this idea, and thus have opted to shadow his approach.

**Theorem 2.12.** *The Weyl group $W$ of a root system $\Phi$ acts simply transitively on the set of basis of $\Phi$*

An element of the Weyl group, say $s$, acts on a base in the obvious way by applying the reflection $s$ to each element of the base. We want to encode the information of the root system into a familiar object to study its properties further. To this end, we will define the Cartan matrix, and then have almost all the tools in our toolbox to define the quantum group associated to a simple Lie algebra.

**Definition 2.13.** Let $\Delta$ be a base of root system $\Phi$. Give the simple roots of the base an arbitrary ordering, writing $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and define the *Cartan matrix* $C$ to be the $n \times n$ matrix with entries

$$C_{ij} = \langle \alpha_i, \alpha_j \rangle$$

Now, since $W$ acts simply transitively on the set of bases by the proceeding theorem, and since the product $\langle s\alpha_i, s\alpha_j \rangle = \langle \alpha_i, \alpha_j \rangle$ for Weyl group element $s$, we see that $C$ is independent of choice of basis. Selecting a different ordering of $\Delta$ just permutes rows and columns of the matrix, so it is quite indifferent to any choices we made. This indifference is inherited by the Dynkin diagram.

**Definition 2.14.** Let $\Phi$ have base $\Delta$ with an ordering of the simple roots$\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, and $C$ the corresponding Cartan Matrix. Construct a graph with vertices labelled 1 through $n$, and connect $max(|C_{ij}|, |C_{ji}|)$ edges between vertices $i$ and $j$. If $C_{ij} \neq C_{ji}$, the edges drawn are given a direction from the shorter root to the longer root, and otherwise they are left undirected. This graph is called the *Dynkin diagram* of a root system. Forgetting the directions on the graph yields the *Coxeter graph*.

From here we confirm that it is indifferent of the choice of base or order given to the roots, and thus encapsulates the essence of the root system in an efficient and unobscured manner. If we could just draw a Dynkin diagram for all the irreducible root systems, we would be done with total classification. We return to this in the following section after finishing the all the necessary background knowledge for our main result.

**Definition 2.15.** Given a root system $\Phi$ with simple roots $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, define the *simple reflections* of the Weyl group $W$ to be the set $S = \{s_1, s_2, \ldots, s_n\}$ where $s_i$ is the reflection over the hyperplane perpendicular to $\alpha_i$. we say $w = s_{i_1} s_{i_2} \cdots s_{i_N}$ is a *word* for $w \in W$, and if the expression is reduced, that it is a *reduced word*. Sometimes, we will index reflections (not necessarily simple), by the root with which the hyperplane is perpendicular, writing $s_\beta$ for the reflection across the hyperplane perpendicular to $\beta$.

As bases aren't unique, neither are selections of simple reflections. In addition, just as the simple roots generate the root system, the simple reflections will generate the Weyl group. We state this as a theorem.

**Theorem 2.16.** *The simple reflections generate the Weyl group.*

We have then that every element of the Weyl group is expressible as a product of simple reflections, which motivates the notion of "length" of an element of the Weyl group with respect to a choice of simple reflections.

**Definition 2.17.** the *length function* $\ell : W \to \mathbb{N}$ is defined $\ell(w) = n$ where $n$ is the smallest integer such that $w$ is writable as a product of $n$ elements in $S$

From the definition of the length function, the preceding theorem, and our knowledge that the Weyl group is finite, it follows that there is a maximal element as well. We will often refer to such an element as the longest element and concern ourselves with a reduced expression for it (and the corresponding order it induces on the positive roots). We write it as a definition.

**Definition 2.18.** The *longest element* of the Weyl group $W$ with respect to choice of simple roots $S$ is an element $w$ of maximal length. Write $w = s_{i_1} s_{i_2} \cdots s_{i_N}$, and denote $\beta_1 = \alpha_{i_1}$, and $\beta_k = s_{i_1} s_{i_2} \cdots s_{i_{k-1}} \alpha_{i_k}$. The ordered set $\{\beta_1, \beta_2, \ldots, \beta_N\}$ is called an *induced ordering* of the roots, and it is said to be induced by the expression for $w$.

With that definition in mind, we should hope that this collection of roots actually contains them all, so that we induce a total ordering. Fortunately, it is the case, and we state it as a theorem.

**Theorem 2.19.** *Let $\Phi$ be a root system with base $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, and $W$ be the corresponding Weyl group, generated by $S$. Let $w = s_{i_1} s_{i_2} \cdots s_{i_N}$ be a reduced expression for the longest element. The collection $\{\beta_i : 1 \leq i \leq N\}$ defined according to $\beta_k = s_{i_1} s_{i_2} \cdots s_{i_{k-1}} \alpha_{i_k}$ is the actually the collection of positive roots with respect to $\Delta$. In particular, $\beta_i \neq \beta_j$ for $i \neq j$*

It is clear by this theorem that the length of the longest element of the Weyl group is actually exactly the number of positive roots. It is also notable that not just any ordering on the positive roots is obtainable in this fashion, and there is a criterion to tell when an ordering may be induced in this fashion. (add reference for this).

Weyl groups are a special case of well studied Coxeter systems, and this more general framework gives us many tools to work with. We briefly define the braid relations of a Coxeter group, which give us better insight into the picture when we lift to the quantum group, specifically telling us why $A_n$ is so well behaved. More details may be found in [Hum90], and we follow the structure of [Per].

**Definition 2.20.** A *Coxeter system* is a pair $(W, S)$, with $W$ a group and $S \subset W$ a set of generators satisfying the following relations:

$$s^2 = 1$$
$$(ss')^{m(s,s')} = 1$$

with $m(s,s') \geq 2$ for $s \neq s'$, where it is possible that $m(s,s') = \infty$ if there is no relation. Elements of $S$ are referred to as *simple reflections*. The above relation may be expressed as

$$ss'ss' \ldots = s'ss's \ldots$$

where there is $m(s,s')$ terms on either side. such relations are called *braid relations*.

Now, going back to our discussion regarding the longest element of the Weyl group, we want to relate all possible reduced words for this element. Thankfully, any two reduced expressions happen to be related in a very nice way.

**Definition 2.21.** Given a reduced expression $w = s_{i_1}s_{i_2} \cdots rur \cdots sts_{i_N}$, with $st = ts$ and $rur = uru$, we say the new reduced expression formed by exchanging the adjacent reflections $s$ and $t$ is obtained by a *commutation move*. The new reduced expression obtained by replacing $rur$ with $uru$ is said to be obtained by a *braid move*.

In general, if the braid relation is more complicated than $uru = rur$ (i.e. $ur \cdots = ru \cdots$ with $m(u,r)$ reflections on each side), the braid move replaces one side of the equality with the other.

We now can state the key theorem relating all the reduced expressions for the longest element of the Weyl group.

**Theorem 2.22** (Matsumoto's theorem)**.** *Any two reduced expressions for the longest word of the Weyl group are related by a sequence of commutation and/or braid moves.*

Matsumoto's theorem relates reduced expressions, but sometimes we will concern ourselves with unreduced expressions (some subset of them anyway). For this, we introduce the concept of a Hurwitz move

**Definition 2.23.** Given a word $(s_{i_1}, s_{i_2}, \ldots, s_{i_j}, s_{i_{j+1}}, \ldots, s_{i_r})$, the word obtained by transforming to $(s_{i_1}, s_{i_2}, \ldots, s_{i_{j+1}}, s_{i_{j+1}}^{-1}s_{i_j}s_{i_{j+1}}, \ldots, s_{i_r})$ is said to be related to the original word by a *Hurwitz move*.

The expressions a given starting reduced word actually have a rich combinatorial structure, one aspect of which may be considered as a simplicial complex in the following manner. We take our reduced word to be the special case of a Coxeter element, say $c = s_n s_{n-1} \cdots s_2 s_1 s$, and consider the collection of all reflections which may be obtained from simple reflections by conjugation a finite number of times (for example, in the symmetric group, these would be transpositions $(i\ j)$: This example is of particular interest to us). From here, find all products of these reflections of length $n$ which are equal to $c$. The sets consisting of these are maximal simplices, and then closing down the simplices generates the simplicial complex. From here, we aim to define the positive cluster fan, albeit with suggestive notation, working directly with formal integer combinations of the reflections themselves at times, instead of with roots. We write some preliminary definitions for ease of explanation.

**Definition 2.24.** Given a set of simple reflections $\{s_\alpha\}_\alpha \in I$, with $I$ an index set (usually a base $\Delta$) a *Hurwitz reflection* is a reflection which is obtained from simple reflections via conjugation (i.e. picking a "seed" reflection and applying conjugation by other reflections some amount of times). The simplicial complex defined above corresponding to Coxeter element $c$ will be called the *c-complex*

**Definition 2.25.** Given a set of simple reflections $\{s_\alpha\}_{\alpha \in I}$ and Coxeter element $c$, the *positive cluster fan* is the set of non-negative linear combinations of Hurwitz reflections such that for each such combination, the set of reflections with non-zero coefficient belongs to the $c$-complex. points where all the coefficients are non-negative integers are called *lattice points* on the cluster fan.

Alternatively, we could associate to each Hurwitz reflection to a positive root (associate $s_\alpha$ to $\alpha$ in the natural way), and have the same positive cluster fan. We illustrate the idea with an example.

**Example 2.26.** Consider the symmetric group $S_{n+1}$ with simple reflections chosen to be simple transpositions $s_i = (i\ i+1)$ for $1 \leq i \leq n$, and take coxeter element $c = s_n s_{n-1} \cdots s_2 s_1$. We fix an order on the Hurwitz reflections corresponding to an order for the roots and find all products of them that are equal to $c$. Our $c$-complex actually has a Catalan number of such maximal simplices, and we close the simplices under subsets to complete the $c$-complex. The positive cluster fan will be all non-negative linear combinations of reflections which lie in the simplicial complex. For example, the element $5 \cdot s_n + 1 \cdot s_{n-1} + 2 \cdot s_1 s_2 s_1$ has non-zero coefficients corresponding to reflections which are a subset of the maximal simplex $s_n s_{n-1} \cdots s_1 (s_1 s_2 s_1)$, so that it is indeed in the positive cluster fan. An element $s_1 + s_1 s_2 s_1 + s_2$ would *not* be in the positive cluster fan, however, since there is no maximal simplex containing all three of those reflections (the Hurwitz move preserves multiplicity of each root $mod 2$, counting them as elements in a product. Here there is not an admissible number of $s_2$'s here, and adding any Hurwitz reflections would only change it by 0 modulo 2). With any Weyl group like this we could instead looked at positive combinations of roots, where in this case $(i\ j)$ corresponds to $\alpha_i + \alpha_2 + \ldots \alpha_{j-1}$ in $A_n$ - We cover this explicitly in the following section.

We are now ready to talk about the quantum group associated to a simple Lie algebra $\mathfrak{g}$, but it is probably better to familiarize ourselves with the irreducible root systems so that the proof of our main result slides through easier.

## 3. The Irreducible Root Systems

All Dynkin diagrams are located at the end of the section.

**Example 3.1** (The $A_n$ root system)**.** Let $\epsilon_i$ for $1 \leq i \leq n+1$ be the standard basis of $\mathbb{R}^{n+1}$, and define

$$A_n = \{\pm(\epsilon_i - \epsilon_j) : 1 \leq i < j \leq n+1\}$$

This collection of vectors satisfies all the properties of a root system. It is simply laced (i.e. all the roots are the same length), and we can quickly see what its Weyl group $W$ looks like after picking an appropriate set of simple roots. To this end, the usual selection of simple roots is

$$\alpha_i = \epsilon_i - \epsilon_{i+1} \text{ for } 1 \leq i \leq n.$$

Reflection through $\alpha_i$'s hyperplane swaps the coordinates $\epsilon_i$ and $\epsilon_{i+1}$, making it correspond to the transposition $(i\ i+1)$ of the symmetric group $S_{n+1}$. Since transpositions generate the whole symmetric group, we see that the Weyl group of the $A_n$ root system is precisely the symmetric group $S_{n+1}$.

An important fact for us in particular is that $\alpha_i + \alpha_{i+1} + \cdots + \alpha_j$ for $i \leq j \leq n$ make up all the positive roots, and that they have corresponding Weyl group element $(i\ j+1)$. This, combined with the relation $(i\ i+1)^{(i+1\ i+2)} = (i\ i+2)$, where the exponent denotes conjugation, actually gives us the complete picture (we have the braid relations $m(s, s') = 3$ for $s \neq s'$ with $s'$ and $s$ permuting an element in common, and $m(s, s') = 2$ otherwise), so that we are able to write reduced expressions for an arbitrary word in the quantum group lifted up to the Weyl group explicitly. It will be clear what this means to us shortly.

Some examples of the possible orderings induced by reduced expression of the longest word would be the linear order, which organizes the roots as $\{\alpha_1, \alpha_1 + \alpha_2, \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3, \alpha_3, \ldots, \alpha_n\}$, or the lexicographical order, organizing the roots as $\{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_n, \alpha_2, \ldots, \alpha_n\}$

**Example 3.2** (The $B_n$ root system)**.** Using the same notation as before for a basis of $\mathbb{R}^n$, we may write

$$\Phi = \{\pm\epsilon_k, \epsilon_i \pm \epsilon_j : k, i, j \in \{1, 2, \ldots, n\}\}$$

This is the collection of all integer vectors in $\mathbb{R}^n$ of length 1 or $\sqrt{2}$. The standard base of simple roots is selected as

$$\alpha_i = \epsilon_i - \epsilon_{i+1} \text{ for } 1 \leq i \leq n-1$$
$$\alpha_n = \epsilon_n$$

The Weyl group of $B_n$ is the group of signed permutations on the set $\{-n, -n+1, \ldots, -1, 1, \ldots, n-1, n\}$, That is, the set

$$W_{B_n} = \{\sigma : \sigma \text{ is a permutation on } \{-n, -n+1, \ldots, -1, 1, \ldots, n-1, n\} \ni \ \sigma(i) = -\sigma(-i)\ \forall i\}$$

We see, upon inspection, that the simple reflection corresponding to roots of the form $\epsilon_i - \epsilon_{i+1}$ performs the same as it does in $A_n$, giving the (signed) transposition $(i\ i+1)$, where the signed condition implies that $-i \to -(i+1)$ and $-(i+1) \to -i$. Finally, the short root corresponds to the signed transposition $(n\ -n)$. From here it is clear we can generate the whole group of signed permutations, and we are ready to work in the quantum group.

However, because we will require an inductive proof of our main result, building the Dynkin diagram for $B_{n+1}$ out of the one for $B_n$, it is actually more convenient for us to flip this base around, selecting

$$\alpha_i = \epsilon_{i+1} - \epsilon_i \text{ for } 1 \le i \le n-1$$
$$\alpha_1 = \epsilon_1$$

Notably, this generates a different set of positive roots, but by our previous work we know not to worry too much about these differences, and of course we see the Weyl group behaves the same.

**Example 3.3** (The $C_n$ root system)**.** The $C_n$ root system is very similar to the $B_n$ root system, but the short root becomes the long root. Again using the same notation as before for a basis of $\mathbb{R}^n$. $C_n$ is the collection of integer vectors in $\mathbb{R}^n$ of length $\sqrt{2}$ together with vectors of the form $2\lambda$, with $\lambda$ an integer vector of length one.

A choice of simple roots is basically the same as $B_n$, selecting

$$\alpha_i = \epsilon_i - \epsilon_{i+1} \text{ for } 1 \le i \le n-1$$
$$\alpha_n = 2\epsilon_n$$

A quick calculation shows the Weyl group is indeed the same as that for $B_n$.

**Example 3.4.** The $D_n$ root system is the collection of vectors from $R_n$ that are of length $\sqrt{2}$. They have choice of simple roots

$$\alpha_i = \epsilon_i - \epsilon_{i+1} \text{ for } 1 \le i \le n-1$$
$$\alpha_n = \epsilon_{n-1} + \epsilon_n$$

By our analysis of the $A_n$ and $B_n$ root systems, we can tell the last simple root performs a negation on the $(n-1)^{st}$ and $n^{th}$ coordinates, and also swaps them as a transposition. In the way we generated the whole group of signed permutations with one negation previously, now we can only generate the group of signed permutations that change signs an even number of times, as our last simple reflection flips two coordinates every time in combination with the other simple reflections.

The Weyl group is thus the group of signed permutations on the set $\{-n, -n+1, \dots, -1, 1, \dots, n-1, n\}$, where each signed permutation changes an even number of signs.

There are more irreducible root systems, but for brevity we just give the Dynkin diagrams for all the irreducible root systems and omit further discussion.

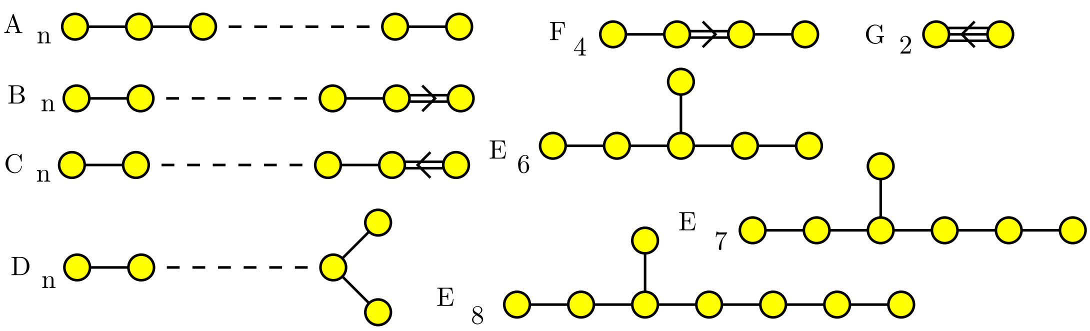


Figure 1. All Irreducible root systems. Image from `https://upload.wikimedia.org/wikipedia/commons/0/0c/Finite_Dynkin_diagrams.svg`

**Theorem 3.5** (Cartan, Killing)**.** *Figure 1 is a complete list of all the irreducible Dynkin diagrams.*

## 4. The quantum group associated to a simple Lie algebra

Finally, we are in a position to build the quantum group associated to a simple Lie algebra $\mathfrak{g}$. We present two formulations, one following Tingley's construction [Tin16], and one used in the QuaGroup package [dGGT22], helpful for explicit computer calculation.

**Definition 4.1.** Let $\nu$ be an indeterminate over $\mathbb{Q}$. For a positive integer $n$, The *symmetric quantum number* $[n]$, also called the *Gaussian number* is defined as

$$[n] = \nu^{n-1} + \nu^{n-3} + \cdots + \nu^{-n+3} + \nu^{-n+1}$$

The Gaussian factorial $[n]!$ is defined as $[n][n-1]\cdots[1]$ for $n > 0$, with $[0]! = 1$ The Gaussian binomial is

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}$$

A quick calculation shows that $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$.

The following definition has one technical detail related to the symmetrizability of the Cartan matrix, but for us is not really important. In particular, given a root system $\Phi$, selecting a base $\Delta = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ and writing the Cartan matrix $C$, this symmetrizability condition implies the existence of a unique sequence of positive integers $d_1, \dots, d_n$ with greatest common divisor 1 such that $d_i C_{ji} = d_j C_{ij}$. We set the following quantity $(\alpha_i, \alpha_j) = d_j C_{ij}$ from here on, and define for $\beta \in \Phi$

$$q_\beta = q^{\frac{(\beta,\beta)}{2}}$$

From here, we define, for non negative integer $n$, $[n]_\alpha = [n]_{\nu = q_\alpha}$, and similarly for the other Gaussian numbers.

**Definition 4.2** (The quantum group, QuaGroup formulation)**.** Let $\mathfrak{g}$ be a Lie algebra, and $\Phi$ be the corresponding root system with base $\Delta = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$. The *quantum group* $U_q(\mathfrak{g})$, or *quantized enveloping algebra*, is the associative algebra with one over $\mathbb{Q}(q)$ generated by $F_\alpha, K_\alpha, K_\alpha^{-1}, E_\alpha$ for $\alpha \in \Delta$, subject to the following relations

$$\begin{aligned} K_\alpha K_\alpha^{-1} &= K_\alpha^{-1} K_\alpha = 1, K_\alpha K_\beta = K_\beta K_\alpha \\ E_\beta K_\alpha &= q^{-(\alpha,\beta)} K_\alpha E_\beta \\ K_\alpha F_\beta &= q^{-(\alpha,\beta)} F_\beta K_\alpha \\ E_\alpha F_\beta &= F_\beta E_\alpha + \delta_{\alpha,\beta} \frac{K_\alpha - K_\alpha^{-1}}{q_\alpha - q_\alpha^{-1}} \end{aligned}$$

together with, for $\alpha \neq \beta$

$$\sum_{k=0}^{1-\langle\beta,\alpha\rangle} (-1)^k \begin{bmatrix} 1 - \langle\beta,\alpha\rangle \\ k \end{bmatrix}_\alpha E_\alpha^{1-\langle\beta,\alpha\rangle-k} E_\beta E_\alpha^k = 0$$

$$\sum_{k=0}^{1-\langle\beta,\alpha\rangle} (-1)^k \begin{bmatrix} 1 - \langle\beta,\alpha\rangle \\ k \end{bmatrix}_\alpha F_\alpha^{1-\langle\beta,\alpha\rangle-k} F_\beta F_\alpha^k = 0$$

We now find an intuitive decomposition for this algebra, viewing it as a vector space, and basically quoting the QuaGroup manual for the theorem statement [dGGT22].

**Theorem 4.3.** *$U_q(g) = U^- \otimes U^0 \otimes U^+$ as vector spaces, where $U^-$ is the subalgebra generated by all the $F_\alpha$, $U^0$ by the $K_\alpha$, and $U^+$ by the $E_\alpha$*

In our case, we will usually look at the lower half generated by the $F_\alpha$'s, but there is an obvious symmetry in any case. We move on to constructing a basis for this space. $U^0$ has basis $K_{\alpha_1}^{r_1} \cdots K_{\alpha_n}^{r_n}$ for $r_i$ integers. Denote $r_{\beta,alpha} = \langle\beta, \alpha\rangle$, and define the algebra automorphism $T_\alpha$ by

$$\begin{aligned} T_\alpha(E_\alpha) &= -F_\alpha K_\alpha \\ T_\alpha(E_\beta) &= \sum_{i=0}^{r_{\beta,\alpha}} (-1)^i q_\alpha^{-i} E_\alpha^{(r_{\beta,\alpha}-i)} E_\beta E_\alpha^{(i)} \text{ if } \alpha \neq \beta \\ T_\alpha(K_\beta) &= K_\beta K_\alpha^{(r_{\beta,\alpha}} \\ T_\alpha(F_\alpha) &= -K_\alpha^{-1} E_\alpha \\ T_\alpha(F_\beta) &= \sum_{i=0}^{r_{\beta,\alpha}} (-1)^i q_\alpha^i F_\alpha^{(i)} F_\beta F_\alpha^{(r_{\beta,\alpha}-i)} \end{aligned}$$

With $E_\alpha^{(k)} = E_\alpha^k / [k]_\alpha!$ and analogously for $F_\alpha^{(k)}$.

**Definition 4.4.** Fixing a reduced expression $w = s_{i_1}s_{i_2}\cdots s_{i_N}$, define root vectors $F_k = T_{\alpha_{i_1}}\cdots T_{\alpha_{i_{k-1}}}(F_{\alpha_{i_k}})$.The monomials $F_1^{(m_1)}F_2^{(m_2)}\cdots F_N^{(m_N)}$ and $E_1^{(n_1)}E_2^{(n_2)}\cdots E_N^{(n_N)}$ with $m_i, n_i$ non-negative integers form *PBW bases* of $U^-$ and $U^+$ respectively. Taking monomials over all the PBW root vectors for the $F_k, E_k, K_\alpha$ in the order $F_1^{(m_1)}F_2^{(m_2)}\cdots F_N^{(m_N)}K_{\alpha_1}^{r_1}\cdots K_{\alpha_n}^{r_n}E_1^{(n_1)}E_2^{(n_2)}\cdots E_N^{(n_N)}$ form the *PBW monomials*.

**Theorem 4.5.** *The PBW basis forms a basis for the quantum group. In particular, PBW monomials for the upper and lower halves form bases for the upper and lower halves respectively.*

A slightly more intuitive definition, particularly in the simpler cases of a Lie algebra of type ADE, comes from [Tin16], which we restate briefly below for convenience.

**Definition 4.6** (The quantum group, Tingley formulation)**.** Let $\mathfrak{g}$ be a Lie algebra of type ADE. Given the Dynkin diagram, index the nodes of the graph, referring to the indexing set as $I$. As the Dynkin diagram is a graph, we can talk about adjacent nodes. From here, Lets $E_i$,$F_i$,$K_i$ for $i \in I$ be generators, subject to the following relations
for $i \neq j \in I$ we have

$$K_iK_i^{-1} = K_i^{-1}K_i = 1, K_iK_j = K_jK_i, K_iE_iK_i^{-1} = q^2E_i$$
$$K_iF_iK_i^{-1} = q^{-2}F_i, E_iF_j - F_jE_i = 0, E_iF_i - F_iE_i = \frac{K_i - K_i^{-1}}{q - q^{-1}}$$

additionally, if $i$ is adjacent to $j$

$$E_i^2E_j + E_jE_i^2 = (q + q^{-1})E_iE_jE_i$$
$$F_i^2F_j + F_jF_i^2 = (q + q^{-1})F_iF_jF_i$$

and otherwise $E_i$ commute with $E_j$, $F_i$ with $F_j$, and $K_iE_jK_i^{-1} = E_j$, $K_iF_jK_i^{-1} = F_j$.

From here we define our algebra automorphism $T_i$ to create the PBW basis.

**Definition 4.7.** The algebra automorphism $T_i$ for $i$ a node of the Dynkin diagram will be the function defined as

$$T_i(F_j) = \begin{cases} F_j & i \text{ not adjacent to } j \\ F_jF_i - qF_iF_j & i \text{ adjacent to } j \\ -K_j^{-1}E_j & i = j \end{cases}$$
$$T_i(E_j) = \begin{cases} E_j & i \text{ not adjacent to } j \\ E_iE_j - qE_jE_i & i \text{ adjacent to } j \\ -F_jK_j & i = j \end{cases}$$
$$T_i(K_j) = \begin{cases} K_j & i \text{ not adjacent to } j \\ K_iK_j & i \text{ adjacent to } j \\ K_j^{-1} & i = j \end{cases}$$

From here in similar fashion, given a fixed reduced expression for the longest element $w = s_{i_1}s_{i_2}\cdots s_{i_N}$ we define root vectors

$$\begin{aligned} F_{w;\beta_1} &:= F_{i_1} \\ F_{w;\beta_2} &:= T_{i_1}F_{i_2} \\ &\vdots \\ F_{w;\beta_N} &:= T_{i_1}T_{i_2}\cdots T_{i_{N-1}}F_{i_N} \end{aligned}$$

When the expression for $w$ is obvious, we suppress it. Finally, building monomials of the root vectors in the same manner as the QuaGroup formulation gives us the PBW basis. Tingley proves many useful facts simply in the case of and ADE type Lie algebra, and we reference some of his lemmas later in the text.

## 5. The $A_n$ root system and associated quantum group

**Theorem 5.1** (Walks in $A_2$). *Let $U_q(\mathfrak{g})$ be the quantum group on $A_2$ with ordering on the positive roots $\alpha_1, \alpha_1+\alpha_2, \alpha_2$. Then the following formula holds.*

$$F_{\alpha_2}^{(n)} F_{\alpha_1}^{(m)} = \sum_{j=0}^{min(n,m)} q^{(n-j)(m-j)} F_{\alpha_1}^{(m-j)} F_{\alpha_1+\alpha_2}^{(j)} F_{\alpha_2}^{(n-j)} \tag{1}$$

To prove this we use induction and one lemma.

**Lemma 5.2.**

$$F_{\alpha_2}^{(n)} F_{\alpha_1} = F_{\alpha_1+\alpha_2} F_{\alpha_2}^{(n-1)} + q^n F_{\alpha_1} F_{\alpha_2}^{(n)} \tag{2}$$

*similarly,*

$$F_{\alpha_2} F_{\alpha_1}^{(n)} = F_{\alpha_1}^{(n-1)} F_{\alpha_1+\alpha_2} + q^n F_{\alpha_1}^{(n-1)} F_{\alpha_2} \tag{3}$$

*Proof.* In $A_2$ with the chosen ordering we have the following basic commutation relations

(i) $F_{\alpha_2} F_{\alpha_1} = F_{\alpha_1+\alpha_2} + q F_{\alpha_1} F_{\alpha_2}$
(ii) $F_{\alpha_1+\alpha_2} F_{\alpha_1} = q^{-1} F_{\alpha_1} F_{\alpha_1+\alpha_2}$
(iii) $F_{\alpha_2} F_{\alpha_1+\alpha_2} = q^{-1} F_{\alpha_1+\alpha_2} F_{\alpha_2}$

We now proceed by induction to prove the lemma, say in the $F_{\alpha_2}^{(n)} F_{\alpha_1}$ case. The base case follows by the first commutation relation, so suppose it holds for $n-1$, and we show it holds for $n$.

$$F_{\alpha_2}^{(n)} F_{\alpha_1} = \frac{F_{\alpha_2}}{[n]} F_{\alpha_2}^{(n-1)} F_{\alpha_1} = \frac{F_{\alpha_2}}{[n]} [F_{\alpha_1+\alpha_2} F_{\alpha_2}^{(n-2)} + q^{n-1} F_{\alpha_1} F_{\alpha_2}^{(n-1)}]$$

This becomes, distributing and applying the basic commutation relations, (and remembering the PBW type monomials exponents wrap the symmetric quantum integers [n] into them)

$$= q^{-1} \frac{[n-1]}{[n]} F_{\alpha_1+\alpha_2} F_{\alpha_2}^{(n-1)} + q^{n-1} \frac{1}{[n]} F_{\alpha_1+\alpha_2} F_{\alpha_2}^{(n-1)} + q^n F_{\alpha_1} F_{\alpha_2}^{(n)}$$

grouping terms, we use that $[n] = q^{-1}[n-1] + q^{n-1}$ to get

$$= F_{\alpha_1+\alpha_2} F_{\alpha_2}^{(n-1)} + q^n F_{\alpha_1} F_{\alpha_2}^{(n)}$$

and the induction is complete. The case for $F_{\alpha_2} F_{\alpha_1}^{(n)}$ is similar, just using explicit calculation and leveraging properties of the symmetric quantum integers. □

It is clear, if we had chosen a different reduced expression for the longest word ($s_2 s_1 s_2$ instead of $s_1 s_2 s_1$), we would exchange the labels $\alpha_1$ and $\alpha_2$, and everything passes through just the same with the analogous statement. We are now ready to move to the proof of the original statement.

*Proof of Theorem 5.1.* Let $n$ be fixed, and $m$ a positive integer with $n \geq m$ in the expression $F_{\alpha_2}^{(n)} F_{\alpha_1}^{(m)}$. We induct on $m$ for fixed $n$. The base case is handled by lemma 5.2, so we move on to the inductive step.

$$F_{\alpha_2}^{(n)} F_{\alpha_1}^{(m)} = F_{\alpha_2}^{(n)} F_{\alpha_1}^{(m-1)} \frac{F_{\alpha_1}}{[m]} = [\sum_{j=0}^{m-1} q^{(n-j)(m-1-j)} F_{\alpha_1}^{(m-1-j)} F_{\alpha_1+\alpha_2}^{(j)} F_{\alpha_2}^{(n-j)}] \frac{F_{\alpha_1}}{[m]}$$

pulling the $\frac{F_{\alpha_1}}{[m]}$ into the sum and applying lemma 5.2 gives

$$= \sum_{j=0}^{m-1} \frac{1}{[m]} q^{(n-j)(m-1-j)} F_{\alpha_1}^{(m-1-j)} F_{\alpha_1+\alpha_2}^{(j+1)} [j+1] F_{\alpha_2}^{(n-j-1)}$$
$$+ \sum_{k=0}^{m-1} \frac{1}{[m]} q^{(n-k)(m-1-k)} q^{n-k} q^{-k} F_{\alpha_1}^{(m-k)} [m-k] F_{\alpha_1+\alpha_2}^{(k)} F_{\alpha_2}^{(n-k)}]$$

To reconcile these sums into one sum, we notice the terms with the same PBW monomial combine when $k = j+1 \iff j = k-1$. plugging in $j = k-1$ in the first sum gives the summand

$$\frac{1}{[m]} q^{(n-k+1)(m-k)} F_{\alpha_1}^{(m-k)} F_{\alpha_1+\alpha_2}^{(k)} [k] F_{\alpha_2}^{(n-k)}$$

and the bottom gives

$$\frac{1}{[m]} q^{(n-k)(m-k)} q^{-k} F_{\alpha_1}^{(m-k)} [m-k] F_{\alpha_1+\alpha_2}^{(k)} F_{\alpha_2}^{(n-k)}$$

Combining them and looking only at the coefficient in front of the PBW monomial gives

$$\frac{q^{(m-k)(n-k)}}{[m]} (q^{m-k}[k] + q^{-k}[m-k])$$

using the form $[m] = \frac{q^m - q^{-m}}{q - q^{-1}}$ simplifies the top expression to

$$q^{(m-k)(n-k)} \frac{q^{m-k}(q^k + q^{-k}) + q^{-k}(q^{m-k} + q^{-m+k})}{q^m - q^{-m}} = q^{(m-k)(n-k)}$$

so the terms combine as they should. Looking at the terms which do not combine, we see $j = m-1$ in the first sum corresponds to

$$F_{\alpha_1+\alpha_2}^{(m)} F_{\alpha_2}^{(n-m)}$$

and $k = 0$ corresponds to the term

$$q^{mn} F_{\alpha_1}^{(m)} F_{\alpha_2}^{(n)}$$

rolling everyone into one sum thus gives

$$F_{\alpha_2}^{(n)} F_{\alpha_1}^{(m)} = \sum_{j=0}^{m} q^{(n-j)(m-j)} F_{\alpha_1}^{(m-j)} F_{\alpha_1+\alpha_2}^{(j)} F_{\alpha_2}^{(n-j)}$$

The case for $m \geq n$ for fixed m follows in similar manner, and since any pair of exponents falls into one of these cases, we are done. □

We see that every possible decomposition of $m \cdot \alpha_1 + n \cdot \alpha_2$ into integer combinations of $\alpha_1, \alpha_1 + \alpha_2, \alpha_2$ shows up in this result, and a similar result will be indeed true in $A_n$ as well. The terms with higher powers of $q$ correspond to "less efficient" combinations, or combinations where we combined less terms together. The generalization to $A_n$ leverages this rank 2 example at each step of the way, using that the Dynkin Diagram for $A_n$ may be constructed from the Diagram for $A_{n-1}$ by just appending a vertex to the right with and edge between the rightmost vertex of $A_{n-1}$, loosely referring to the diagram drawn as usual.

From this point forward, to have a more concise and meaningful notation, we will write $F_{(i\ j+1)}$ for $F_{\alpha_i+\cdots+\alpha_j}$, indexing the element of the quantum group by the Weyl group element corresponding to the positive root normally indexing it.

**Theorem 5.3** (Basic commutation relations in $A_n$)**.** *We have the following relations in the lexicographical order on the quantum group, where $(k\ l)$ is always written with $k < l$.*

*(i)* $F_{(j\ j+1)} F_{(i\ j)} = F_{(i\ j+1)} + q F_{(i\ j)} F_{(j\ j+1)}$
*(ii)* $F_{(j\ j+1)} F_{(i\ j+1)} = q^{-1} F_{(i\ j+1)} F_{(j\ j+1)}$
*(iii)* $F_{(i\ j+1)} F_{(i\ j)} = q^{-1} F_{(i\ j)} F_{(i\ j+1)}$

*Proof.* The reduced expression that induces this ordering is $s_1 s_2 \cdots s_n s_1 s_2 \cdots s_{n-1} s_1 \cdots s_1 s_2 s_1$. From here, the defining relations of the quantum group, and the definition of the root vectors, the relations follow. In particular, we can use lemma 3.3 of [Tin16] combined with the fact that we may move any 3 vectors that fit the form of Theorem 5.3 to be adjacent via commutation moves (which don't change anything), and braid moves that will impact the composition of root vectors uninvolved (they don't change root vectors indexed by simple roots). □

From here, knowing the effects of the commutation and braid moves on our root vectors, we see the linear and lexicographical order are interchangeable. We also see that these relations are exactly the same shape as the case of $A_2$ in Lemma 5.2. We hope to generalize this idea to any admissible orderings in a simple manner, stated in the final section.

**Definition 5.4.** Given a positive integer combination of simple roots $m_1 \cdot \alpha_1 + m_2 \cdot \alpha_2 + \cdots + m_n \cdot \alpha_n$ and an ordering on the roots $\{\beta_1, \ldots, \beta_N\}$, we say a *walk with respect to order $\{\beta_1, \ldots, \beta_N\}$* on the root lattice from 0 to $m_1 \cdot \alpha_1 + m_2 \cdot \alpha_2 + \cdots + m_n \cdot \alpha_n$ is a finite sequence of positive roots written in order such that the sum of all the entries is $m_1 \cdot \alpha_1 + m_2 \cdot \alpha_2 + \cdots + m_n \cdot \alpha_n$.

When the order is apparent, we will just refer to such a sequence as a walk without mentioning it. For us, this order will be induced by the reduced word for the longest element taken when formulating the PBW basis of the quantum group.

When we discard the order on the walk, we end up with something referred to as a *Kostant partition*, which just repackages the simple roots with multiplicity into a collection of positive roots whose sum preserves multiplicity.

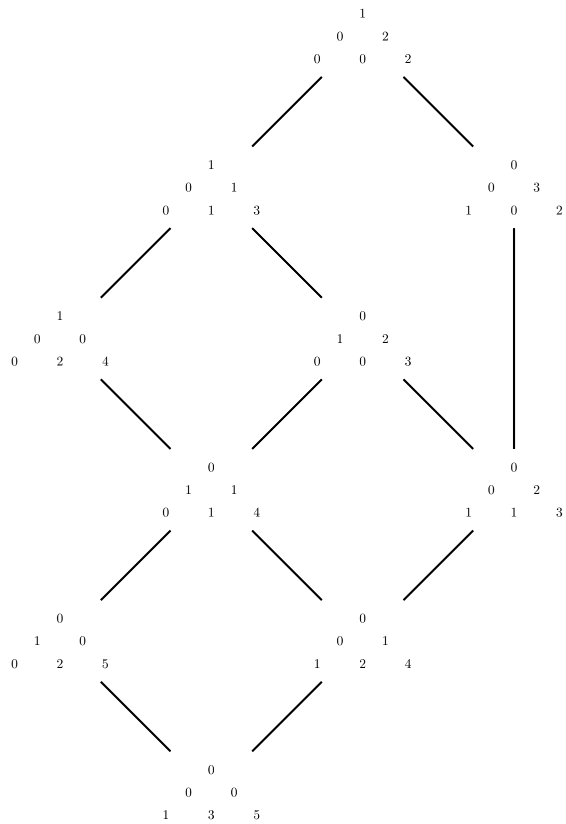


FIGURE 2. The set of Kostant partitions $K(\alpha_1+3\alpha_2+5\alpha_3) \in Q_4^+$, partially ordered using the canonical basis of the quantum group $U_q^+(\mathfrak{sl}_4)$. Coefficients of roots are arranged by dominance ordering proposed by Professor Williams of UTD.

**Example 5.5.** A walk in $A_3$ from 0 to $4 \cdot \alpha_1 + 6 \cdot \alpha_2 + 5 \cdot \alpha_3$ could be

$$(\alpha_1,\ \alpha_1+\alpha_2,\ \alpha_1+\alpha_2,\ \alpha_1+\alpha_2+\alpha_3,\ \alpha_2, \alpha_2+\alpha_3,\ \alpha_2+\alpha_3,\ \alpha_3,\ \alpha_3,\ \alpha_3)$$

Where the order is apparent from the order the sequence is written in.

When we write a product of quantum group elements indexed by the simple roots in the reverse order with multiplicities, we will associate with it a point on the root lattice. Every possible walk to this point will be realized in the PBW straightening of this expression, and the associated power of $q$ that a given realized walk comes with will be determined as follows:

Each element in the walk will look behind him and see how many terms he could have combined with to make a "more efficient" walk, and it will happen that the number of such terms will correspond to the power of $q$ contributed in that PBW monomial straightening of the expression of the simple roots taken in the totally wrong order. We give a more precise definition followed by an example.

**Definition 5.6.** The *inversion number* of a walk $\omega = (\gamma_1, \gamma_2, \ldots, \gamma_l)$ ($\gamma_i$ are positive roots, usually with repetitions, and $l$ is the length of the sequence) with respect to order $\{\beta_1, \ldots, \beta_N\}$ on the positive roots is the number

$\mathrm{inv}(\boldsymbol{\omega}) = |\{i < j : \gamma_i + \gamma_j$ admits a Kostant partition with element of height $> \max(h(\gamma_i), h(\gamma_j))\}|$

**Example 5.7.** We may use Gap's QuaGroup package to straighten the small expression below in $A_n$, with $n \geq 3$, and interpret its results ourselves. In this case, we have, rather than the lexicographical order, the linear order.

$$\begin{aligned} F^{(2)}_{\alpha_3}F^{(3)}_{\alpha_2}F^{(1)}_{\alpha_1} = & F_{\alpha_2}F_{\alpha_1+\alpha_2+\alpha_3}F_{\alpha_2+\alpha_3} + q^2F^{(2)}_{\alpha_2}F_{\alpha_1+\alpha_2+\alpha_3}F_{\alpha_3} + \\ & q^2F_{\alpha_1+\alpha_2}F^{(2)}_{\alpha_2+\alpha_3} + q^3F_{\alpha_1}F_{\alpha_2}F^{(2)}_{\alpha_2+\alpha_3} + \\ & q^3F_{\alpha_1+\alpha_2}F_{\alpha_2}F_{\alpha_2+\alpha_3}F_{\alpha_3} + q^5F_{\alpha_1}F^{(2)}_{\alpha_2}F_{\alpha_2+\alpha_3}F_{\alpha_3} + \\ & q^6F_{\alpha_1+\alpha_2}F^{(2)}_{\alpha_2}F^{(3)}_{\alpha_3} + q^9F_{\alpha_1}F^{(3)}_{\alpha_2}F^{(2)}_{\alpha_3} \end{aligned}$$

Let us examine the $q^3F_{\alpha_1}F_{\alpha_2}F^{(2)}_{\alpha_2+\alpha_3}$ term, and see how it got that $q^3$. The $F_{\alpha_1}$ sees no one behind him, and contributes no $q$'s. The $F_{\alpha_2}$ looks behind him and notices he could have combined with $F_{\alpha_1}$ to form $F_{\alpha_1+\alpha_2}$, and thus gives us one $q$. Each $F_{\alpha_2+\alpha_3}$ looks behind him and notices he could have combined with $F_{\alpha_1}$ to form $F_{\alpha_1+\alpha_2+\alpha_3}$, giving us one $q$ each so that they give two $q$'s. The total number of $q$'s comes out to be three.

Writing out the walk corresponding here is the real picture of interest, and we pick another term to further illustrate the point. Consider the $q^5F_{\alpha_1}F^{(2)}_{\alpha_2}F_{\alpha_2+\alpha_3}F_{\alpha_3}$ term, and write the corresponding walk:

$$(\alpha_1,\ \alpha_2,\ \alpha_2,\ \alpha_2+\alpha_3,\ \alpha_3)$$

$\alpha_1$ sees no one behind him as usual, each $\alpha_2$ sees the $\alpha_1$ behind them and contribute a $q$ each, the $\alpha_2+\alpha_3$ only sees the one $\alpha_1$ behind him and contributes a $q$, and the $\alpha_3$ sees the two $\alpha_2$'s behind him and contributes one q for each $\alpha_2$, for a total of $2+1+2=5$ $q$'s.

We prove this statement generally with a main theorem of interest.

**Definition 5.8.** Let $\omega$ be a walk to a given root lattice point $p = m_1\cdot\alpha_1 + m_2\cdot\alpha_2 + \cdots + m_n\cdot\alpha_n$. Denote by the symbol $F_\omega$ the PBW type monomial with exponent of each $F_x$ for $x$ a positive root to be the number of times $x$ shows up in the walk $\omega$.

**Definition 5.9.** Let $\omega_p$ be the set of all walks to a root lattice point $p$. Denote the *counting function f* to be the function mapping walks to powers of $q$ in the following manner.

$$f\colon G_p \to \mathbb{Z} \quad \omega \mapsto \mathrm{inv}(\omega) \tag{4}$$

We now state the theorem

**Theorem 5.10** (Walks in $A_n$)**.** *Endow the positive roots of $A_n$ with their lexicographic order with respect to their form as reflections in the Weyl group and consider the quantum group with this choice of PBW basis, and let $p = m_1\cdot\alpha_1 + m_2\cdot\alpha_2 + \cdots + m_n\cdot\alpha_n$. Then*

$$F^{(m_n)}_{\alpha_n}F^{(m_{n-1})}_{\alpha_{n-1}}\dots F^{(m_2)}_{\alpha_2}F^{(m_1)}_{\alpha_1} = \sum_{\omega\ walk\ to\ p} q^{f(\omega)}F_\omega \tag{5}$$

*Proof.* We prove by induction on $n$, using our previous exposition regarding the mini copies of $A_2$ in $A_n$. The base case is precisely Theorem 5.1, so we move to the inductive step. Suppose the theorem holds for $A_{n-1}$, and call point $p^*$ the point obtained from point $p$ by setting $m_n = 0$ we have

$$F^{(m_n)}_{\alpha_n}F^{(m_{n-1})}_{\alpha_{n-1}}\dots F^{(m_2)}_{\alpha_2}F^{(m_1)}_{\alpha_1} = F^{(m_n)}_{\alpha_n}\sum_{\omega^*\ walk\ to\ p^*} q^{f(\omega^*)}F_{\omega^*}$$

Pulling $F^{(m_n)}_{\alpha_n}$ into the sum must be treated very carefully, but one part at least is clear - from the basic commutation relations, each time an $F_{\alpha_n}$ runs into someone he does not commute with, he splits into two terms, one with the combination applied, and one with the order changed (at the expense of a $q$). From here it is clear that every possible walk to point $p$ must occur, and that these walks are the only terms in the sum (the multiplicity of each $\alpha_i$ is conserved at each step). Alternatively, remembering Theorem 5.1, and applying it successively from right to left in the original straightened expression, we see that the only coefficients in front of each PBW monomial can be a power of $q$. Putting these two observations together gives the following result:

$$F^{(m_n)}_{\alpha_n}F^{(m_{n-1})}_{\alpha_{n-1}}\dots F^{(m_2)}_{\alpha_2}F^{(m_1)}_{\alpha_1} = \sum_{\omega\ walk\ to\ p} q^{g(\omega)}F_\omega$$

for some function $g$ which outputs a positive integer given a walk to $p$. By linear independence of the PBW monomials, all that remains is to find this well defined function $g$, and to prove that is agrees with $f$ for every walk $\omega$ to $p$.

We show how $F^{(m_n)}_{\alpha_n}$ acts on each summand, splitting it into a collection of new walks. The element $F_{(n\ n+1)}$ will commute at no $q$ cost with all the elements $F_{(i\ j)}$ for $j < n$, and each $F_{(n\ n+1)}$ costs one

$q$ to go past each $F_{(i\ n)}$ without combining, remembering our basic commutation relations. we write an arbitrary summand $F_{(n\ n+1)}^{(m_n)} \cdot F_{(1\ 2)}^{(j_1)} \dots F_{(i\ n)}^{(j_k)} \dots F_{(n-1\ n)}^{(j_l)}$, where we have written it so that $F_{(i\ n)}$ is the leftmost term in the product that has an $n$ in it. The $F_{(n\ n+1)}$'s then go past all the terms to the left of this element for free, and meet the first crossroads here. By our previous argument, every walk will show up for sure with exactly some power of $q$ as its coefficient, so we don't need to do the bookkeeping of splitting at each step, and only need to see how much $q$ cost we incurred bringing $(n\ n+1)$'s past the other terms without combining for each summand to find the power of q associated to it.

One important fact in this is that the lexicographical order is not disturbed by combining a $F_{(n\ n+1)}$ with $F_{(i\ n)}$, for terms to the right of $F_{(i\ n)}$ must be of the form $F_{(j\ k)}$ with $i < j < k$, so that $F_{(i\ n+1)}$ is still ahead of $F_{(j\ k)}$. This makes the bookkeeping significantly easier. An $F_{(n\ n+1)}$ combining with a fixed $F_{(i\ n)}$ comes at no $q$ cost, and when the newly minted $F_{(i\ n+1)}$ looks behind him in the walk sequence, he sees the same people who $F_{(i\ n)}$ saw previously, but also sees those terms of the form $F_{(j\ n)}$ with $j < i$. But then the number of $q$'s incurred by pulling the the $F_{(n\ n+1)}$ through the terms of the form $F_{(j\ n)}$ before finally combining with $F_{(i\ n)}$ plus the number of $q$'s from the original walk sequence to $p^*$ is precisely the number of $q$'s determined by the counting function $f$ to point $p$, so that the induction is complete. □

It is quite interesting to see that exactly one PBW monomial has no $q$'s as a coefficient in front of them. This is clear by just applying the straightening from Theorem 5.1 repeatedly and noting that combining the most terms possible iteratively from right to left is the only way to not incur a $q$. This leading term is special, and motivates the following discussion.

The act of combining as many powers of adjacent terms together in the quantum group for $A_n$ has a striking resemblance to a Hurwitz move performed on the corresponding element in the Weyl group, but in this case depends upon the relative sizes of exponents of adjacent terms in a product for the quantum group. We illustrate with an example, and formalize it afterwards.

**Example 5.11.** Consider $F_{\alpha_2}^{(n)} F_{\alpha_1}^{(m)}$ in $A_2$, and write out the positive cluster fan element $n \cdot s_2 + m \cdot s_1$ corresponding to this. One can perform the Hurwitz move leading to $s_1(s_1 s_2 s_1)$ on the Weyl group element $s_2 s_1$, or the move leading to $(s_2 s_1 s_2) s_2$, and these two cases become the cases for $m > n$ and $n < m$ respectively when thinking in the quantum group according to Lemma 5.2 and writing out the leading term. In the positive cluster fan, $n \cdot s_2 + m \cdot s_1$ gets sent to $min(n, m) s_1 s_2 s_1 + |m - n| s_i$ for $i = 1, 2$ based on sign of $m - n$. When they are equal, we instead get the subword $s_1 s_2 s_1 = s_2 s_1 s_2$ by the braid relations, and in the positive cluster fan in this case we end up on a diagonal rather than a maximal simplex.

This discussion holds in $A_n$ as well, by the same token of considering the small copies of $A_2$. We state this explicitly.

**Definition 5.12.** Assign to each root $\alpha_r + \ldots + \alpha_t$ its Hurwitz reflection to be $s_r$ conjugated by $s_{r+1}$ and successively conjugated by each $s_l$ in order until finally conjugating by $s_t$ . Given a monomial in PBW root vectors $F_{\beta_{i_1}}^{(m_{i_1})} F_{\beta_{i_2}}^{(m_{i_2})} \cdots F_{\beta_{i_j}}^{(m_{i_j})}$ with each $m_{i_k} > 0$, the *corresponding Weyl group element* will be the product of reflections $s_{\beta_{i_1}} s_{\beta_{i_2}} \cdots s_{\beta_{i_j}}$, where each $s_{\beta_{i_k}}$ is written as its assigned Hurwitz reflection in terms of simple roots. The *corresponding fan lattice element* will be $m_{i_1} \cdot s_{\beta_{i_1}} + m_{i_2} \cdot s_{\beta_{i_2}} + \ldots + m_{i_j} \cdot s_{\beta_{i_j}}$

**Definition 5.13** (Quantum Hurwitz Move)**.** Given a PBW monomial $F_{\beta_{i_1}}^{(m_{i_1})} F_{\beta_{i_2}}^{(m_{i_2})} \cdots F_{\beta_{i_j}}^{(m_{i_j})}$ with each $m_{i_k} > 0$, the straightening of two adjacent root vectors via an application of Theorem 5.1 performs a *quantum Hurwitz move* on the monomial, where the leading term is the only part of interest. The leading term has corresponding Weyl group element which has been changed by a usual Hurwitz move applied to the adjacent terms straightened, and corresponding fan lattice element will have the changed Weyl group element replacing the minimum of the two from the original fan lattice element.

**Example 5.14.** The rightmost to leftmost combination of terms to get a unique leading term in the expression $F_{\alpha_n}^{(m_n)} F_{\alpha_{n-1}}^{(m_{n-1})} \ldots F_{\alpha_2}^{(m_2)} F_{\alpha_1}^{(m_1)}$ performs a quantum Hurwitz move at each step, where the original corresponding Weyl group element $s_n s_{n-1} \cdots s_2 s_1$ gets sent to a subword of $s_n s_{n-1} \cdots (s_2 s_1 s_2) s_2$ or $s_n s_{n-1} \cdots s_1 (s_1 s_2 s_1)$ in the first iteration, and continues the combining process from right to left until completion.

In the case of a totally reversed monomial of simple roots, we see that at each step, the only terms with non-zero coefficient in the fan lattice point change in such a way to maintain the collection of vectors with non-zero coefficient to be a subword of the coxeter element $c = s_n s_{n-1} \cdots s_1$, as the corresponding Weyl group element is related by Hurwitz move.

In particular, since the leading term after all the untangling corresponds to an expression in the Weyl group that is a subword of successive applications of Hurwitz moves on the initial word $s_n s_{n-1} \cdots s_2 s_1$, we have almost proven the following theorem first conjectured by Professor Williams.

**Theorem 5.15** (Hurwitz moves in the quantum group)**.** *The leading term on the right hand side expression from Theorem 5.10 has fan lattice point on the positive cluster fan. This correspondence is bijective, so that for every element in the positive cluster fan, there is a unique unstraightened element in the quantum group that straightens to it.*

*Proof.* That every unstraightened monomial straightens to have a unique leading term which corresponds to an element of the positive cluster fan is known. Injectivity follows from preservation of the multiplicity of each root. Surjectivity follows from the fact that each each positive cluster fan element may be repackaged backwards to unstraighten in a unique way, just reversing the straightening process from Theorem 5.10. We conclude they are in bijection. □

We have established our two important results, one regarding walks, and one regarding the nature of the leading term in the expansion in the simplest case. Specifically, we chose $A_n$ with a chosen ordering on the positive roots, untangling an expression in the exact wrong order, but these results are likely generalizable with the heavy lifting done prior.

## 6. Generalizations and Future Research

The linear order is special in some sense, but we conjecture the following result, which would follow immediately if the shape of the relations from Theorem 5.3 remain unchanged under braid moves (the only change that we predict would occur is reversing the order of the three adjacent root vectors, and changing the compositions of the middle). If the braid move does not impact relations of the form Theorem 5.3 aside from possible exchanging the places of the non-commuting root vectors, we immediately have the following proposition.

**Proposition 6.1.** *let $w = s_{i_1} s_{i_2} \cdots s_{i_N}$ be a reduced expression for the longest word, and construct the PBW basis for the quantum group. Let $\{\gamma_i\}_{i=1}^n$ be the simple roots, re-indexed so that they are in the order induced by $w$. Then*

$$F_{\gamma_n}^{(m_n)} F_{\gamma_{n-1}}^{(m_{n-1})} \dots F_{\gamma_2}^{(m_2)} F_{\gamma_1}^{(m_1)} = \sum_{\omega \text{ walk to } p} q^{f(\omega)} F_\omega$$

*Where lattice point $p = m_1 \cdot \gamma_1 + m_2 \cdot \gamma_2 + \cdots + m_n \cdot \gamma_n$. Furthermore, the LHS straightens to have a unique leading term which corresponds to a lattice point on the positive cluster fan corresponding to the c-complex with Coxeter element c which multiplies the simple roots in the reverse order from the induced ordering.*

The technique of examining the rank 2 case of $A_n$ and applying it successively applies in the simply laced cases, perhaps believable so (after all, all of Tingley's lemmas hold in these cases), but also indeed applies to the non-simply laced cases as well, with a caveat accounting for the different length roots. We show the idea in $B_2$, and perform a similar induction to get to $B_n$

To address the differing length roots, we wish to ascribe a weight depending on the normalized length of the vectors normally counted in the inversion number.

**Definition 6.2** ($B_n$ inversion number)**.** for shorthand, two positive roots $\gamma_i, \gamma_j$ satisfy condition $\star$ if $\gamma_i + \gamma_j$ admits a Kostant partition with element of height $> \max(h(\gamma_i), h(\gamma_j))$. Given a walk $\omega = (\gamma_1, \gamma_2, \dots, \gamma_l)$ in $B_n$, we defined the *$B_n$ inversion number*

$$\begin{aligned} \operatorname{inv}(\boldsymbol{\omega}) =& |\{i < j : \gamma_i \text{ and } \gamma_j \text{ satisfy condition } (\star)\}| \\ &+ |\{i < j : \gamma_i \text{ and } \gamma_j \text{ satisfy condition } (\star) \text{ and } |\gamma_i| = \sqrt{2}\}| \end{aligned}$$

Basically, we want to double count those pairs which have a preceding longer root in $B_n$. We could see how one may generalize this to an arbitrary root system by counting pairs multiple times based on the length of the preceding root (normalized to the shortest root). The counting function definition does not change (except that now it uses this generalized inversion number), nor does the definition for $F_\omega$ from the previous section. We illustrate the idea with an small example in $B_3$, again using the QuaGroup package.

**Example 6.3.**

$$\begin{aligned} F_{\alpha_3}^{(2)} F_{\alpha_2}^{(1)} F_{\alpha_1}^{(2)} =& F_{\alpha_1} F_{\alpha_1+\alpha_2+2\alpha_3} + q F_{\alpha_1} F_{\alpha_1+\alpha_2+\alpha_3} F_{\alpha_3} + q^4 F_{\alpha_1}^{(2)} F_{\alpha_2+2\alpha_3} \\ &+ q^4 F_{\alpha_1} F_{\alpha_1+\alpha_2} F_{\alpha_3}^{(2)} + q^5 F_{\alpha_1}^{(2)} F_{\alpha_2+\alpha_3} F_{\alpha_3} + q^8 F_{\alpha_1}^{(2)} F_{\alpha_2+\alpha_3} F_{\alpha_3} \end{aligned}$$

Inspection shows the counting process with the new definition of inversion number checks out.

**Proposition 6.4.** *In $B_n$, with standard selection of roots in linear ordering $\alpha_1, \alpha_1 + \alpha_2, \alpha_2, \ldots, \alpha_{n-1} + \alpha_n, \alpha_{n-1} + 2\alpha_n, \alpha_n$, we have*

$$F_{\alpha_n}^{(m_n)} F_{\alpha_{n-1}}^{(m_{n-1})} \ldots F_{\alpha_2}^{(m_2)} F_{\alpha_1}^{(m_1)} = \sum_{\omega \text{ walk to } p} q^{f(\omega)} F_\omega$$

*Proof sketch:* We will have a rank two case similar to Theorem 5.1 for $B_2$, proving with induction and properties of symmetric quantum integers. From here, we will effectively use our non-standard choice of positive roots to imagine what happens when we glue on a long root to the end. Loosely, this is like gluing a root $\alpha_0 = \epsilon_0 - \epsilon_1$ on the left and pulling it through in the induction step, utilizing the rank two case to split it into every possible walk, and again having a unique term with no $q$'s obtained by combining at every step. The only roots for which this new root talks to are those of the form $\alpha_1 + \ldots$, and the elementary relations for combining this vector to $\alpha_0 + \alpha_1 + \ldots$ will encode exactly a $q^2$ for every time we do not combine to form such a vector, which remembers the new lattice point precisely as it should in the walk interpretation. □

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