SemiDistrim Lattices

Never in the history of mathematics has a mathematical theory been the object of such vociferous vituperation as lattice theory.

— Gian-Carlo Rota

Colin Defant

Nathan Williams

BIRS DAC 5 Nov 2021

WHY LATTICES?

(over general posets)

Posets ARE algebra

$a \leq b$ iff $a \vee b = b$

WHERE LATTICES?

- submodules/subgroups
- closed subspaces of topological spaces
- torsion classes
- hyperplane arrangements/matroids
- cluster algebras
- ⋮

A HISTORY OF LATTICES

1880 — **C.S. Peirce**

"All lattices are distributive."

"easily proved ... but the proof is too tedious to give."

1890 — **Schröder**

"Not all lattices are distributive."

1900 — **Dedekind**

- Lattices defined "Dualgruppen"
- discovers 28 (the number)

1933 — **Birkhoff** establishes lattice theory as "a thing"

1939 — **Dov Tamari** imprisoned for explosives

"a scatterbrain, like all mathematicians"

1954, 8AM — **Ore**

"I think lattice theory is played out."

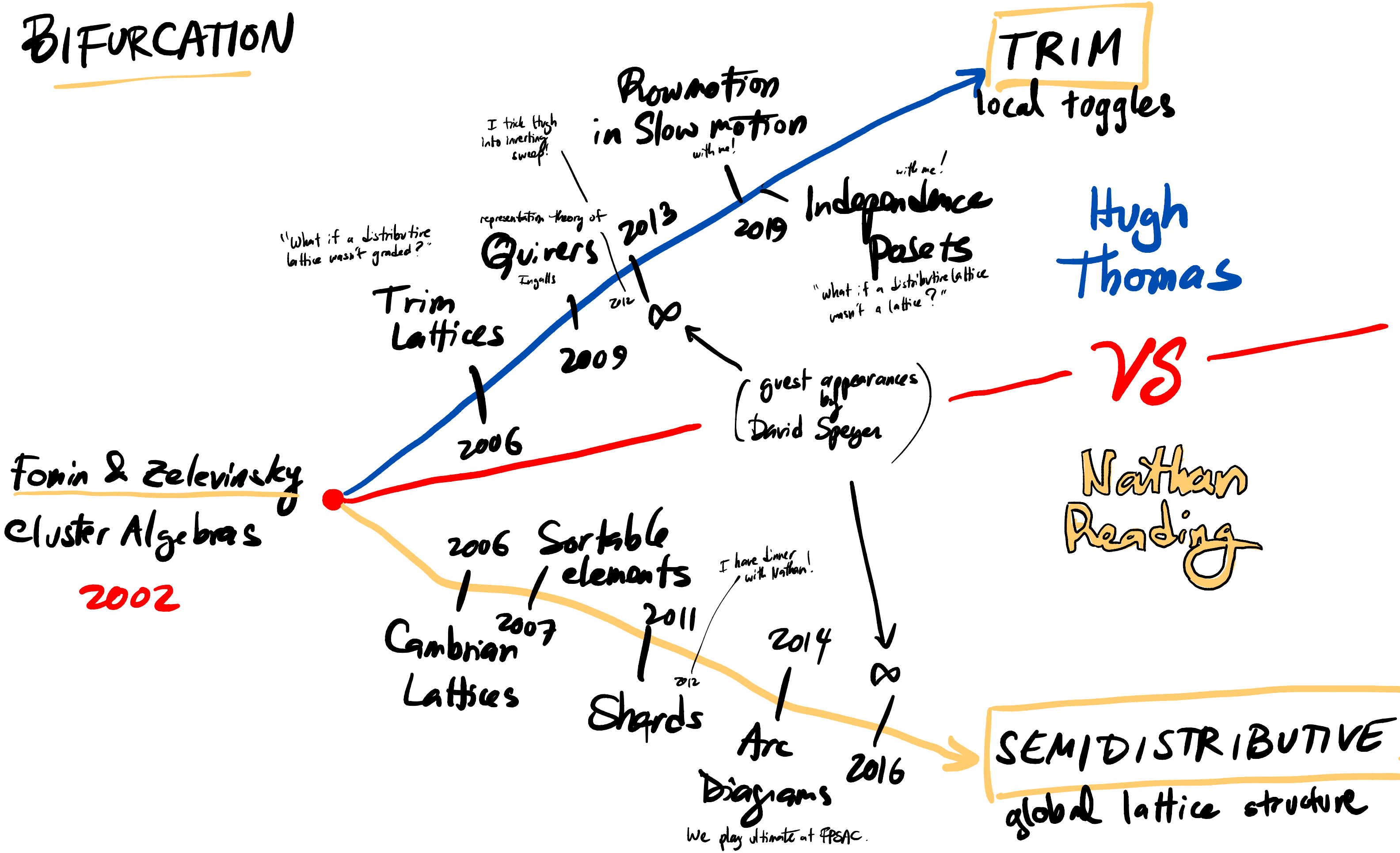
1972 — All lattice theorists move to Hawaii

2002 — **Fomin & Zelevinsky**

cluster Algebras

Rota. "The many lives of lattice theory"
Grätzer. "Two problems that shaped a century of lattice theory"
Schlimm. "On the creative role of axiomatics"
Nation. "Jónsson's contributions to lattice theory"
"Tamari Memorial Festschrift"
Freese, Ježek, Nation. "Free Lattices"
Bilová. "Lattice theory -- its birth and life"

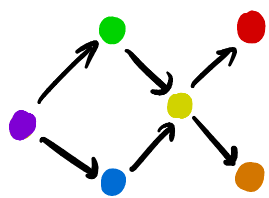
BIFURCATION



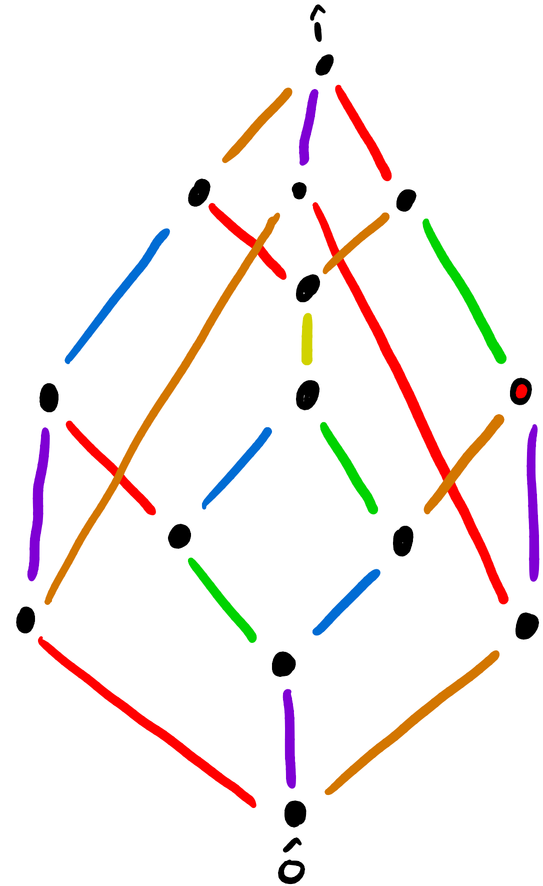
The Hugh Thomas school of TRIM lattices

(local toggles)

① Quiver : 

② Indecomposable Modules : 

③ Lattice of torsion classes :

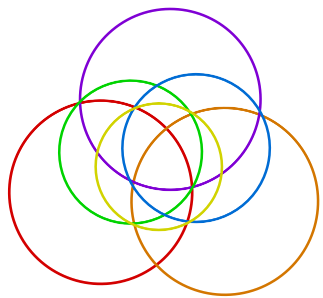


④ Rowmotion : 

the Nathan Reading school of SEMIDISTRIBUTIVE lattices.

(global lattice structure)

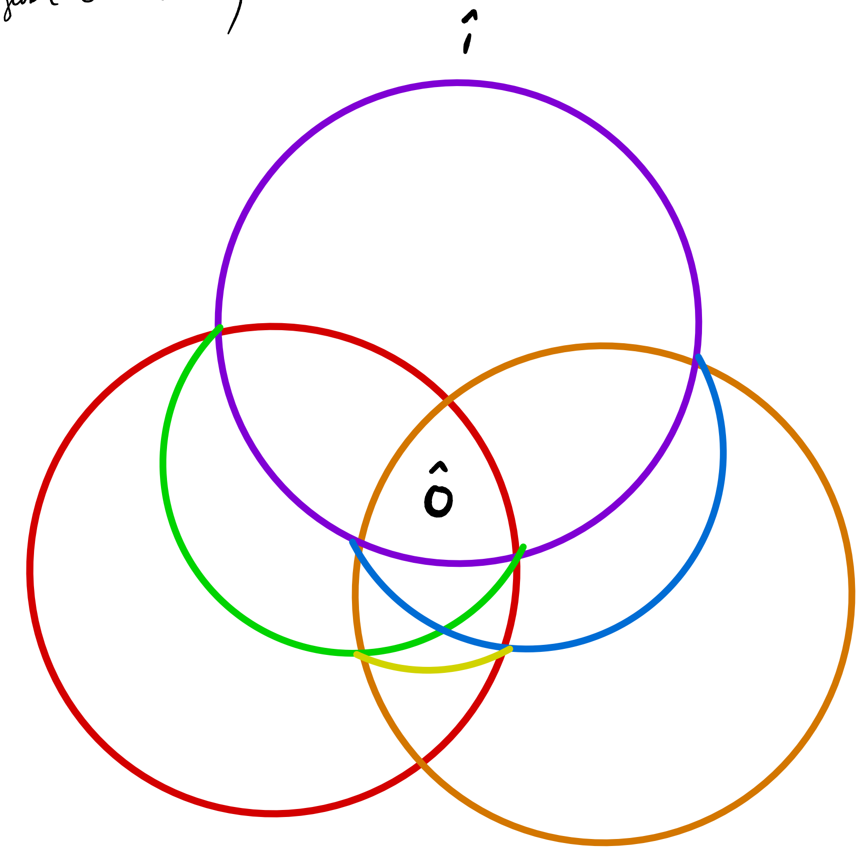
① Hyperplane arrangement:



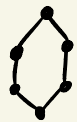
② Shards:

③ Lattice of regions:

④ Rowmotion: $D \to U$



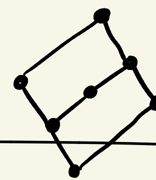
TODAY: RESOLUTION.



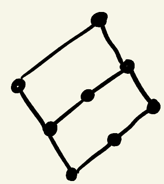
SEMIDISTRIBUTIVE

\wedge $\vee$

TRIM



SEMIDISTRIM



LATTICE VARIETIES (ignore cardinality considerations)

DEF: Mod : a set of lattice equations $\mathcal{E}$ $\longrightarrow$ $\underbrace{\text{the set of lattices } \mathcal{V} \text{ satisfying } \mathcal{E}}_{\text{a lattice variety.}}$

DEF: Id : a set of lattices $\mathcal{V}$ $\longrightarrow$ the set of lattice equations $\mathcal{E}$ satisfied by all lattices in $\mathcal{V}$

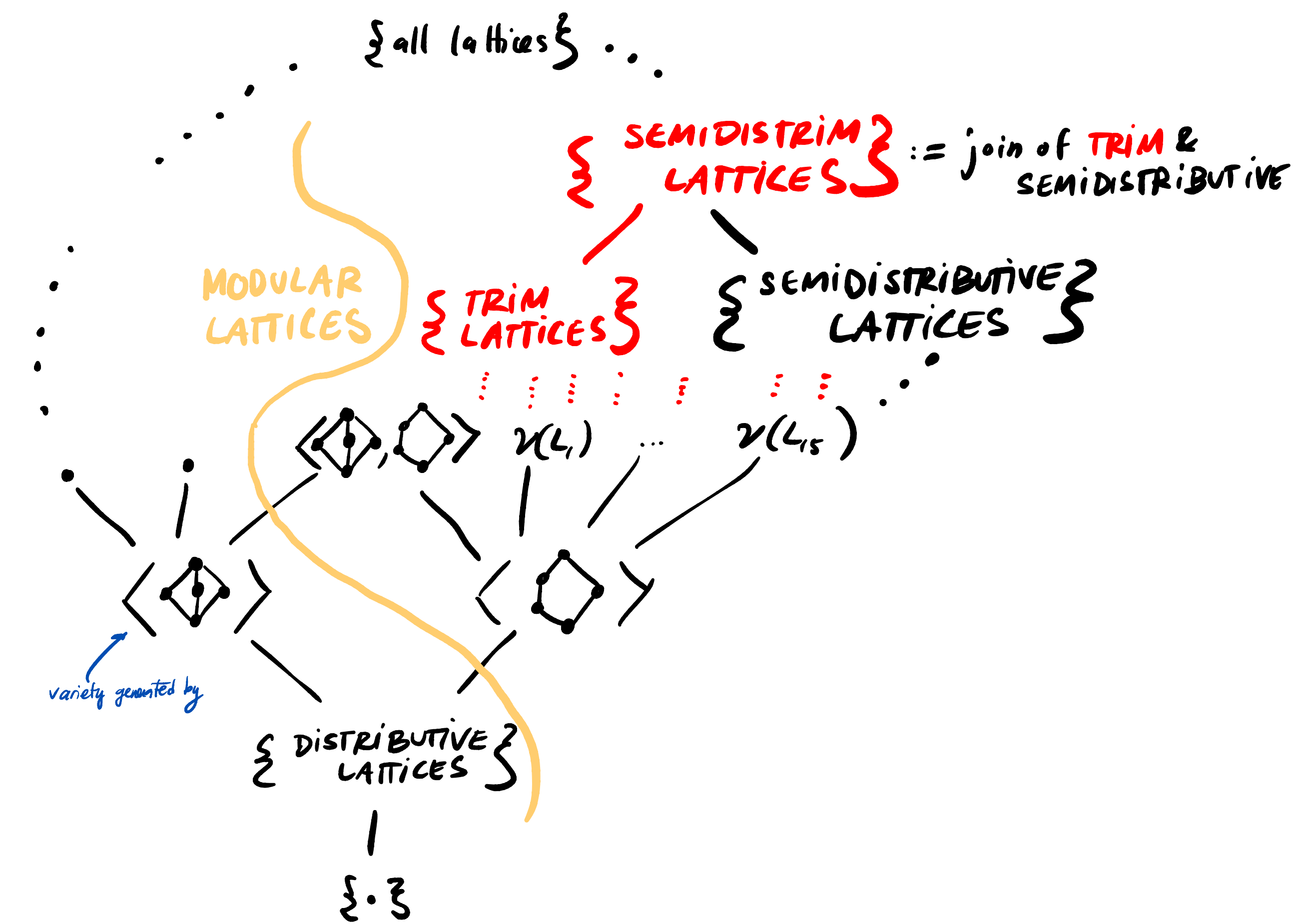
if $\mathcal{V}$ a lattice variety, $\mathcal{V} = \mathrm{Mod}\,\mathrm{Id}\,\mathcal{V}$.

EX: $\mathrm{Mod}\{x = y\} = \{\bullet\}$

$\mathrm{Mod}\{x(y+z) = xy + xz\} = \{\text{DISTRIBUTIVE LATTICES}\}$

Let's order lattice varieties by inclusion!

THE LATTICE OF LATTICE VARIETIES



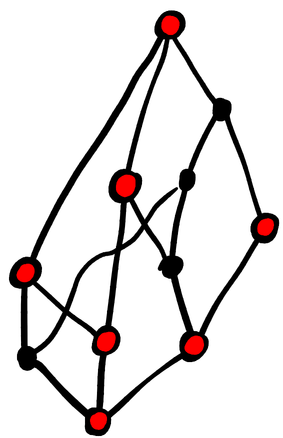
Nation. "Jónsson's contributions to lattice theory"

PROBLEM: the set of trim lattices is not a lattice variety!

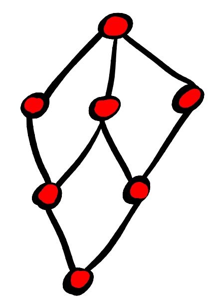
THM (Birkhoff): Lattice varieties are the subsets of lattices closed under:

- homomorphic images
- sublattices
- direct products

The trim lattice



has the untrim sublattice



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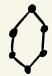
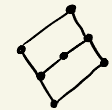
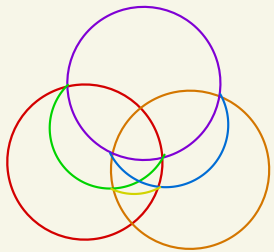
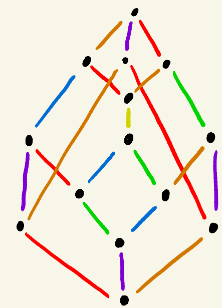
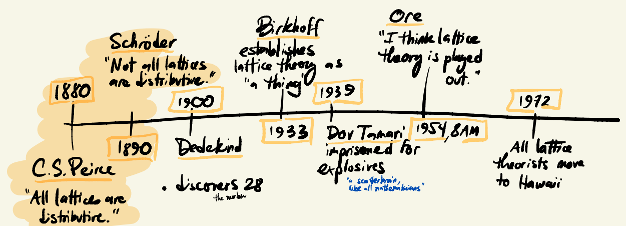
 SEMIDISTRIBUTIVE ∨ TRIM 

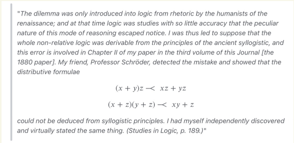
Let's identify what TRIM & SEMIDISTRIBUTIVE lattices have in common:



ROWMOTION






"The dilemma was only introduced into logic from rhetoric by the humanists of the renaissance; and at that time logic was studies with so little accuracy that the peculiar nature of this mode of reasoning escaped notice. I was thus led to suppose that the whole non-relative logic was derivable from the principles of the ancient syllogistic, and this error is involved in Chapter II of my paper in the third volume of this Journal [the 1880 paper]. My friend, Professor Schröder, detected the mistake and showed that the distributive formulae

$$(x + y)z \prec xz + yz$$

$$(x + z)(y + z) \prec xy + z$$

could not be deduced from syllogistic principles. I had myself independently discovered and virtually stated the same thing. (Studies in Logic, p. 189.)"

https://hsm.stackexchange.com/questions/13018/in-what-sense-was-it-thought-at-one-point-that-every-lattice-was-distributive

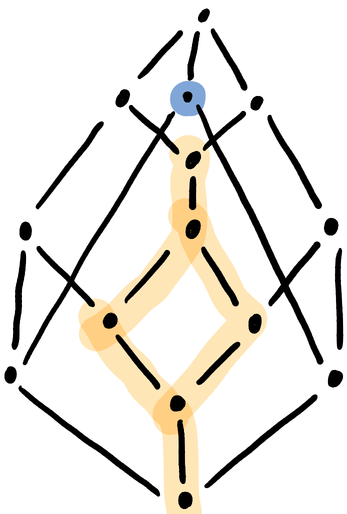
What if we "build rowmotion in"?

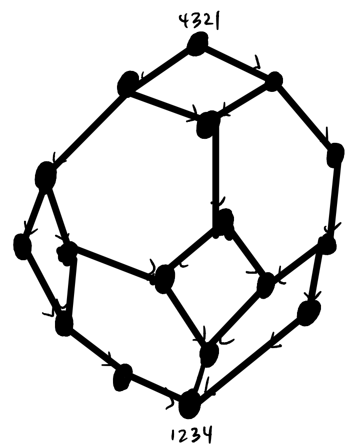
THM (Defant–W.): If L is semidistributive, then

"pop = row"

$$\overset{D\to U}{\mathrm{Row}}(x) = \max\Big\{y\in L : \overbrace{\mathrm{Pop}^{\downarrow}(x)}^{x\wedge\bigwedge_{z\lessdot x} z} = x\wedge y\Big\}$$

$$\underset{U\to D}{\mathrm{Row}'}(x) = \min\Big\{y\in L : \underbrace{\mathrm{Pop}^{\uparrow}(x)}_{x\vee\bigvee_{x\lessdot z} z} = x\vee y\Big\}.$$





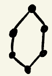
What if we "build rowmotion in"?

DEF: L is *uniquely completely paired* if there is a unique bijection $\mathrm{Row}: L \to L$ so that

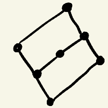
$$\mathrm{Row}(x) \in \max\{y \in L : \mathrm{Pop}^{\downarrow}(x) = x \wedge y\}$$
$$\mathrm{Row}^{-1}(x) \in \min\{y \in L : \mathrm{Pop}^{\uparrow}(x) = x \vee y\}.$$

PROBLEM: No structure!
We can't prove anything about these!

"too many lattices"

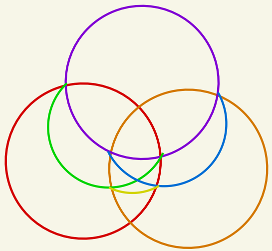


SEMIDISTRIBUTIVE ∨ TRIM

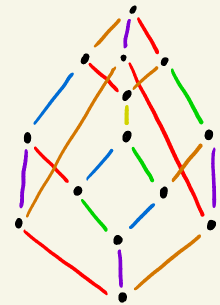


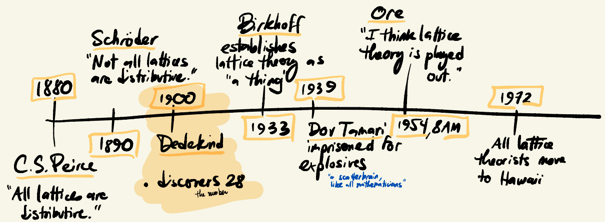
Let's identify what TRIM & SEMIDISTRIBUTIVE lattices have in common:

1

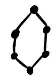
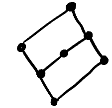


$J \leftrightarrow M$



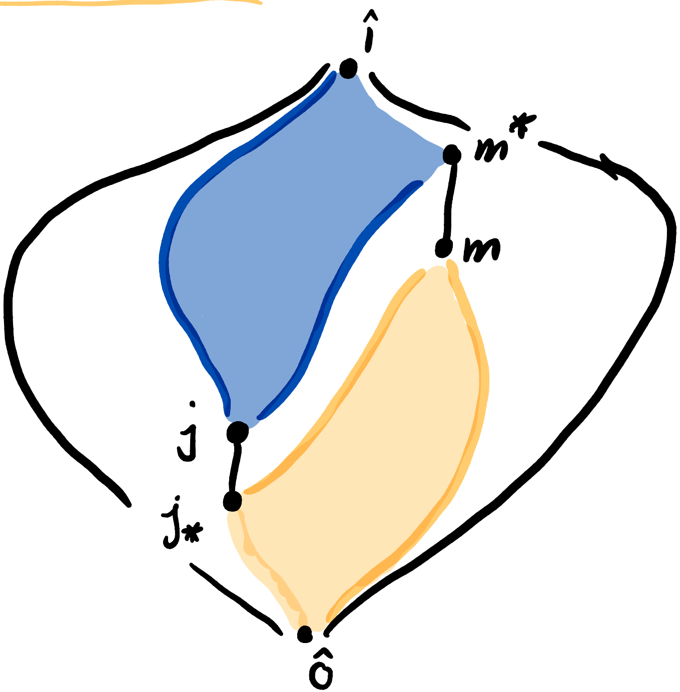


https://blogs.ams.org/visualinsight/2016/01/01/free-modular-lattice-on-3-generators/

 SEMIDISTRIBUTIVE ∨ TRIM 

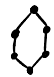
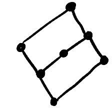
① A canonical bijection between $\overbrace{\text{join-}}^{J}$ and $\overbrace{\text{meet-irreducibles}}^{M}$.

SEMIDISTRIBUTIVE



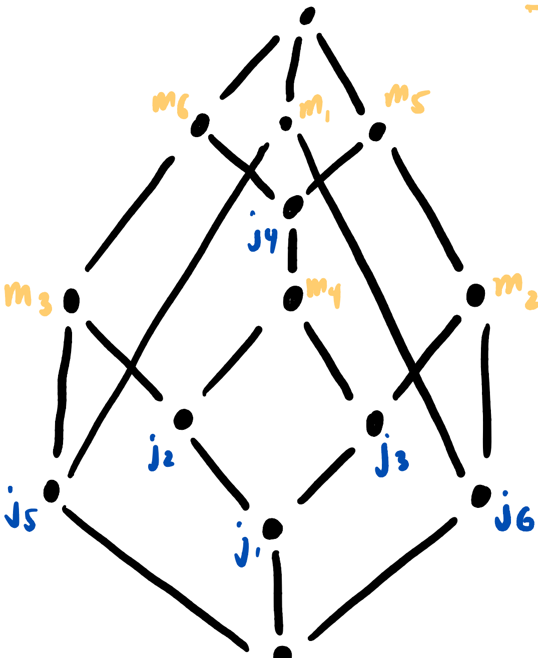
$$m = \max\left\{x \in L : \begin{array}{l} x \geq j_* \\ x \not\geq j \end{array}\right\}$$

$$j = \min\left\{x \in L : \begin{array}{l} x \leq m^* \\ x \not\leq m \end{array}\right\}$$

 SEMIDISTRIBUTIVE ∨ TRIM 

① A canonical bijection between $\overbrace{\text{join-}}^{J}$ and $\overbrace{\text{meet-irreducibles}}^{M}$.

SEMIDISTRIBUTIVE



$$m = \max\left\{ x \in L : \begin{array}{l} x \geq j_* \\ x \not\geq j \end{array} \right\}$$

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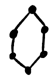
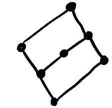
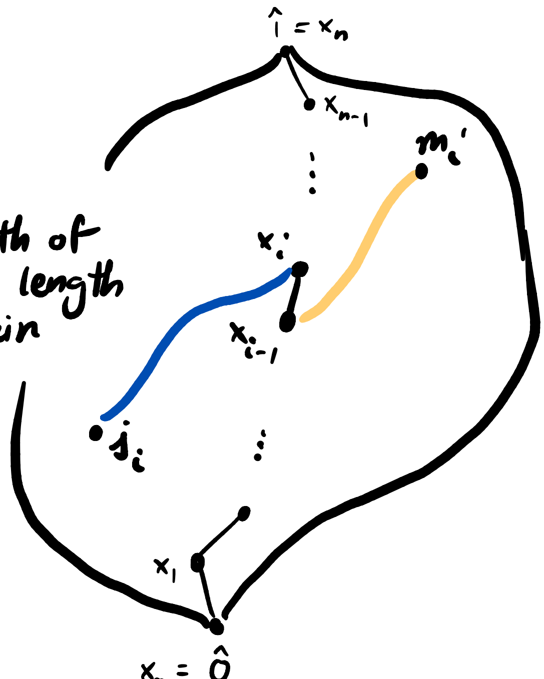
 SEMIDISTRIBUTIVE $\vee$ TRIM 

① A canonical bijection between $\overbrace{\text{join-}}^{J}$ and $\overbrace{\text{meet-}}^{M}$ irreducibles.

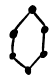
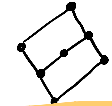
TRIM

$$\underbrace{|J| = |M|}_{n} = \text{length of max length chain}$$



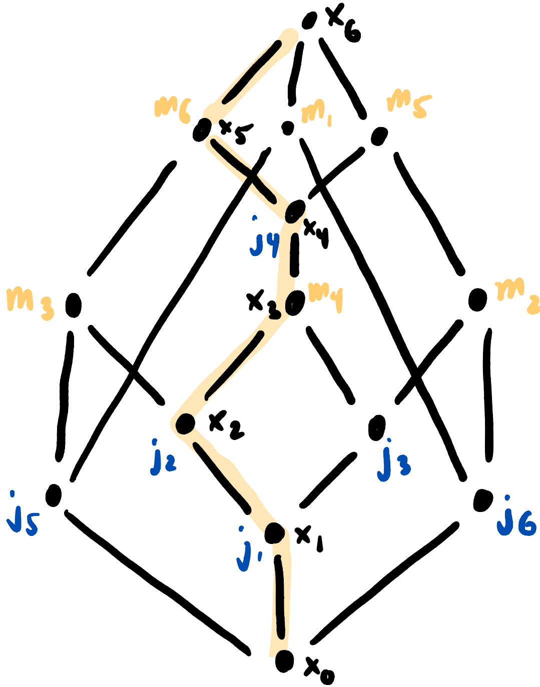
$$x_{i-1} = x_i \wedge m_i$$

$$x_i = x_{i-1} \vee j_i$$

 SEMIDISTRIBUTIVE $\vee$ TRIM 

① A canonical bijection between $\overbrace{\text{join-}}^{J}$ and $\overbrace{\text{meet-irreducibles}}^{M}$.

TRIM



$$x_{i-1} = x_i \wedge m_i$$

$$x_i = x_{i-1} \vee j_i$$

① A canonical bijection between join- and meet-irreducibles.

DEF: A lattice L is uniquely paired if there is a unique bijection

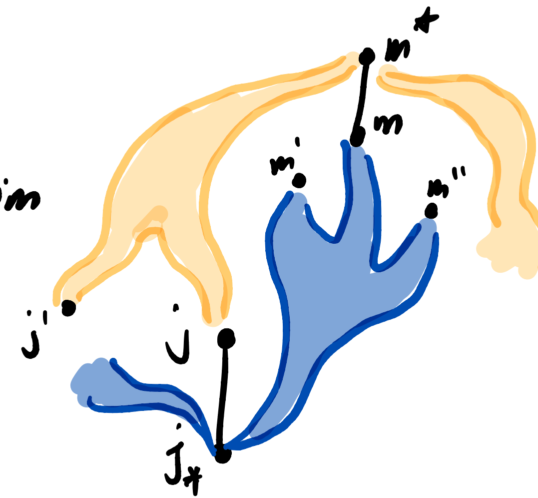
$\kappa : J \to M$ with

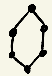
$$\kappa(j) \in \max\{x \in L : j_* = j \wedge x\}$$

$$\kappa^{-1}(m) \in \min\{x \in L : m^* = m \vee x\}.$$

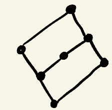
a "pairing"

PROP (Defant-W.): Semidistributive & trim lattices are uniquely paired.

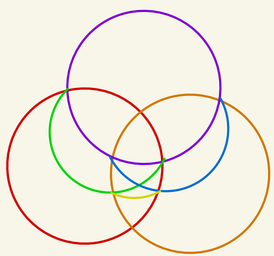




SEMIDISTRIBUTIVE $\vee$ TRIM

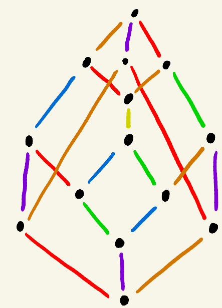


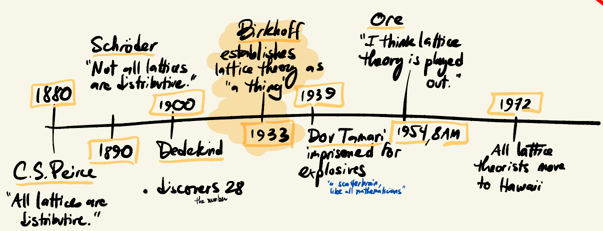
Let's identify what TRIM & SEMIDISTRIBUTIVE lattices have in common:



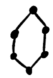
2

PROOFS!

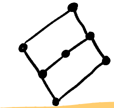




Bilová. "Lattice theory — its birth and life"

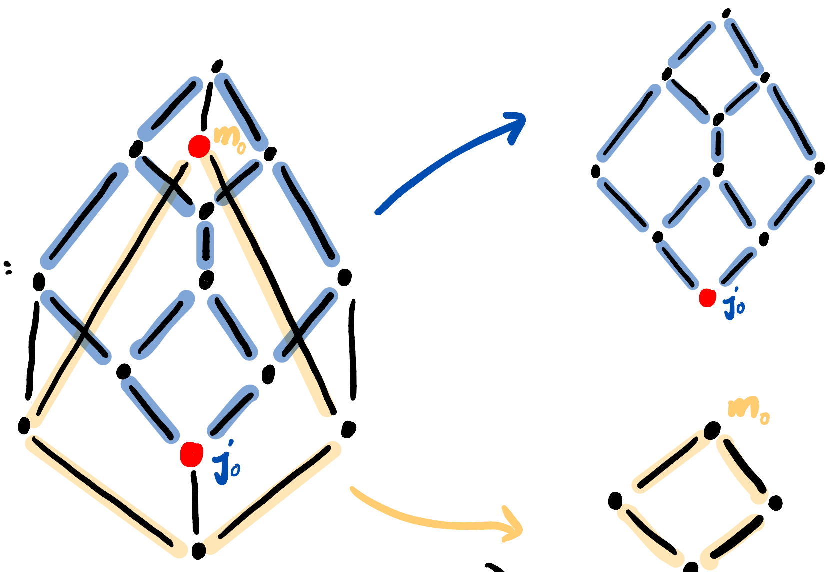


SEMIDISTRIBUTIVE $\vee$ TRIM



2) You can prove things about them!

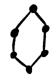
Decomposition:



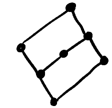
also semidistributive/trim

also semidistributive/trim

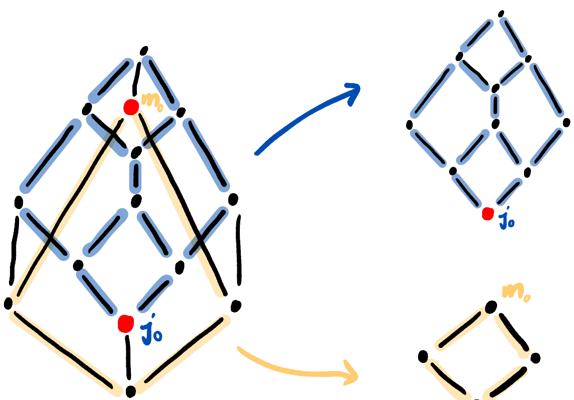
(prototype: Cambrian recurrence)



SEMIDISTRIBUTIVE v TRIM



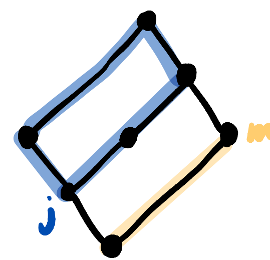
② You can prove things about them!

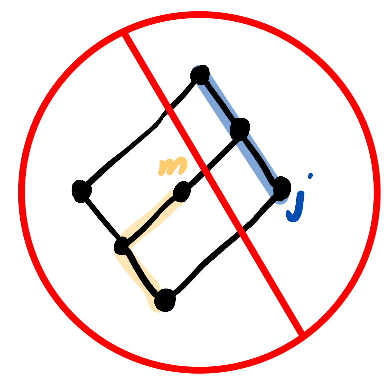


Semidistributive: any atom gives such a decomposition

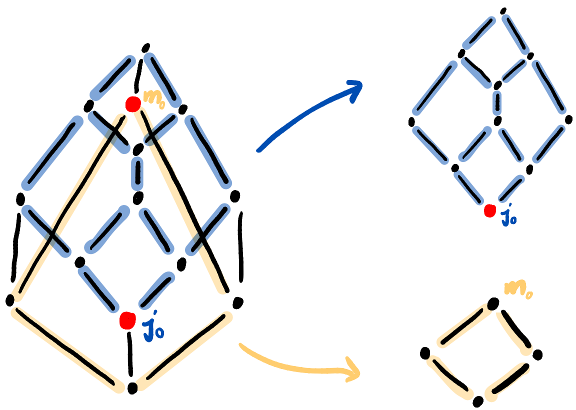
Trim: some atom (covers $\hat{0}$) gives such a decomposition.

Ex: Trim





② You can prove things about them!



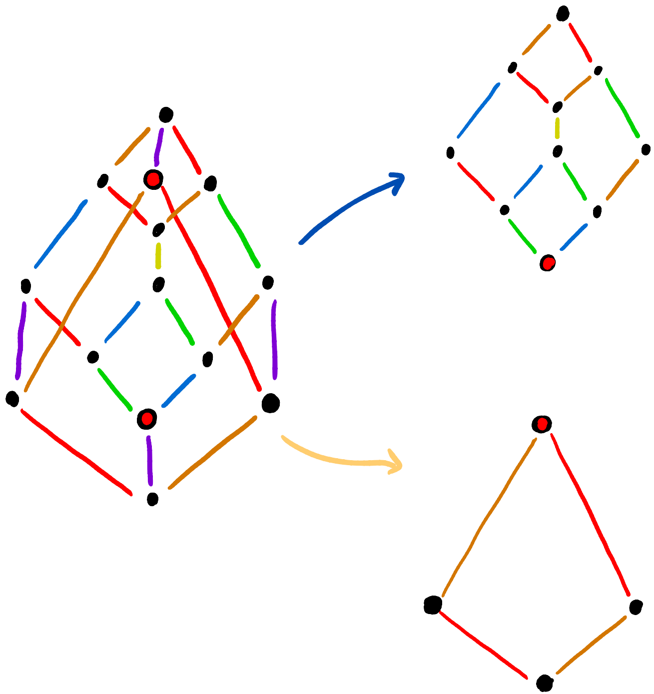
usual: $x \vee y \geq j_0 \Rightarrow x \geq j_0$ or $y \geq j_0$

DEF: An element $j_0 \in L$ is join-prime if there exists $m_0 \in L$ such that

$$L = [\hat{0}, m_0] \sqcup [j_0, \hat{1}].$$

PROP: If $\kappa: J \to M$ is a pairing and j_0 is join-prime, then $\kappa(j_0) = m_0$.

② You can prove things about them!



DEF (Adaricheva, Hyndman, Lempp, Nation): L is internal dismantlable if

$|L| = 1$ or

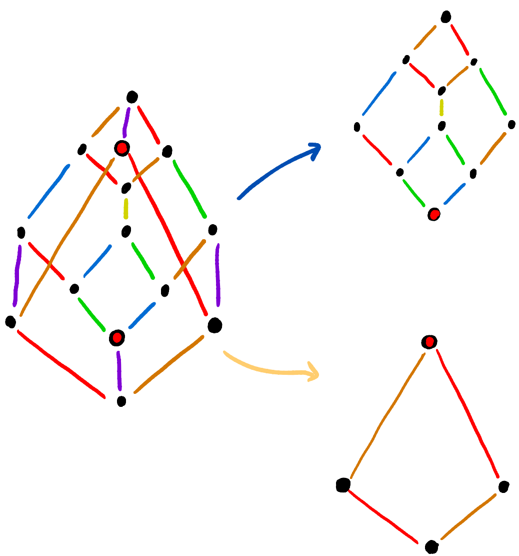
$L = [\hat{0}, m_0] \cup [j_0, \hat{1}]$ with both

$[\hat{0}, m_0]$ & $[j_0, \hat{1}]$ internal dismantlable.

But what about colors?

② You can prove things about them!

But what about colors?



DEF (Defant-W.): A uniquely paired lattice L is *compatibly dismantlable* if

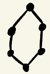
$|L| = 1$ or

$$L = [\hat{0}, m_0] \sqcup [j_0, \hat{1}]$$

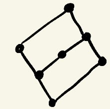
with $[\hat{0}, m_0]$ compatibly dismantlable and if $j \leq m_0$, then j pairs with $m_0 \wedge \kappa(j)$ in $[\hat{0}, m_0]$

& $[j_0, \hat{1}]$ compatibly dismantlable and if $m \geq j_0$, then m pairs with $j_0 \vee \kappa^{-1}(m)$ in $[j_0, \hat{1}]$

PROP (Defant-W.) Semidistributive & trim lattices are compatibly dismantlable.

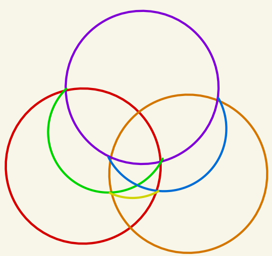


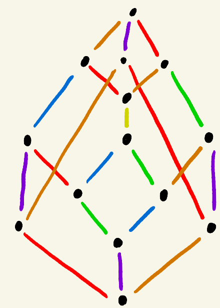
SEMIDISTRIBUTIVE $\vee$ TRIM



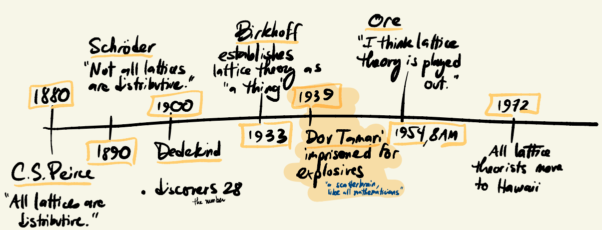
Let's identify what TRIM & SEMIDISTRIBUTIVE lattices have in common:

3





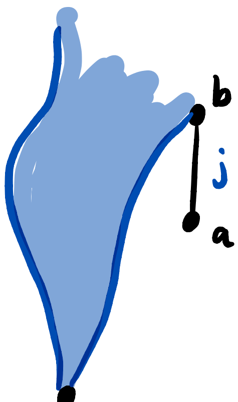
EDGE LABELS



Müller-Hoissen & Walther. "Dov Tamari (formerly Bernhard Teitler)" in "Tamari Memorial Festschift"

③ Canonical edge labelings by join-irreducibles

SEMIDISTRIBUTIVE



$j = \min\{x \in L : a \vee x = b\}$

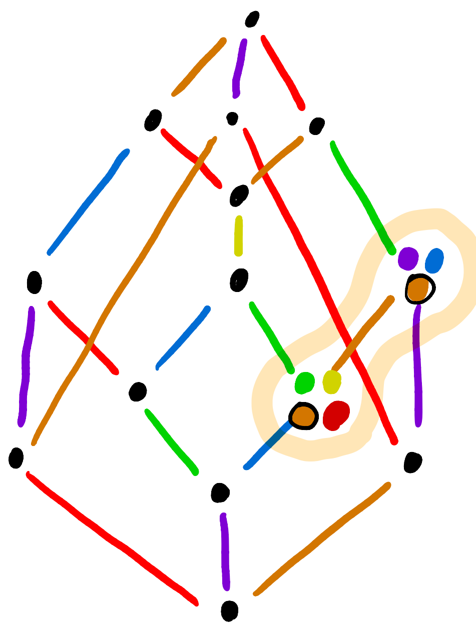
TRIM

$j_1, j_2, \cdots, j_n$

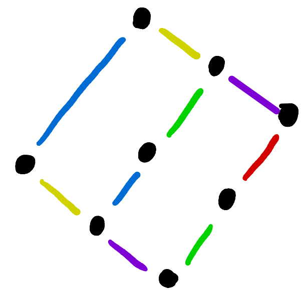
i minimal such that $a \vee j_i = b$.

③ Canonical edge labelings by join-irreducibles

DEF: A uniquely paired lattice L is overlapping if for every cover $x \lessdot y$,
$$\{ j \in J : j \leq y \} \cap \{ j \in J : \kappa(j) \geq x \} = \{ j_{xy} \}.$$



PROP (Defant-W.): Compatibly dismantlable lattices are overlapping.



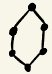
Just as with semidistributive & trim lattices, edge labels specify elements:

DEF: write $D(x) = \{ j_{yx} : y \lessdot x \}$ and $U(x) = \{ j_{xy} : x \lessdot y \}$.

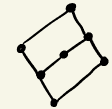
THM: Let L be compatibly dismantlable. Then for $x \in L$,

$$x = \bigvee D(x) = \bigwedge \kappa(U(x)).$$

PROBLEM: intervals in compatibly dismantlable are not necessarily compatibly dismantlable.

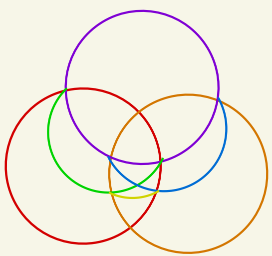


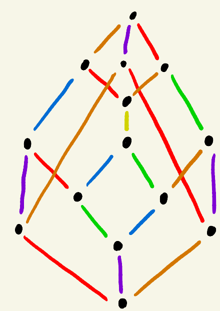
SEMIDISTRIBUTIVE $\vee$ TRIM



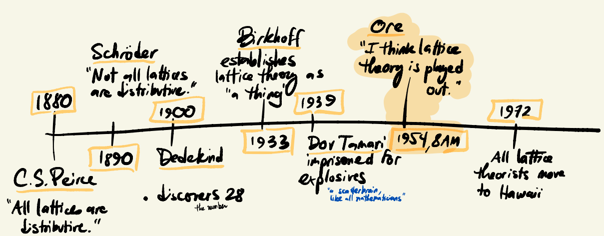
Let's identify what TRIM & SEMIDISTRIBUTIVE lattices have in common:

4

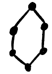
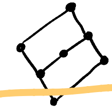




CANONICAL JOIN REP

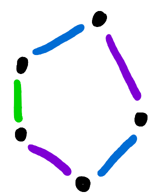
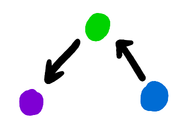


Rota. "The many lives of lattice theory"

 SEMIDISTRIBUTIVE $\vee$ TRIM 

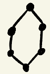
4 Canonical join representations

DEF: Let L be uniquely paired. Its (directed) Galois graph G_L has vertices J and edges $j \to j'$ iff $j \not\leq \kappa(j')$ and $j \neq j'$.

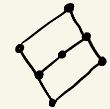
$L =$  $\qquad G_L =$ 

Write $\mathrm{Ind}(G_L)$ for the set of independent sets in G_L.

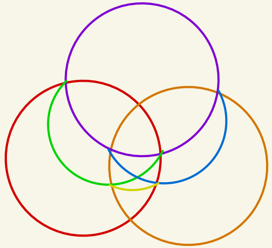
THM: If L is semidistributive or trim, $x \longmapsto D(x)$ & $x \longmapsto U(x)$ are bijections from L to $\mathrm{Ind}(G_L)$.

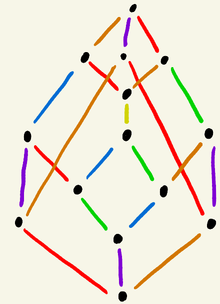


SEMIDISTRIBUTIVE ∨ TRIM



5





SEMIDISTRIM

- **1880** C.S. Peirce: "All lattices are distributive."
- **1890** Schröder: "Not all lattices are distributive."
- **1900** Dedekind
  - discovers 28 (the number)
- **1933** Birkhoff establishes lattice theory as "a thing"
- **1939** Dov Tamari imprisoned for explosives ("a scatterbrain, like all mathematicians")
- **1954, 8AM** Ore: "I think lattice theory is played out."
- **1972** All lattice theorists move to Hawaii

"Algebras and Lattices in Hawai'i: a conference in honor of Ralph Freese, William Lampe, and J.B. Nation."

DEF (Defant-W.):

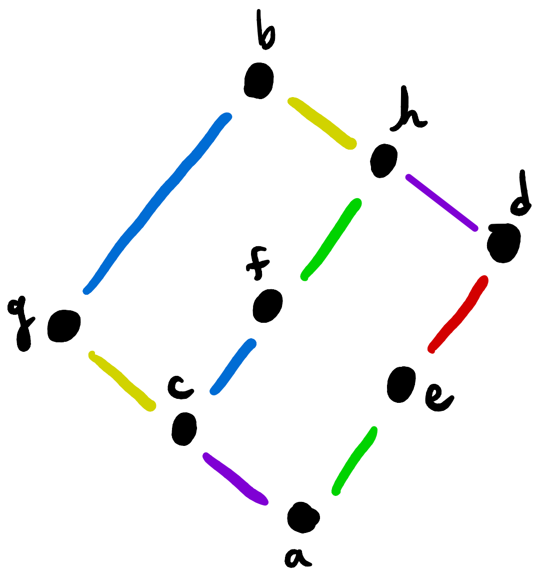
A lattice L is semidistrim if it is $\overbrace{\text{compatibly dismantlable}}^{\text{"can prove things"}}$ **and** if $\underbrace{D(x), U(x) \in \mathrm{Ind}(G_L}_{\text{"canonical join rep"}})$ for all $x \in L$.

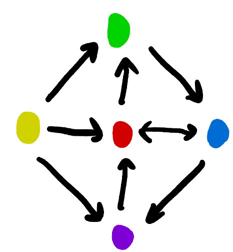
THM (Defant-W.) Products and intervals of semidistrim lattices are semidistrim.
(but not sublattices)

THM (Defant-W.) The order complex of a semidistrim lattice is either contractible or homotopy equivalent to a sphere.

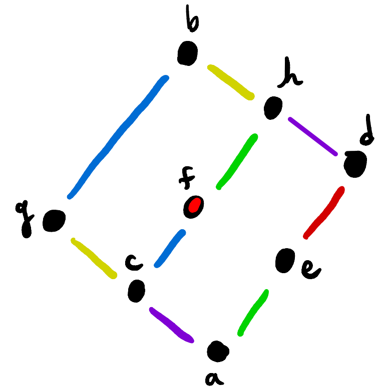
THM (Defant-W.) For L semidistrib, $D: L \to \mathrm{Ind}(G_L)$, $U: L \to \mathrm{Ind}(G_L)$ are bijections.

So we can define $\mathrm{Row}(x) = y$ for $D(x) = U(y)$.





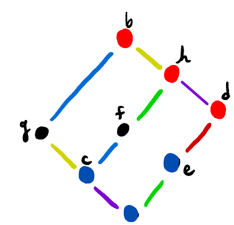
THM (Defant–W.) For L semidistrim, $\mathrm{Pop}^{\downarrow}(x) = x \wedge \mathrm{Row}(x)$.

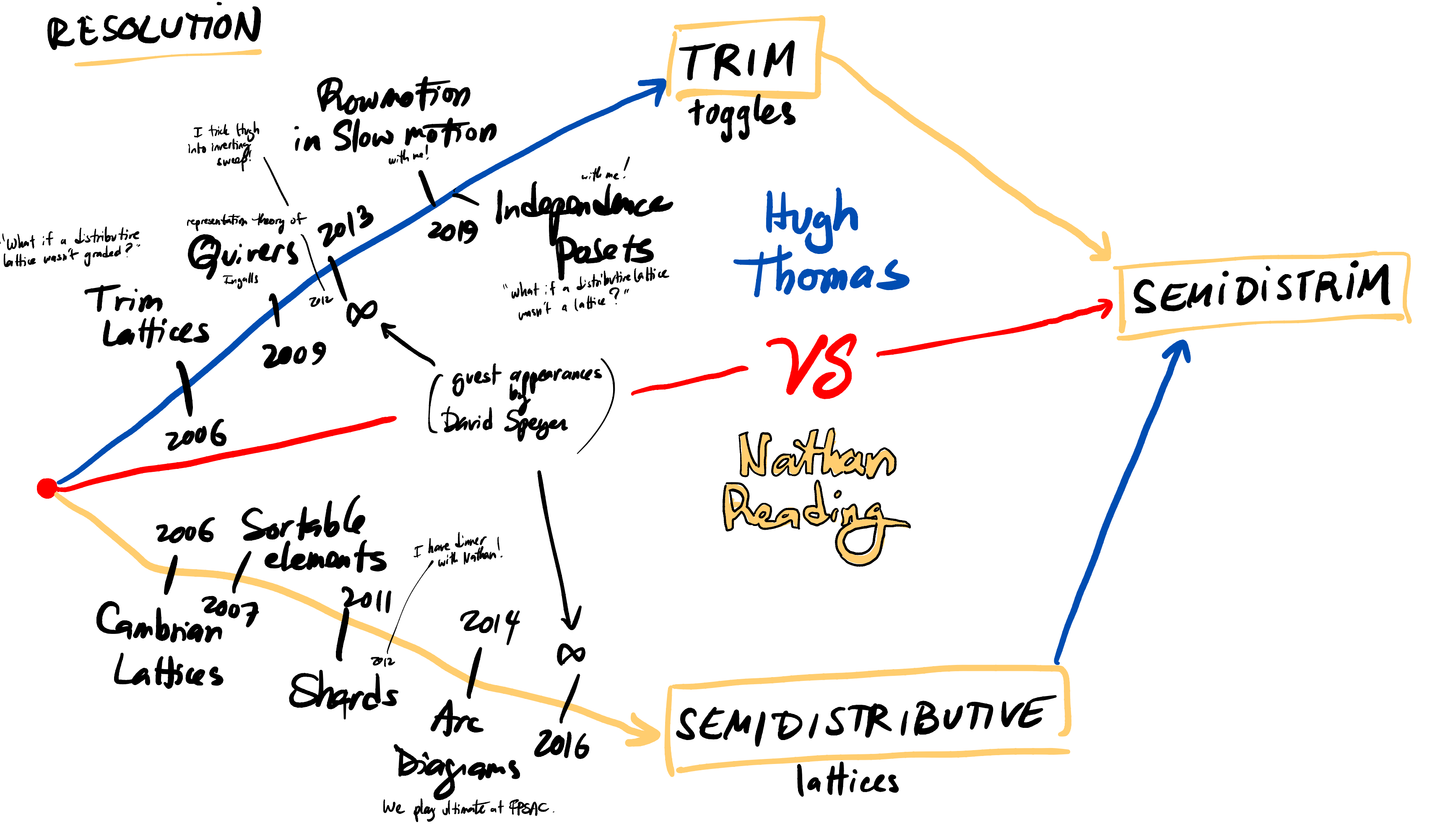


DEF: An independent dominating set is an independent set I such that every vertex of G is in I or adjacent to a vertex in I.

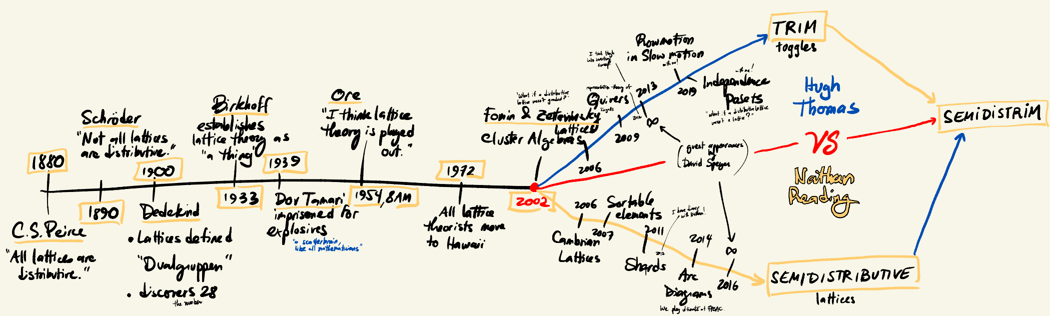
THM (Defant–W.) For L semidistrim,

$$\left|\{x \in L : \mathrm{Row}(x) \le x\}\right| = \left|\mathrm{Pop}(L)\right| = \left|\mathrm{Ind}^{\mathrm{dom}}(G_L)\right|.$$




RESOLUTION
TRIM
toggles
Rowmotion
in Slow motion
with me!
I trick Hugh
into inverting
sweep!
2019
with me!
Independence
Posets
"what if a distributive lattice
wasn't a lattice?"
representation theory of
Quivers
Ingalls
2013
2012
∞
"What if a distributive
lattice wasn't graded?"
Trim
Lattices
2009
2006
guest appearances
by
David Speyer
Hugh
Thomas
VS
Nathan
Reading
SEMIDISTRIM
2006
Sortable
elements
2007
Cambrian
Lattices
2011
I have dinner
with Nathan!
2012
Shards
2014
Arc
Diagrams
We play ultimate at FPSAC.
∞
2016
SEMIDISTRIBUTIVE
lattices

THANK YOU!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SEMIDISTRIBUTIVE
lattices

Open problems

- closed under quotients?
- characterize galois graphs?
- toggles?
- interesting families?