COMBINATORIAL TOOLS IN REPRESENTATION THEORY

1. Introduction

This proposal would fund a research program on new combinatorial tools with applications to representation theory, Macdonald theory, and Schubert Calculus. There are many natural algebraic objects whose Hilbert or Frobenius series are known or expected to be indexed by purely combinatorial objects—the study of such indexing sets is a rich field at the intersection of representation theory and algebraic combinatorics, with connections to many different areas of mathematics (for example, cohomology of Lie groups, the Hilbert scheme of points in the plane, Deligne-Lusztig varieties,...). There are, however, many remaining problems that lack a combinatorial understanding. For example, the *Macdonald polynomials* P_μ are an important family of symmetric functions that specialize to both the Hall-Littlewood polynomials and to the Jack polynomials. As conjectured by Garsia and Haiman and proved by Haiman, the *modified Macdonald polynomials* $\widetilde{H}_\mu(\mathbf{x}_n; q, t)$ are the Frobenius series of an $\mathfrak{S}_n$-module, and so are Schur positive:

$$\widetilde{H}_\mu(\mathbf{x}_n; q, t) = \sum_{\lambda \vdash n} \widetilde{K}_{\lambda,\mu}(q, t) s_\lambda(\mathbf{x}_n),$$

where the $\widetilde{K}_{\lambda,\mu}(q, t)$ are the (modified) *(q, t)-Kostka polynomials*. Although Haglund found a combinatorial monomial expansion for $\widetilde{H}_\mu$, it remains an outstanding problem to find a combinatorial interpretation for their Schur expansion. One of the main goal of this proposal is to attack this (q, t)-Kostka problem using the PI's past experience with *zeta and sweep maps*. There are many related and partial results to be proven along the way:

- Study Hilbert series of coinvariant rings of complex reflection groups, finding combinatorial interpretations (Problem 1) and invariant-theoretic refinements (Problems 2 and 3).
- Study zeta/sweep maps from the point of view of Dynamical Algebraic Combinatorics (Problem 4).
- Search for bigraded $\mathfrak{S}_n$-modules that correspond to combinatorial Hilbert series (Problem 7), with applications to rational Catalan combinatorics.
- Generalize zeta/sweep maps and related theorems (Problems 8 and 9), with a goal of extending the combinatorics to other root systems (Problem 12), studying (q, t)-symmetry (Problem 5), and finding the Frobenius expansion of ∇e_n (Problem 10).

The PI has introduced new techniques to resolve related problems, and has an original toolkit and perspective that has yielded substantial new progress. This point of view yields an interconnected library of concrete combinatorial problems especially suitable for graduate and undergraduate students (see Section 7), while its broad perspective allows for consequential results and relevance to other fields.

2. Coinvariant Rings and Combinatorial Statistics

2.1. **Coinvariant Ring of the Symmetric Group.** The Hilbert series for the space of coinvariants is the generating function for two important statistics on the $n!$ permutations in $\mathfrak{S}_n$:

$$\mathrm{Hilb}\left(\mathbb{C}[\mathbf{x}_n]/\langle\mathbb{C}[\mathbf{x}_n]_+^{\mathfrak{S}_n}\rangle; q\right) = \sum_{w\in\mathfrak{S}_n} q^{\mathsf{inv}(w)} = \sum_{w\in\mathfrak{S}_n} q^{\mathsf{maj}(w)}, \tag{1}$$

where $\mathbb{C}[\mathbf{x}_n]$ is shorthand for a polynomial ring in n variables and $\langle\mathbb{C}[\mathbf{x}_n]_+^{\mathfrak{S}_n}\rangle$ is the ideal of $\mathbb{C}[\mathbf{x}_n]$ generated by symmetric polynomials with no constant term.

Artin gave a basis for this space using the code of a permutation to reflect the first generating function of Equation (1) [Art44], while Garsia and Stanton found a basis using the descents of a permutation to explain the second [GS84]. In particular, for a permutation $\pi \in \mathfrak{S}_n$, we write $c(\pi)_i = |\{j < i : \pi_j > \pi_i\}|$ and $\mathrm{Des}(\pi) = \{i : \pi_i > \pi_{i+1}\}$. Then the two bases—each indexed by permutations in $\mathfrak{S}_n$ are

$$x_1^{c(\pi)_1} x_2^{c(\pi)_2} \cdot \cdots \cdot x_n^{c(\pi)_n} \qquad \text{and} \qquad \prod_{i\in\mathrm{Des}(\pi)} x_{\pi_1}x_{\pi_2}\cdot\cdots\cdot x_{\pi_i}$$

A statistic with the same distribution as inv or maj is called *mahonian*, after MacMahon [Mac13], but Foata gave the first bijection sending one statistic to the other [Foa68], explaining the equality in Equation (1). Exploiting the fact that this bijection preserves descents of the inverse permutation, Foata and Schützenberger later found a series of compositions of maps leading to an involution that *interchanges* inv and maj [FS78], combinatorially proving

$$\sum_{w\in\mathfrak{S}_n} q^{\mathsf{inv}(w)} t^{\mathsf{maj}(w)} = \sum_{w\in\mathfrak{S}_n} t^{\mathsf{inv}(w)} q^{\mathsf{maj}(w)}. \tag{2}$$

Although inv generalizes naturally to all finite Coxeter groups using the corresponding hyperplane arrangement and root system, there is no satisfactory definition of maj in general—and no notion of inv for complex reflection groups.

2.2. **Hilbert Series for Coinvariant Rings.** Fix V a complex vector space of dimension n. A more general ring studied by, for example, Chevalley and Shephard-Todd is the *coinvariant algebra* of a finite *complex reflection group* $G \leq \mathrm{GL}(V)$—that is, a subgroup of $\mathrm{GL}(V)$ generated by unitary reflections. Associated to this group is the following ring. Let $\mathbb{C}[V] := \mathrm{Sym}(V^*)$ be the symmetric algebra on the dual vector space V^*, and write $\mathbb{C}[V]^G$ for its G-invariant subring. The coinvariant algebra is defined as the quotient $\mathbb{C}[V]_G := \mathbb{C}[V]/\langle\mathbb{C}[V]_+^G\rangle$, where $\langle\mathbb{C}[V]_+^G\rangle$ is the ideal generated by all G-invariants with no constant term. The Hilbert series for the coinvariant ring for any finite complex reflection group G satisfies

$$\mathrm{Hilb}\left(\mathbb{C}[V]/\langle\mathbb{C}[V]_+^G\rangle; q\right) = \prod_{i=1}^{n} \frac{1-q^{d_i}}{1-q}, \tag{3}$$

where $d_1 \leq d_2 \leq \cdots \leq d_n$ are the *degrees* of the homogenous generators of the G-invariant ring $\mathbb{C}[V]^G$. As an ungraded G-representation, the coinvariant algebra is isomorphic to the regular representation of G (of dimension $|G|$)—as such, its Hilbert series should be expressable as a sum over G:

$$\text{Hilb}\left(\mathbb{C}[V]_G; q\right) = \sum_{i\geq 0} \dim \mathbb{C}[V]^i_G q^i = \sum_{g\in G} q^{\text{stat}(g)}, \tag{4}$$

where, in the case of real reflection groups, $\text{stat}(g)$ can be taken to be the number of inversions of g.

Problem 1. Find a combinatorial proof of Equation (4) for all complex reflection groups by defining a statistic on group elements.

A natural place to start would be to mirror the definition of inversions in real groups using the construction of complex root systems for complex reflection groups, as in Cohen's reclassification of finite complex reflection groups [Coh76]. Another approach which has seen some success for the infinite family is the construction of normal forms for words in the "simple" braid generators of Bessis et al. It would also be interesting to study this problem for the restriction to Shephard groups, which satisfy some additional properties shared in the real case (they have an associated Coxeter group and an analogue of the Coxeter complex). A first special case would certainly be to study complex reflection groups whose braid group coincides with the braid group of a real reflection group (and so has a well-defined length function for the positive monoid). In this case, it would be reasonable to introduce a many-to-one surjection from the complex reflection group to the real one.

2.3. **Reflexponents.** The *fake degree* of an m-dimensional simple G-module M is the polynomial encoding the degrees M occurs in the coinvariant ring $\mathbb{C}[V]_G$:

$$f_M(q) = \sum_i (\mathbb{C}[V]^i_G, M) q^i = \sum_{i=1}^m q^{e_i(M)},$$

where $e_1(M) \leq e_2(M) \leq \cdots \leq e_m(M)$ are nonnegative integers called *M-exponents*. We write $e_i := e_i(V)$ and $e_i^* := e_i(V^*)$ for the *exponents* and *co-exponents*. For $g \in G$, write $\mathscr{M}_M(g)$ for the codimension of the subspace of the G-module M fixed by g. The following is well-known [ST54, Sol63, OS80].

Theorem 1. *For G a finite irreducible complex reflection group,*

$$\sum_{g\in G} x^{\mathscr{M}_V(g)} = \prod_{i=1}^n (1 + e_i x).$$

As explained by Solomon and then Orlik and Solomon [Sol63, OS80], this follows from a careful analysis of the space of differential invariants $(\mathbb{C}[V] \otimes \wedge M^*)^G$ (for any *amenable* representation M; for example, Galois conjugates of the reflection representation), which turns out to be an exterior algebra over $\mathbb{C}[V]$. This allows for the specialization of the identity

$$\text{Hilb}\left(\left(\mathbb{C}[V] \otimes \bigwedge M^*\right)^G; x, y\right) = \prod_{i=1}^n \left(\frac{1 + x^{e_i(M)} y}{1 - x^{d_i}}\right), \tag{5}$$

where the x-variable keeps track of the grading from $\mathbb{C}[V]$, while the y-variable keeps track of the grading from $\bigwedge M^*$.

My main result in [Wil19] gives an analogue of Theorem 1 incorporating different G-orbits of reflecting hyperplanes. The reflecting hyperplanes $\mathscr{H}$ of the group G are broken into (at most three) G-orbits $\mathscr{H}/G = \{\mathscr{H}_\epsilon\}_{\epsilon\in\{s,t,u\}}$. Choose one such orbit $\mathscr{H}_\epsilon$, and let $\mathscr{R}_\epsilon$ be the associated set of reflections. For *well-generated* complex reflection groups (those groups that are generated by n of their reflections), there always exists a unique simple G-module V_ϵ that restricts to the reflection representation of a parabolic subgroup of G supported on $\mathscr{R}_\epsilon$.

Theorem 2. *For G a finite irreducible well-generated complex reflection group and $\mathscr{H}_\epsilon$ a hyperplane orbit, there is a reindexing of the ϵ-(co)reflexponents (with undefined ϵ-(co)reflexponents taken to be zero) giving the factorizations*

$$\sum_{g\in G} (x/y)^{\mathscr{M}_{V_\epsilon}(g)}\, y^{\mathscr{M}_V(g)} = \prod_{i=1}^{n} \Big(1+\epsilon_i x + (e_i-\epsilon_i)y\Big).$$

One can forgo the well-generated hypothesis at the expense of some technical difficulties. More generally, for any Galois twist $\sigma \in \mathrm{Gal}(\mathbb{Q}(|G|/\mathbb{Q})$, we may twist the reflection representation V to obtain V^σ. Orlik and Solomon proved that [OS80]

$$\sum_{g\in G}\left(\prod_{\lambda_i(g)\neq 1} \frac{1-\lambda_i(g)^\sigma}{1-\lambda_i(g)}\right) q^{\mathscr{M}_V(g)} = \prod_{i=1}^{n}\left(1+e_i(V^\sigma)q\right), \tag{6}$$

where the product is over all eigenvalues of g not equal to 1 (in the reflection representation). Already in [Wil19], I have extended this for $V^\sigma = V^*$ the complex conjugate of V. Following a suggestion of T. Douvropoulos, preliminary computations with the infinite family $G(ab,b,n)$ and a complete set of calculations for the exceptional types suggests the following problem.

Problem 2. Find an invariant-theoretic framework in the style of Orlik and Solomon that extends Equation (6) to orbits of hyperplanes.

I believe that I have a reasonable approach to this problem, obtained by generalizing beyond the level of hyperplane orbits. B. Rhoades has asked for a characterization of those irreducible representations U for which we have a factorization

$$\sum_{g\in G} (x/y)^{\mathscr{M}_U(g)}\, y^{\mathscr{M}_V(g)} = \prod_{i=1}^{n}\Big(1+e_i(U)x+(e_i(V)-e_i(U))y\Big).$$

Call such a representation *factorizing*.

Problem 3. Classify all factorizing representation U for complex reflection groups G.

Up to Galois automorphisms—other than the previously-considered representations corresponding to hyperplane orbits—it seems that factorizing representations are rare and are linked to normal reflection subgroups of reflection groups. My colleague Carlos Arreche and I are working on resolving Problem 2 and Problem 3 by extending Equation (5) to nested chains of normal reflection subgroups $N_s \lhd \cdots \lhd N_1 \lhd G$. Some classification work in this direction has been done by Bessis, Bonnafé, and Rouquier [BBR02]. Using this insight, we have already constructed 4-variable generalizations of Theorem 2 for certain normal chains of exceptional groups (ex: $G(4,2,2) \lhd G_6 \lhd G_7 \lhd G_{10}$), which suggests that we are on the right track to finding a uniform statement and proof—for a single normal subgroup $N \triangleleft G$ with $H = G/N$, the idea is to refine Orlik and Solomon's approach by taking successive invariants

$$(\mathbb{C}[V] \otimes \wedge M^*)^G \simeq \left((\mathbb{C}[V] \otimes \wedge M^*)^N \right)^H. \tag{7}$$

In particular, we will produce weighted versions of the usual coinvariant ring of a Coxeter group, giving an invariant-theoretic proof of some results of Macdonald for non-simply-laced Coxeter groups obtained by weighting different simple reflections differently [Mac72]. As a bonus, such methods may also give a more algebraic (and less combinatorial) approach to Problem 1.

3. Diagonal Coinvariant Rings

Motivated by the rich combinatorics of coinvariant spaces for Weyl and Coxeter groups, Garsia and Haiman introduced the space of *diagonal* coinvariants [Hai94, GH96], which has since been an extremely active area of research. Write

$$\mathbb{C}[\mathbf{x}_n, \mathbf{y}_n] := \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n].$$

The ring of diagonal invariants $\mathbb{C}[\mathbf{x}_n, \mathbf{y}_n]^{\mathfrak{S}_n}_+$ is the ring of $\mathfrak{S}_n$-invariant polynomials (with no constant term) in two sets of commuting variables, where $\mathfrak{S}_n$ acts diagonally (permuting the x and y variables simultaneously). The space of *diagonal coinvariants* is the quotient

$$\mathrm{DH}_n := \mathbb{C}[\mathbf{x}_n, \mathbf{y}_n] / \left\langle \mathbb{C}[\mathbf{x}_n, \mathbf{y}_n]^{\mathfrak{S}_n}_+ \right\rangle.$$

By a result of Weyl, it is known that $\left\langle \mathbb{C}[\mathbf{x}_n, \mathbf{y}_n]^{\mathfrak{S}_n} \right\rangle$ is generated by the polarized power sums $p_{k,l} = \sum_{i=1}^n x_i^k y_i^l$ for $k + l > 0$.

The problem is to give combinatorial interpretations for the Hilbert or Frobenius series of DH_n. It is known that the Frobenius series related to the modified Macdonald polynomials of Section 4 by:

$$\mathrm{Frob}\,(\mathrm{DH}_n; q, t) = \sum_{\mu \vdash n} \widetilde{H}_\mu t^{n(\mu)} q^{n(\mu')} (1-q)(1-t) \left(\sum_{x \in \mu} q^{a'} t^{l'} \right) \prod_{x \in \mu} \frac{(1 - q^{a'} t^{l'})}{(q^a - t^{l+1})(t^l - q^{a+1})}, \tag{8}$$

where a (resp. a') and l (resp. l') stand for the *(co)arm* and *(co)leg* lengths of the partition. But what we really want is a Schur or monomial expansion in terms of familiar combinatorial objects. To relate back to usual coinvariants: while the space of coinvariants $\mathbb{C}[\mathbf{x}_n]/\langle \mathbb{C}[\mathbf{x}_n]^{\mathfrak{S}_n}_+ \rangle$ is related to the symmetric group $\mathfrak{S}_n$, the diagonal coinvariants turn out to be related to the affine symmetric group $\widetilde{\mathfrak{S}}_n$.

3.1. Zeta and sweep maps on lattice paths.

3.1. Zeta and sweep maps on lattice paths. Perhaps due to the relative complexity of the underlying combinatorial objects, the combinatorics of diagonal coinvariants was first understood, and generalized for the alternating subspace DH_n^ϵ of the space of diagonal coinvariants [KOP02, Hag03, GH02, ALW15, TW18b].

Let $\mathcal{D}_{a,b}$ be the set of lattice paths from $(0,0)$ to (b,a) that stay above the main diagonal; write $\mathcal{D}_n = \mathcal{D}_{n+1,n}$. The classical *zeta map* ζ is a bijection from $\mathcal{D}_n$ to itself developed by Garsia, Haglund, and Haiman to explain the equidistribution of area with

Haglund's statistic bounce and Haiman's statistic dinv in the combinatorial expansion of the Hilbert series of DH_n^ϵ [GH02, Hai02, CM15, HX17]:

$$\mathrm{Hilb}\left(\mathrm{DH}_n^\epsilon; q, t\right) = \sum_{\mathsf{d}\in\mathfrak{D}_n} q^{\mathsf{dinv}(\mathsf{d})} t^{\mathsf{area}(\mathsf{d})} = \sum_{\mathsf{d}\in\mathfrak{D}_n} q^{\mathsf{area}(\mathsf{d})} t^{\mathsf{bounce}(\mathsf{d})},$$

where q records the degree of the variables $\mathbf{x}$ and t the degree of $\mathbf{y}$. Specifically, ζ has the pleasant property of translating Haglund's and Haiman's (inspired) statistics into the simple statistic area, so that [AKOP02, Hag03]:

$$\mathrm{Hilb}\left(\mathrm{DH}_n^\epsilon; q, t\right) = \sum_{w\in\mathfrak{D}_n} \overbrace{q^{\mathrm{area}(\zeta^{-1}(w))}}^{\mathrm{dinv}} t^{\mathrm{area}(w)} = \sum_{w\in\mathfrak{D}_n} q^{\mathrm{area}(w)} \overbrace{t^{\mathrm{area}(\zeta(w))}}^{\mathrm{bounce}}.$$

As Dyck paths have been generalized, so too have these zeta maps [Loe03, Egg03, GM14, LLL14, ALW15]—but proving invertibility of these generalized zeta maps has been a traditionally difficult problem [Xin15, CDH16b]. We note that the zeta map has been rediscovered many times (often by accident)—perhaps most recently, it appeared as an answer to a question on MathOverflow [Vat13, Stu14].

To state one reasonably general version, the *sweep map* from $\mathfrak{D}_{a,b} \to \mathfrak{D}_{a,b}$ rearranges the steps of a path in $\mathfrak{D}_{a,b}$ according to the order in which they are encountered by a line of slope a/b sweeping down from above [ALW15, Section 3.4]. Figure 1 computes the sweep map on a lattice path in $\mathfrak{D}_{4,7}$. My main result is the following.

Theorem 3 ([TW18b])**.** *For $a, b \in \mathbb{N}$, the sweep map is a bijection on $\mathfrak{D}_{a,b}$.*

In fact, we prove a substantially more general form of the theorem above, which has already inspired several related papers, including [GX16a, GX16b]. In particular, our theorem covers the traditional case when m and n are coprime, but also the more recently considered (and more difficult) case when $\gcd(m, n) > 1$ [GMV17]. It would be very interesting to study these results from the point of view of dynamical algebraic combinatorics.

Problem 4. Study instances of homomesy and other dynamical algebraic combinatorial notions with respect to the sweep map.

Wide open remains the problem of combinatorially explaining (q, t)-symmetry.

Problem 5. Combinatorially prove (q, t)-symmetry of $\mathrm{Hilb}\left(\mathrm{DH}_n; q, t\right)$ or the alternating subspace $\mathrm{Hilb}\left(\mathrm{DH}_n^\epsilon; q, t\right)$.

Although it has proven to be notoriously difficult, I would like to try to apply our techniques and perspective to this problem. There are a number of approaches to Problem 5 that seem fruitful. Most people concentrate on limiting the number of rows or columns of a path; this approach seems to run out of steam around 4 or 5 rows. It is also very tempting to try to construct an $\mathfrak{sl}_2$ action that essentially give a symmetric chain decomposition with respect to the q and t statistics. But the approach that seems to have the most merit is to try to generalize the Foata-Schützenberger involution by composing zeta with its inverse, with a few other nontrivial involutions thrown in. This approach leads, for example, to the construction of an area-preserving involution for rational Dyck paths (which is not simply transposition of core partitions) [CDH16a].

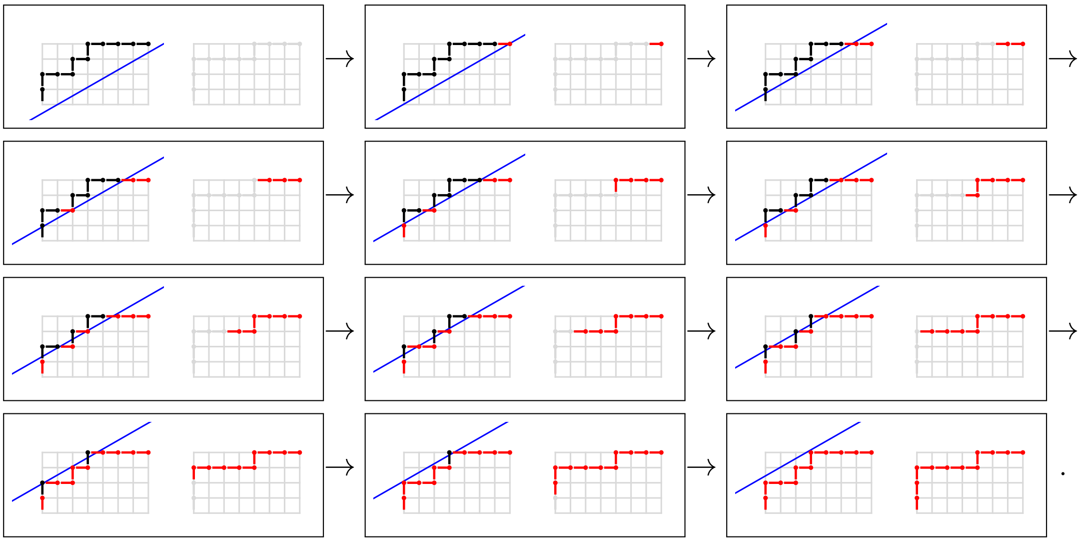

FIGURE 1. An illustration of the geometric interpretation of sweep. To form the right path, the steps of the left path are rearranged according to the order in which they are encountered by a line of slope 4/7 sweeping down from above.

3.2. **Rational Parking Spaces.** The most general *rational* (m, n) version of the theory of diagonal coinvariants comes from Hikita's study of the Borel-Moore homology of affine type A Springer fibers, which has a natural basis indexed by the m^{n-1} elements of the affine symmetric group $\widetilde{\mathfrak{S}}_n$ lying inside an m-fold dilation of the fundamental alcove [Hik14, Che03, Shi87, CP02, Hai94, Som05, GMV16a, Thi16].

There is another indexing combinatorial set for this space. For $m, n \in \mathbb{N}$, the *(m,n)-parking functions* $\mathscr{P}_m^n$ are those words $\mathsf{p} = \mathsf{p}_0 \cdots \mathsf{p}_{n-1} \in [m]^n = \{0, 1, \ldots, m-1\}^n$ such that

$$\left|\left\{j : \mathsf{p}_j < i\right\}\right| \geq \frac{in}{m} \text{ for } 1 \leq i \leq m. \tag{9}$$

Write $\mathscr{P}_n = \mathscr{P}_{n+1}^n$. Just as Dyck paths encoding the Hilbert series of the alternating subspace of the space of diagonal coinvariants, the full Hilbert series of DH_n is encoded by parking functions.

$$\mathrm{Hilb}\left(\mathrm{DH}_n; q, t\right) = \sum_{\mathsf{p} \in \mathscr{P}_n} q^{\mathsf{dinv}(\mathsf{p})} t^{\mathsf{area}(\mathsf{p})} = \sum_{\mathsf{p} \in \mathscr{P}_n} q^{\mathsf{area}(\mathsf{p})} t^{\mathsf{dinv}(\mathsf{p})}, \tag{10}$$

where area and dinv are certain statistics on parking functions. Carlsson and Mellit's proof of the shuffle conjecture and very recent work of Carlsson and Oblonkov on an explicit basis for the ring of diagonal harmonics [HHL$^+$05, CM15, HX17, CO18] implies the long-suspected fact that the bigraded Hilbert series of the space of diagonal coinvariants is encoded as a positive sum over the $(n+1)^{n-1}$ parking functions $\mathscr{P}_n$ [Hai02, HL05]. In fact, Carlsson and Oblonkov's basis is a wonderful interpolation between the Artin and Garsia-Stanton bases for the usual coinvariant ring, which is recovered when one of the sets of variables $\mathbf{x}_n$ or $\mathbf{y}_n$ are forgotten (or, equivalently, one of q or t is set to 0), expressing the equality

$$\text{Hilb}\left(\text{DH}_n; q, t\right) = \sum_{\pi \in \mathfrak{S}_n} t^{\text{maj}(\pi)} \prod_{i=1}^{n} [w_i(\pi)]_q, \tag{11}$$

where $w_i(\pi)$ counts the number of elements in the jth consecutive increasing subsequence of π larger than π_i. It would be very interesting if their basis actually is predicted by zeta maps.

Problem 6. Check if the Carlsson–Oblonkov basis for DH_n can be replaced with the set of monomials $\left\{\mathbf{x}_n^{\mathsf{p}} \mathbf{y}_n^{\zeta(\mathsf{p})}\right\}_{\mathsf{p} \in \mathscr{P}_n}$, where $\mathbf{x}_n^{\mathsf{p}} := \prod_i x_i^{\mathsf{p}_i}$ and $\mathbf{y}_n^{\zeta(\mathsf{p})} := \prod_i y_i^{\zeta(\mathsf{p})_i}$.

The upshot of such a result would be to suggest a reasonable basis for hypothetical more general bigraded rings (see Problem 7).

The classical parking words $\mathscr{P}_n$, their statistics area and dinv, and the shuffle conjecture have all been (at least combinatorially) generalized to the (m,n)-parking words $\mathscr{P}_m^n$ [BGLX15, ALW16, GMV16a, GN15, Thi16, GMV17]. Armstrong found natural interpretations of area and dinv in terms of affine permutations for the Fuss case [Arm13], and his work was extended to the rational case by Gorsky, Mazin, and Vazirani [GMV16a, GMV17]. Gorsky and Negut formulated the rational shuffle conjecture in [GN15]—that Hikita's polynomial was given by an operator from an elliptic Hall algebra (see also [BGLX15]). This operator formulation leads to a q,t-symmetric bivariate polynomial generalizing Equation (10):

$$\sum_{\mathsf{p} \in \mathscr{P}_m^n} q^{\mathsf{area}(\mathsf{p})} t^{\mathsf{dinv}(\mathsf{p})} = \sum_{\mathsf{p} \in \mathscr{P}_m^n} q^{\mathsf{dinv}(\mathsf{p})} t^{\mathsf{area}(\mathsf{p})}. \tag{12}$$

Something is lost, however, in the rational case: one statistic remains the degree, but the second statistic now appears only using a filtration.

Problem 7. Find a bigraded $\mathfrak{S}_n$-module whose Hilbert series is given by Equation (12).

Note that the *algebraic parking spaces* of [ARR15]—defined using a homogeneous system of parameters $(\theta_1, \ldots, \theta_n)$ coming from the theory of rational Cherednik algebras and work of Gordon and Griffeths give bigraded [GG09]—give singly-graded such $\mathfrak{S}_n$-modules. Related to this is the problem of giving uniform constructions of *rational noncrossing objects*. As the Catalan numbers of Gordon and Griffeth seem to depend on Galois twists, it is natural to wonder if (at least in the real case) there isn't a definition involving *powers* of a Coxeter element. If Problem 6 works out, we would have a guess for a basis of such a space and might be able to reverse engineer the underlying ideal.

3.3. **Zeta map on rational parking functions.** We recently developed simple combinatorics governing the (q,t)-statistics on rational parking functions [MTW17]. The previous state-of-the-art was work of Gorsky, Mazin, and Vazirani, who used the affine symmetric group to define the zeta map on $\mathscr{P}_m^n$, which takes area to dinv. They conjectured that it was a bijection by providing what they believed to be an inverse map [GMV16b]. In [MTW17], we invert their zeta by having parking functions act on $V := 1_m\backslash\mathbb{R}^m/\mathfrak{S}_m$ (that is, $\mathbb{R}^m$ up to permutation of coordinates and addition of multiples of the all-ones vector) and applying the Brouwer fixed point theorem—a letter $i \in [m]$ acts on $\mathbf{x} \in V$ by adding m to the ith smallest coordinate of $\mathbf{x}$, and a word $\mathsf{w} \in [m]^n$ acts on $\mathbf{x} \in V$ by acting by its letters from left to right. Writing $\mathbf{y} := \mathrm{sort}(\mathbf{x})$ for the increasing rearrangement of a point $\mathbf{x} \in \mathbb{R}^m_t$, define $i(\mathbf{x}) := \mathrm{sort}(\mathbf{x} + \mathbf{e}_i - 1_m)$ for $\mathbf{x} \in V$.

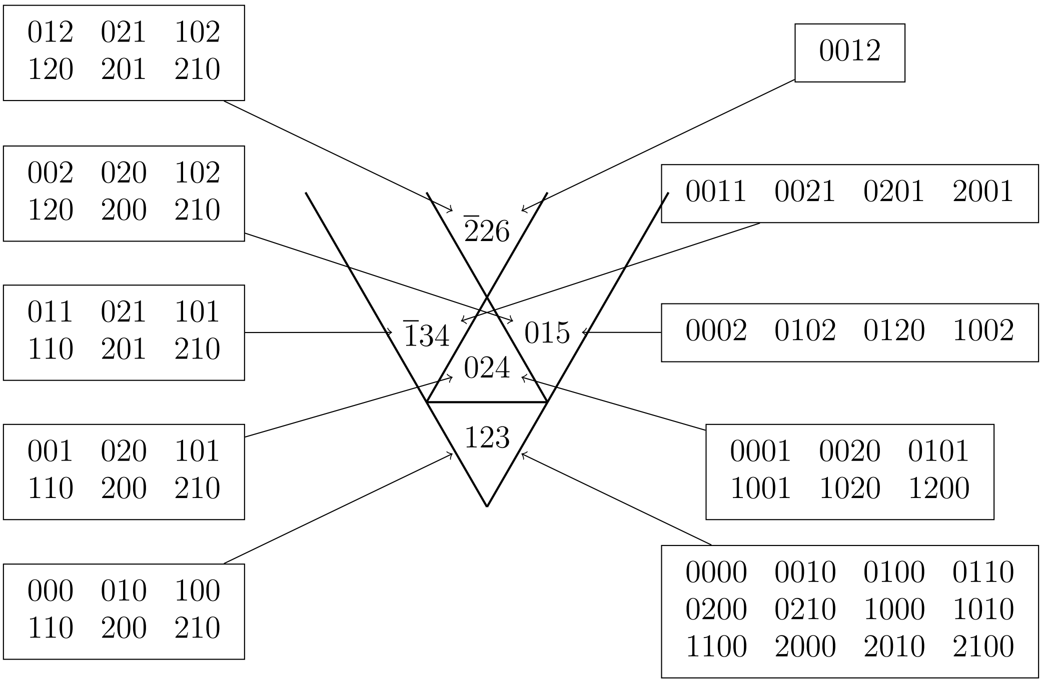

Figure 2. The dominant part of the $\widetilde{\mathfrak{S}}_3$ Shi arrangement. Each region is labeled by a coordinate corresponding to the one-line notation of the affine permutation whose alcove is lowest in the region. The words on the left are the $(3,3)$-parking words that fix every point of the (closed) region to which they point; the words on the right are the $(4,3)$-parking words that fix precisely the coordinate to which they point.

Theorem 4 ([MTW17])**.** *The action of* $\mathsf{w} \in [m]^n$ *on* V*:*

- *has a* **unique** *fixed point iff* $\mathsf{w} \in \mathscr{P}^n_m$ *and* $\gcd(m,n) = 1$*;*
- *has* **infinitely many** *fixed points iff* $\mathsf{w} \in \mathscr{P}^n_m$ *and* $\gcd(m,n) > 1$*; and*
- *has* **no** *fixed points iff* $\mathsf{w} \in [m]^n \setminus \mathscr{P}^n_m$*.*

In fact, one can even speak about the "dimension" of the various fixed spaces. As a corollary of Theorem 4, we show that dinv and area are equidistributed on coprime (m,n)-parking functions.

Theorem 5. *For m and n relatively prime,*

$$\sum_{\mathsf{p}\in\mathscr{P}^n_m} q^{\mathsf{dinv}(\mathsf{p})} = \sum_{\mathsf{p}\in\mathscr{P}^n_m} q^{\mathsf{area}(\mathsf{p})}.$$

3.4. **Non-coprime case.** We have recently begun work on extending Theorem 5 to understand what happens when $\gcd(m,n) > 1$, generalizing the setup in [GMV17] to parking functions (our work on sweep maps of lattice paths already provides the inverse to the zeta map on paths in the non-coprime case). Interestingly, the regions of the Shi arrangement and its Fuss generalizations may be described as the points fixed by some (n,kn)-parking function; more generally, we have a $\gcd(m,n)$-dimensional collection of fixed points living in $\mathbb{R}^m$ that warrants further investigation.

Problem 8. Generalize the Shi arrangement by explicitly describing the points in V fixed by some element of $\mathscr{P}_m^n$ for $\gcd(m,n) > 1$.

Loehr and Warrington's sweep maps on lattice paths are quite general, while the zeta maps on parking functions (thought of as labeled Dyck paths) seem rather more specialized—for example, sweep maps have no restriction on the number of different directions for steps, while zeta maps only allow two so that the paths must lie in a plane.

Problem 9. Find a common generalization of sweep maps on lattice paths, and the zeta map on rational parking functions.

3.5. **Frobenius Expansion.** Although we now know the monomial expansion of the Frobenius series of DH_n as well as its Hilbert series, an outstanding problem remains to find its Schur expansion (stated here only for the case of usual parking spaces).

Problem 10. Find a combinatorial intepretation of $\mathrm{Frob}\,(\mathrm{DH}_n; \mathbf{x}; q, t)$.

My experience with zeta maps should provide a useful perspective, as I now explain. The special case of the coefficient $s_{(1,1,\ldots,1)}$ recovers the (q,t)-Catalan numbers, and it is reasonable to search for related zeta maps. It is not hard to check that at $q = t = 1$, the coefficient of s_λ is counted by $\frac{1}{n+1} s_{\lambda'}(1,1,\ldots,1)$. This is counted by the number of semistandard tableaux of shape λ whose content is a parking function—that is, when rearranged, the content forms the columns of a Dyck path. The q-weight of such a parking tableau is given by $\binom{n+1}{2}$ minus its content. For example, for $n = 3$ and $\lambda = (2,1)$, we have the five parking tableaux

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\;.$$

From the Catalan case for $\lambda = (1,1,\ldots,1)$, we know the zeta map is given by acting by the letters of the parking tableau from left to right. It makes sense for more general shapes to act using the columns of the tableau (in the same way as in Theorem 4), realizing the tyep A crystal V_λ for $\lambda = (\lambda_1,\ldots,\lambda_k)$ as a connected component of $V_{\lambda_1} \otimes V_{\lambda_2} \otimes \cdots \otimes V_{\lambda_k}$ (which is typically picked out by applying RSK). Small preliminary calculations suggest that this generalized zeta map is not only a bijection, but also gives correct statistics—at least for certain special shapes (like hooks). One can still prove an analogous fixed-point theorem; the problem now is to match this up with the representation theory by establishing suitable recursions on both the ring of diagonal harmonics and the combinatorial objects.

3.6. **Other Cartan Types.** The definition of rational parking functions as the b^{a-1} alcoves in the b-fold dilation of the fundamental alcove in $\widetilde{\mathfrak{S}}_a$ easily extends to other Cartan types. And yet, the combinatorics of parking functions for other root systems is almost completely undeveloped (see also Problem 1).

Problem 11. Find statistics to explain (q,t)-Catalan numbers in other Cartan types.

As suggested by the interpretation in Section 3.5, our Theorem 4 shows that rational parking functions in type A may also be characterized as those words of length $a-1$ whose action on $\mathbb{R}^b$ has a fixed point. This new characterization suggests a novel way to approach Problem 11—by finding the right set of steps and the right space in which to act. Since $\widetilde{\mathfrak{S}}_n$ is interchanged with $\widetilde{\mathfrak{S}}_m$, there may be some sort of Howe duality involved. More generally, one might expect a generalization of Theorem 4 to hold.

Problem 12. Fix a complex simple Lie algebra $\mathfrak{g}$ with weight lattice $\Lambda \subset V$. Let $(p_1,\ldots,p_k) \in \Lambda^k$ be a path in V, and write $\mathrm{wt}(p) = \sum p_i$. The path p acts on a dominant point $x \in V$: for $1 \leq i \leq k$, add p_i to x and reflect whenever a simple hyperplane is crossed. Show that p has a fixed point if and only if $(\mathrm{wt}(p),\lambda_i) \leq 0$ for all fundamental weights λ_i.

A second approach comes from a different characterization of rational parking fucntions I have found. The classical cycle lemma can be generalized to other Cartan types using the image Ω in the Weyl group of the quotient group of the coroot by the coweight lattice. In more detail, let Λ_{min} be the set of fundamental weights in the orbit of 0 (the *minuscule weights*). Define the usual dot action of Ω on V by $g \cdot x = g(x+\rho/h) - \rho/h$, where ρ is the half sum of the positive roots and h is the Coxeter number.

Theorem 6 (The Cycle Lemma)**.** *A fundamental domain for the dot action of* Ω *on* V *is given by* $\{x \in V : (x,\omega) \geq 0, \omega \in \Lambda_{min}\}$.

By applying this theorem to the natural Weyl group action on crystals, I am able to give a unified framework for many combinatorial results in the literature. For example, let V_{ω_1} be the fundamental representation for $\mathfrak{sl}_b$. Then for a coprime to b, $\mathfrak{sl}_b(\omega_1)^{\otimes a}$ has a tableau model in bijection with arbitrary words of length a with entries in $[b]$. The action of Ω on $\mathfrak{sl}_b(\omega_1)^{\otimes a}$ has free orbits and each orbit contains exactly one of the b^{a-1} rational parking functions. It is reasonable to wonder if this construction can be extended to give a definition of parking functions in other Cartan types.

4. Macdonald Polynomials

The *modfied Macdonald polynomials* $\widetilde{H}_\mu(\mathbf{x}_n;q,t)$ are the unique polynomials in the algebra of symmetric functions in infinitely many variables $\mathbf{x} = \{x_1,x_2,\ldots\}$ with coefficients in $\mathbb{Q}(q,t)$ satisfying the triangularity conditions

(1) $\widetilde{H}_\mu[X(1-q);q,t] = \sum_{\lambda \geq \mu} a_{\lambda,\mu}(q,t)s_\lambda$,
(2) $\widetilde{H}_\mu[X(1-t);q,t] = \sum_{\lambda \geq \mu'} b_{\lambda,\mu}(q,t)s_\lambda$, and
(3) $\left\langle \widetilde{H}_\mu, s_{(n)} \right\rangle = 1$,

where we use the usual plethystic notation with $X = x_1 + x_2 + \cdots$, $\langle\cdot,\cdot\rangle$ is the *Hall inner product* defined by orthonormality of Schur functions, $\geq$ is *dominance order* on integer partitions, and μ' is the tranpose of the partition μ. They are related to the usual Macdonald polynomials P_μ by $\widetilde{H}_\mu(\mathbf{x}_n;q,t) = t^{n(\mu)}P_\mu\left[\frac{X}{1-1/t};q,t^{-1}\right]$.

Lascoux and Schutzenberger's famous charge/cocharge statistic on semistandard tableaux gives the coefficients $K_{\lambda,\mu}(t)$ in the expansion of the Hall-Littlewood polynomials P_λ into Schur polynomials s_μ:

$$P_\lambda(\mathbf{x}_n;q) := H_\lambda(\mathbf{x}_n;q,0) = \sum_\mu K_{\lambda,\mu}(t)s_\mu(\mathbf{x}_n).$$

For the modified Macdonald polynomials, we have a similar expansion

$$\widetilde{H}_\lambda(\mathbf{x}_n;q,1) = \sum_\mu \widetilde{K}_{\lambda,\mu}(q)s_\mu(\mathbf{x}_n),$$

where the expansion $\widetilde{K}_{\lambda,\mu}(q) = \sum_{T\in\mathrm{SYT}(\mu)} q^{\mathrm{comaj}_\lambda(T)}$ is due to Macdonald. It makes sense that the t-statistic ought to come from some sort of zeta map—by symmetry, the t-statistic matches the q-satistics in the coefficient $\widetilde{K}_{\lambda,\mu'}(q,t)$.

Problem 13. Define a *zeta map* ζ on the set of tableaux of shape λ so that

$$\widetilde{K}_{\lambda,\mu}(q,t) = \sum_{T\in\mathrm{SYT}(\mu)} q^{\mathrm{comaj}_\lambda(T)}t^{\mathrm{comaj}_\lambda(\zeta(T))}.$$

Although some partial progress has been made on finding combinatorial expansions for $\widetilde{K}_{\lambda,\mu'}(q,t)$ (for example, Fishel has an expansion for tableaux with at most two rows), it would be a natural first step to try partitions μ of the form (k,k), since there are Catalan-many such standard Young tableaux—thus, there ought to be a connection with the zeta map coming from the alternating component of the ring of diagonal coinvariants DH_n. Again, this is a notoriously difficult open problem, but I think that this zeta map apporoach is novel; since such a technique works in the closely related problem of studying diagonal harmonics, it seems reasonable to try it here—as usual, one statistic was easy to find, suggesting that the correct way to proceed is not to produce a difficult second statistic, but rather a bijection under which the second statistic becomes mapped to the first statistic.

4.1. **k-Schur Expansion.** By Haiman's n! theorem, the modified Macdonald polynomials expand positively into the Schur basis. One of the reasons that k-Schur polynomials were introduced is that modified Macdonald polynomials still expand positively into k-Schur polynomials $s_\mu^{(k)}$:

$$\widetilde{H}_\lambda(\mathbf{x}_n;q,t) = \sum_\mu K^{(k)}_{\lambda_\mu}(q,t)s^{(k)}_\mu(\mathbf{x}_n),$$

and when k is taken to be big enough, this recovers the usual Schur expansion. By work of Morse and Lapointe, $K_{\lambda_\mu}(1,1)$ counts the number of reduced words for the dominant affine symmetric group element whose corresponding k-bounded partition is λ. This generalizes the result of Macdonald using standard Young tableaux, since all elements will be fully commutative for k large, so that their reduced words correspond to tableaux. In the general case, these reduced words are *chains* of elements of the affine symmetric group for which Hugh Thomas and I constructed a zeta map in [TW14]. A first guess to try to extend our zeta map would be to apply it to each element of such a chain separately. Recent work of Ceballos, Mühle, and Fang proves bijectivity for a two-step chain (motivated by work on Hopf algebras of subword complexes by Bergeron, Ceballos, and Pilaud [BCP18]), and it would be interesting to consider extensions to longer chains [CFM19].

5. Prior Support: Not Applicable

I have not held an NSF grant before.

6. Intellectual Merit

My research is in algebraic combinatorics, with a broad interest in motivation from other areas of mathematics such as Lie theory, geometric group theory and Artin/braid groups, and reflection groups. I believe that my research has had a positive effect on the combinatorics community, and many results have applied to research problems outside of the context in which they originally arose. I have a record of producing problems and research areas accessible to beginning researchers. I have given over 60 invited seminars and talks. I have been selected five times as a speaker at the week-long refereed international conference Formal Power Series and Algebraic Combinatorics (FPSAC). While at FPSAC in 2019, I was asked by the executive committee to give one of the plenary talks, as the original speaker was unable to attend. I was one of four invited speakers at the University of Michigan for ALGECOM 2018, an invited speaker at the 50th anniversary conference of the Centre de Recherches Mathématiques, and I will be an invited speaker at the 10th Discrete Geometry and Algebraic Combinatorics conference in 2019 as well as Open Problems in Algebraic Combinatorics 2020.

My work with J. Striker in [SW12] has served as a catalyst for the involvement of undergraduate and young graduate students in cutting-edge research at REUs and doctoral programs—there were many developments motivated by the appearance of our paper [SW12]: [CHHM15, EP13, EFG+15, Had14, Hop16, GR14, GR15, GR16, PR15, Rob16, RS13, RW15, Rus16, DPS17, Str15, Str16]. In 2015, Striker, Propp, Roby and I organized an AIM workshop that launched a new field of combinatorics that J. Propp has termed "Dynamical Algebraic Combinatorics", and many papers have resulted from and been inspired by our workshop, including [DPS17, EFG+15, JR17, STWW17, HMP16, GHMP17b, GHMP17a, GP17]. We organized a successful session at the Joint Mathematics Meetings in 2018, and J. Striker and M. Arnold and I organized an AMS special session in Hawaii in Spring 2019; M. Arnold and I are organizing a follow-up special session at the 2020 JMM in Denver. I revisited this area with Thomas this past year in two papers [TW17, TW18a].

My work with Z. Hamaker, R. Patrias, and O. Pechenik [HPPW16]—using K-theoretic Schubert calculus to resolve a long-standing open bijective problem involving plane partitions—led to two separate REU projects over the past two years: one at Morrow's REU at the University of Washington mentored by Hamaker and Griffith, and one supervised by Pechenik [BHK16, BHK17].

My work with H. Thomas inverting sweep and zeta maps [TW18b] solved a long-standing problem in the field of diagonal coinvariants, and has already found applications outside of the field [HV17, Proposition 4.4]. Our follow-up project extending this work to resolving conjectures from [GMV16b] has led to further interesting problems [LLP12].

As detailed in my proposal, I have recently been working on some projects related to invariant theory of reflection groups (recently with C. Arreche) [Wil19], and I am organizing two minisymposia at the 2nd Annual Meeting of the SIAM Texas-Louisiana Section in November 2019 on this topic.

7. Broader Impacts

I have two Ph.D. students (Amit Kaushal and Priyojit Palit) who are currently pursuing research with me. As the only combinatorialist at UT Dallas, I have designed new undergraduate and graduate courses in combinatorics; due to the success of my undergraduate Discrete Math and Combinatorics class, I was asked by the honors college to teach an honors reading course in fall 2019.

I served on the program committe of FPSAC, organized the Graduate Student Combinatorics Conference at UT Dallas in 2018 (which hosted over 75 graduate students from around the country) and appeared as a mathematical consultant in a 2018 televised report (WFAA) regarding the NCAA basketball bracket, which since aired in over 15 cities nationwide. I am interested in continuing to increase the visibility and participation of women in mathematics at UT Dallas by establishing an AWM chapter.

7.1. **Mentoring and REUs.** Because of its many elementary problems, combinatorics is a discipline in which undergraduate and graduate students can immediately become involved in research-level mathematics. I have formulated a large interconnected library of concrete combinatorial problems especially suitable for graduate and undergraduate students, and I have substantial past experience in involving students and underrepresented students in research. I will continue to seek out such opportunities with the goal to eventually build a strong combinatorics program at UT Dallas.

While at UT Dallas I have worked with **graduate students** in the following ways:

- Currently the **thesis advisor** of Amit Kaushal (since Fall 2018);
- Currently the **thesis advisor** of Priyojit Palit (since Spring 2019);
- Organized the 2018 **Graduate Student Combinatorics Conference**;
- Supervised two **independent study** courses with graduate students. In Spring 2019, I worked with Barbara Melillio, suggesting a reading program of recent papers and textbooks to learn material required to solve the problem. In Fall 2017, I supervised a reading course with Austin Marstaller on the theory of root systems.

While at UT Dallas I have worked with **undergraduates** in the following ways:

- Supervised Kevin Zimmer's **senior honors thesis** in spring 2018;
- Mentored rising senior Robert Hubbard for eight weeks in the summer of 2018 as part of the **Pioneer REU program**;
- Supervised **independent research** with junior Joshua Marsh in the spring semester of 2019;

- Supervised **independent research** with undergraduates Christian Kondor and Michelle Patten in the spring semester and summer of 2019; and
- Due to the success of the Discrete Math and Combinatorics course I designed for the new BS in Data Science program, I was asked by the honors college to teach a **reading course** in fall 2019 for 10 students.

My past experience in involving undergraduate students in research includes:

- In 2016, I co-mentored Florence Maas-Gariepy on a research/study project involving finite reflection groups, which led to her project report (in French) being featured on the funding agency's website [MG16].
- In 2014, I mentored Stephanie Schanack, Fatiha Djermane, and Sarah Ouahib on an original research problem involving the characterization of the fixed points of a certain combinatorial set under a cyclic group action. I guided them through a case-by-case analyses which the three wrote up (in French) [SSD14].
- At the 2011 University of Minnesota REU, I provided support to David B Rush and XiaoLin Shi [RS13], who found a generalization of my work in [SW12].
- For the 2010 Minnesota REU, I helped direct Gaku Liu's research in partition identities [Liu] and helped a second group formulate and computationally test conjectures on a combinatorial reformulation of the four-color theorem [CSS14].

7.2. **Conferences and Workshops Organized.** I have also been very active in **organizing conferences and workshops**:

- I organized a week-long workshop at the American Institute of Mathematics;
- I organized the 2018 Graduate Student Combinatorics Conference at UT Dallas, with over 75 attendees (also obtaining $20,000 of NSF funding);
- I served on the program committee for FPSAC in 2019 and I will serve in 2020–2022 as its NSF/NSA PI for funding;
- I have organized four AMS special sessions. My UT Dallas colleague M. Arnold and I organized a special session in Hawaii in 2019, and we are organizing another special session at the 2020 Joint Mathematical Meetings in Denver, both relating to the interactions between dynamical systems and combinatorics;
- I am also currently organizing a minisymposium on "Coinvariant Spaces and Parking Functions" for the 2019 SIAM Texas Louisiana Section at Southern Methodist University under the meta-organization of Dr. F. Sottile;
- In October 2018, I organized a two-week "research-in-pairs" program at Oberwolfach, resulting in a 132-page preprint; and
- My workshop proposal with Drs. J. Propp, T. Roby, and J. Striker for a week-long program on "Dynamical Algebraic Combinatorics" was accepted by BIRS for 2020.

7.3. **Referee Activities.** As a member of the **mathematical community**, I have **refereed** for over twenty journals, including Proceedings of the American Mathematical Society, Selecta Mathematica, and Transactions of the AMS. I also served as a referee for UT Dallas' own undergraduate research journal *The Exley*.

In 2019, I refereed two mathematical grants: one for the state of Texas in ConTex, and one for the French National Research Agency.

References

[AKOP02] George Andrews, Christian Krattenthaler, Luigi Orsina, and Paolo Papi, *ad-nilpotent $\mathfrak{b}$-ideals in $sl(n)$ having a fixed class of nilpotence: combinatorics and enumeration*, Transactions of the American Mathematical Society **354** (2002), no. 10, 3835–3853.

[ALW15] Drew Armstrong, Nicholas Loehr, and Gregory Warrington, *Sweep maps: A continuous family of sorting algorithms*, Advances in Mathematics **284** (2015), 159–185.

[ALW16] Drew Armstrong, Nicholas A Loehr, and Gregory S Warrington, *Rational parking functions and Catalan numbers*, Annals of Combinatorics **1** (2016), no. 20, 21–58.

[Arm13] Drew Armstrong, *Hyperplane arrangements and diagonal harmonics*, Journal of Combinatorics **4** (2013), no. 2.

[ARR15] Drew Armstrong, Victor Reiner, and Brendon Rhoades, *Parking spaces*, Advances in Mathematics **269** (2015), 647–706.

[Art44] E Artin, *Galois theory*.

[BBR02] David Bessis, Cédric Bonnafé, and Raphaïl Rouquier, *Quotients et extensions de groupes de réflexion*, Mathematische Annalen **323** (2002), no. 3, 405–436.

[BCP18] Nantel Bergeron, Cesar Ceballos, and Vincent Pilaud, *Hopf dreams*, arXiv preprint arXiv:1807.03044 (2018).

[BGLX15] Francois Bergeron, Adriano Garsia, Emily Sergel Leven, and Guoce Xin, *Compositional (km, kn)-shuffle conjectures*, International Mathematics Research Notices **2016** (2015), no. 14, 4229–4270.

[BHK16] Thomas Browning, Max Hopkins, and Zander Kelley, *Doppelgangers: Revisiting recurrences and the ur-operation*.

[BHK17] ______, *Doppelgangers: the ur-operation and posets of bounded height*, arXiv preprint arXiv:1710.10407 (2017).

[CDH16a] Cesar Ceballos, Tom Denton, and Christopher Hanusa, *Combinatorics of the zeta map on rational Dyck paths*, Journal of Combinatorial Theory, Series A **141** (2016), 33–77.

[CDH16b] Cesar Ceballos, Tom Denton, and Christopher RH Hanusa, *Combinatorics of the zeta map on rational Dyck paths*, Journal of Combinatorial Theory, Series A **141** (2016), 33–77.

[CFM19] Cesar Ceballos, Wenjie Fang, and Henri Mühle, *The steep-bounce zeta map in parabolic Cataland*, arXiv preprint arXiv:1903.08515 (2019).

[Che03] Ivan Cherednik, *Double affine hecke algebras and difference Fourier transforms*, Inventiones mathematicae **152** (2003), no. 2, 213–303.

[CHHM15] Melody Chan, Shahrzad Haddadan, Sam Hopkins, and Luca Moci, *The expected jaggedness of order ideals*, arXiv preprint arXiv:1507.00249 (2015).

[CM15] Erik Carlsson and Anton Mellit, *A proof of the shuffle conjecture*, arXiv preprint arXiv:1508.06239 (2015).

[CO18] Erik Carlsson and Alexei Oblomkov, *Affine Schubert calculus and double coinvariants*, arXiv preprint arXiv:1801.09033 (2018).

[Coh76] Arjeh M Cohen, *Finite complex reflection groups*, Annales scientifiques de l'École Normale Supérieure, vol. 9, 1976, pp. 379–436.

[CP02] Paola Cellini and Paolo Papi, *ad-nilpotent ideals of a Borel subalgebra II*, Journal of Algebra **258** (2002), no. 1, 112–121.

[CSS14] Sebastian A Csar, Rik Sengupta, and Warut Suksompong, *On a subposet of the Tamari lattice*, Order **31** (2014), no. 3, 337–363.

[DPS17] Kevin Dilks, Oliver Pechenik, and Jessica Striker, *Resonance in orbits of plane partitions and increasing tableaux*, Journal of Combinatorial Theory, Series A **148** (2017), 244–274.

[EFG+15] David Einstein, Miriam Farber, Emily Gunawan, Michael Joseph, Matthew Macauley, James Propp, and Simon Rubinstein-Salzedo, *Noncrossing partitions, toggles, and homomesies*, arXiv preprint arXiv:1510.06362 (2015).

[Egg03] Eric Egge, *A Schröder generalization of Haglund's statistic on Catalan paths*, The Electronic Journal of Combinatorics **10** (2003), no. 1, 21.

[EP13] David Einstein and James Propp, *Combinatorial, piecewise-linear, and birational homomesy for products of two chains*, arXiv preprint arXiv:1310.5294 (2013).

[Foa68] Dominique Foata, *On the netto inversion number of a sequence*, Proceedings of the American Mathematical Society **19** (1968), no. 1, 236–240.

[FS78] Dominique Foata and Marcel-Paul Schützenberger, *Major index and inversion number of permutations*, Mathematische Nachrichten **83** (1978), no. 1, 143–159.

[GG09] I. Gordon and S. Griffeth, *Catalan numbers for complex reflection groups*, arXiv preprint arXiv:0912.1578 (2009).

[GH96] Adriano Garsia and Mark Haiman, *A remarkable q,t-Catalan sequence and q-Lagrange inversion*, Journal of Algebraic Combinatorics **5** (1996), no. 3, 191–244.

[GH02] Adriano Garsia and James Haglund, *A proof of the q,t-Catalan positivity conjecture*, Discrete Mathematics **256** (2002), no. 3, 677–717.

[GHMP17a] Pavel Galashin, Sam Hopkins, Thomas McConville, and Alexander Postnikov, *Root system chip-firing I: Interval-firing*, arXiv preprint arXiv:1708.04850 (2017).

[GHMP17b] ———, *Root system chip-firing II: Central-firing*, arXiv preprint arXiv:1708.04849 (2017).

[GM14] Evgeny Gorsky and Mikhail Mazin, *Compactified Jacobians and q,t-Catalan numbers, II*, Journal of Algebraic Combinatorics **39** (2014), no. 1, 153–186.

[GMV16a] Eugene Gorsky, Mikhail Mazin, and Monica Vazirani, *Affine permutations and rational slope parking functions*, Transactions of the American Mathematical Society **368** (2016), no. 12, 8403–8445.

[GMV16b] ———, *Affine permutations and rational slope parking functions*, Transactions of the American Mathematical Society (2016).

[GMV17] ———, *Rational dyck paths in the non relatively prime case*, arXiv preprint arXiv:1703.02668 (2017).

[GN15] Eugene Gorsky and Andrei Neguţ, *Refined knot invariants and Hilbert schemes*, Journal de Mathématiques Pures et Appliquées **104** (2015), no. 3, 403–435.

[GP17] Pavel Galashin and Pavlo Pylyavskyy, *R-systems*, arXiv preprint arXiv:1709.00578 (2017).

[GR14] Darij Grinberg and Tom Roby, *The order of birational rowmotion*, DMTCS Proceedings (2014), no. 01, 753–764.

[GR15] ———, *Iterative properties of birational rowmotion II: Rectangles and triangles*, The Electronic Journal of Combinatorics **22** (2015), no. 3, P3–40.

[GR16] ———, *Iterative properties of birational rowmotion I: generalities and skeletal posets*, The Electronic Journal of Combinatorics **23** (2016), no. 1, P1–33.

[GS84] Adriano Garsia and Dennis Stanton, *Group actions on Stanley-Reisner rings and invariants of permutation groups*, Advances in Mathematics **51** (1984), no. 2, 107–201.

[GX16a] Adriano Garsia and Guoce Xin, *Inverting the rational sweep map*, arXiv preprint arXiv:1602.02346 (2016).

[GX16b] Adriano M Garsia and Guoce Xin, *Dinv and area*, arXiv preprint arXiv:1609.04480 (2016).

[Had14] Shahrzad Haddadan, *Some instances of homomesy among ideals of posets*, arXiv preprint arXiv:1410.4819 (2014).

[Hag03] James Haglund, *Conjectured statistics for the q,t-Catalan numbers*, Advances in Mathematics **175** (2003), no. 2, 319–334.

[Hai94] Mark Haiman, *Conjectures on the quotient ring by diagonal invariants*, Journal of Algebraic Combinatorics **3** (1994), no. 1, 17–76.

[Hai02] ———, *Vanishing theorems and character formulas for the Hilbert scheme of points in the plane*, Inventiones mathematicae **149** (2002), no. 2, 371–407.

[HHL$^+$05] J. Haglund, M. Haiman, N. Loehr, JB Remmel, and A. Ulyanov, *A combinatorial formula for the character of the diagonal coinvariants*, Duke Mathematical Journal **126** (2005), no. 2, 195–232.

[Hik14] Tatsuyuki Hikita, *Affine springer fibers of type a and combinatorics of diagonal coinvariants*, Advances in Mathematics **263** (2014), 88–122.

[HL05] J. Haglund and N. Loehr, *A conjectured combinatorial formula for the Hilbert series for Diagonal Harmonics*, Discrete Mathematics **298** (2005), no. 1, 189–204.

[HMP16] Sam Hopkins, Thomas McConville, and James Propp, *Sorting via chip-firing*, arXiv preprint arXiv:1612.06816 (2016).

[Hop16] Sam Hopkins, *The CDE property for minuscule lattices*, arXiv preprint arXiv:1606.06248 (2016).

[HPPW16] Zachary Hamaker, Rebecca Patrias, Oliver Pechenik, and Nathan Williams, *Doppelgängers: bijections of plane partitions*, International Mathematics Research Notices (2016).

[HV17] Paul Hamacher and Eva Viehmann, *Irreducible components of minuscule affine Deligne-Lusztig varieties*, arXiv preprint arXiv:1702.08287 (2017).

[HX17] James Haglund and Guoce Xin, *Lecture notes on the Carlsson-Mellit proof of the shuffle conjecture*, arXiv preprint arXiv:1705.11064 (2017).

[JR17] Michael Joseph and Tom Roby, *Toggling independent sets of a path graph*, arXiv preprint arXiv:1701.04956 (2017).

[KOP02] Christian Krattenthaler, Luigi Orsina, and Paolo Papi, *Enumeration of ad-nilpotent b-ideals for simple Lie algebras*, Advances in Applied Mathematics **28** (2002), no. 3-4, 478–522.

[Liu] G. Liu, *A few approaches to solving the Borwein conjecture.*

[LLL14] Kyungyong Lee, Li Li, and Nicholas Loehr, *Combinatorics of certain higher q,t-Catalan polynomials: chains, joint symmetry, and the Garsia–Haiman formula*, Journal of Algebraic Combinatorics **39** (2014), no. 4, 749–781.

[LLP12] Cédric Lecouvey, Emmanuel Lesigne, and Marc Peigné, *Random walks in Weyl chambers and crystals*, Proceedings of the London Mathematical Society **104** (2012), no. 2, 323–358.

[Loe03] Nicholas Loehr, *Multivariate analogues of Catalan numbers, parking functions, and their extensions*, University of California, San Diego, 2003.

[Mac13] Percy Alexander MacMahon, *The indices of permutations and the derivation therefrom of functions of a single variable associated with the permutations of any assemblage of objects*, American Journal of Mathematics **35** (1913), no. 3, 281–322.

[Mac72] I.G. Macdonald, *The Poincaré series of a Coxeter group*, Mathematische Annalen **199** (1972), no. 3, 161–174.

[MG16] Florence Maas-Gariepy, *Rapport de stage été 2016*, https://personal.utdallas.edu/∼nxw170830/docs/Papers/Maas-Gariepy - Rapport de stage ISM 2016.pdf, 2016.

[MTW17] Jon McCammond, Hugh Thomas, and Nathan Williams, *Fixed points of parking functions.*

[OS80] Peter Orlik and Louis Solomon, *Unitary reflection groups and cohomology*, Inventiones mathematicae **59** (1980), no. 1, 77–94.

[PR15] James Propp and Tom Roby, *Homomesy in products of two chains*, Electronic Journal of Combinatorics **22** (2015), no. 3, P3–4.

[Rob16] Tom Roby, *Dynamical algebraic combinatorics and the homomesy phenomenon*, Recent Trends in Combinatorics, Springer, 2016, pp. 619–652.

[RS13] David B Rush and XiaoLin Shi, *On orbits of order ideals of minuscule posets*, Journal of Algebraic Combinatorics **37** (2013), no. 3, 545–569.

[Rus16] David B Rush, *On order ideals of minuscule posets III: The cde property*, arXiv preprint arXiv:1607.08018 (2016).

[RW15] David B Rush and Kelvin Wang, *On orbits of order ideals of minuscule posets II: Homomesy*, arXiv preprint arXiv:1509.08047 (2015).

[Shi87] Jian-Yi Shi, *Sign types corresponding to an affine Weyl group*, Journal of the London Mathematical Society **2** (1987), no. 1, 56–74.

[Sol63] Louis Solomon, *Invariants of finite reflection groups*, Nagoya Mathematical Journal **22** (1963), 57–64.

[Som05] Eric Sommers, *B-stable ideals in the nilradical of a Borel subalgebra*, Canadian Mathematical Bulletin **48** (2005), no. 3, 460–472.

[SSD14] Stéphanie Schank, Sarah Sekheri, and Fatiha Djermane, *Un CSP sur des tableaux décalés en forme d'escalier*, https://personal.utdallas.edu/∼nxw170830/docs/Papers/SekheriSchanckDjermane2014.pdf, 2014.

[ST54] Geoffrey C Shephard and John A Todd, *Finite unitary reflection groups*, Canad. J. Math **6** (1954), no. 2, 274–301.

[Str15] Jessica Striker, *The toggle group, homomesy, and the razumov-stroganov correspondence*, arXiv preprint arXiv:1503.08898 (2015).

[Str16] ———, *Rowmotion and generalized toggle groups*, arXiv preprint arXiv:1601.03710 (2016).

[Stu14] Christian Stump, *On a new collection of words in the Catalan family*, Journal of Integer Sequences **17** (2014), no. 2, 3.

[STWW17] Anne Schilling, Nicolas M Thiéry, Graham White, and Nathan Williams, *Braid moves in commutation classes of the symmetric group*, European Journal of Combinatorics **62** (2017), 15–34.

[SW12] Jessica Striker and Nathan Williams, *Promotion and rowmotion*, European Journal of Combinatorics **33** (2012), no. 8, 1919–1942.

[Thi16] Marko Thiel, *From Anderson to zeta*, Advances in Applied Mathematics **81** (2016), 156–201.

[TW14] Hugh Thomas and Nathan Williams, *Cyclic symmetry of the scaled simplex*, Journal of Algebraic Combinatorics **39** (2014), no. 2, 225–246.

[TW17] ———, *Rowmotion in slow motion*, arXiv preprint arXiv:1712.10123 (2017).

[TW18a] ———, *Independence posets*, arXiv preprint arXiv:1805.00815 (2018).

[TW18b] ———, *Sweeping up zeta*, Selecta Mathematica **24** (2018), no. 3, 2003–2034.

[Vat13] Vincent Vatter, *A family of words counted by the Catalan numbers*, http://www.mathoverflow.net/questions/131585, 2013.

[Wil19] Nathan Williams, *Reflexponents*, arXiv:1902.06264 (2019).

[Xin15] Guoce Xin, *An efficient search algorithm for inverting the sweep map on rational Dyck paths*, arXiv preprint arXiv:1505.00823 (2015).