Project Description

1. Introduction

Dynamical algebraic combinatorics (DAC) is a relatively new field pioneered by the PI. DAC extends classical enumerative combinatorics to accommodate group actions—well-behaved actions hint at deep connections between combinatorial objects and other, more algebraic, constructions (such as integrable systems, bases in quantum groups, or cluster variables in cluster algebras). Conceptual proofs often exploit these connections and have led to fruitful interchanges between combinatorics, representation theory, and algebraic geometry.

1.1. **A first example.** The distributive lattice on the product of two chains of length 2 is illustrated on the left of Figure 1. This same lattice appears on the right of Figure 1 in a more algebraic context as the crystal graph on the canonical basis for a certain fundamental highest-weight representation of $\mathfrak{sl}_3$ (here indexed by semistandard tableaux of shape ⊟).

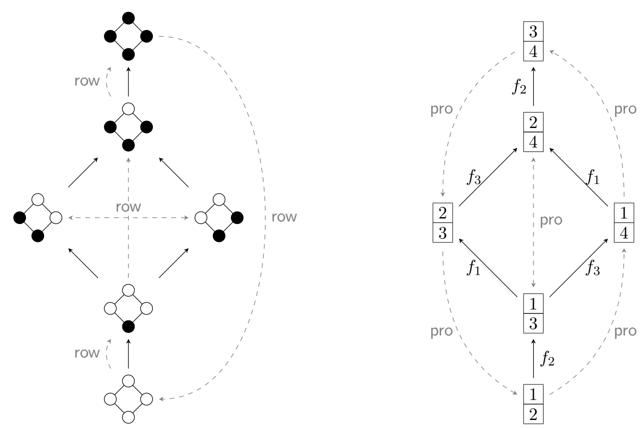


Figure 1. *Left:* the distributive lattice of order ideals in the product of two chains, with rowmotion denoted by dashed gray arrows. *Right:* a crystal graph for $\mathfrak{sl}_3$, with promotion denoted by dashed gray arrows.

We now introduce cyclic group actions on these sets. *Rowmotion* on an order ideal I of a poset P is the order ideal $\mathsf{row}(I)$ generated by the minimal elements of P *not* in I [SW12]. The action of rowmotion is illustrated with dashed gray edges on the left in Figure 1. On the other hand, Schutzenberger's classical *promotion* operator $\mathsf{pro}(T)$ on a semistandard tableau T can be expressed as a series of jeu-de-taquin slides: subtract one from all labels (replacing 1 by n) and slide the boxes labeled by n past the other boxes to recover semistandardness. The action of promotion is illustrated with dashed gray edges on the right in Figure 1. Perhaps surprisingly, promotion and rowmotion have the same orbit structure in this example (though they aren't equivariant with respect to the poset-preserving bijection), which is a special case of a more general phenomenon relating a piecewise-linear (PL) version of rowmotion to the action of the cactus group via Lusztig involutions on type A crystals [SW12, EP14].

This project proposes new approaches to several open problems in DAC by bridging the PI's recent conjectural definition of independence polytopes with the algebraic rigidity provided by quantum groups and cluster algebras: Section 2 presents a unified combinatorial framework for many DAC results using the PI's novel definition of **independence polytopes**. Although theoretical properties of the construction have not yet been proven, the PI has already implemented his definition as Sage code, and he proposes to further develop the theory as part of this proposal; Section 3 proposes a framework using **quantum groups** for periodic piecewise-linear actions on certain independence posets related to the representation theory of Lie algebras; and Section 4 addresses problems related to **cluster algebras** coming from independence posets related to the representation theory of quivers.

1.2. **Overview.** We give a historical overview of certain recent and classical work related to DAC that motivates the three interrelated areas of this proposal: independence polytopes, quantum groups, and cluster algebras.

1.2.1. *Rowmotion and DAC. Rowmotion* was introduced by Duchet in [Duc74]; studied for the Boolean lattice (and the product of two chains) by Brouwer and Schrijver [BS74, Bro75]; and (still for the Boolean lattice) related to matroid theory by Deza and Fukuda [DF90]. Cameron and Fon-der-Flaass considered rowmotion on the product of two and then three chains [FDF93, CFDF95]. Its study then apparently lay dormant for over a decade until Panyushev resurrected it in the form of a series of conjectures coming from Lie theory [Pan09]. The focus then shifted to finding equivariant bijections to natural combinatorial objects, and Stanley and Thomas completely characterized the orbit structure of rowmotion on the product of two chains combinatorially (using the Stanley-Thomas word) [Sta09]. Striker and the PI unified and extended various results by relating rowmotion to *jeu-de-taquin* and made terminological innovations to the theory [SW12]. This popularization of rowmotion led to a swell of related work falling under Propp's heading of *dynamical algebraic combinatorics*.

1.2.2. *From Rowmotion to Independence Polytopes.* Motivated by Berenstein and Kirillov's piecewise-linear (PL) Bender-Knuth involutions on Gelfand-Tsetlin patterns [KB96], Einstein and Propp considered a PL-lifting of rowmotion to the order polytope of a poset [EP13, EP14]. Einstein and Propp [EP14] (and Hopkins [H^+20, Appendix A]) elucidated the connection between PL-rowmotion on rectangular plane partitions and promotion of rectangular semistandard Young tableaux, further solidifying the representation-theoretic connections. Thus, while PL-rowmotion on plane partitions recovers promotion of semistandard tableaux. Section 2 proposes a novel extension of these constructions from distributive lattices to a generalization of a wide class of posets defined by Thomas and the PI on independent sets [TW19].

1.2.3. *From Rowmotion on Minuscule Posets to Quantum Groups.* Building on the PI's work with Striker [SW12], Rush and Shi placed rowmotion in a natural representation-theoretic setting, giving a partial explanation for the reappearance of certain posets with preferred properties [RS13]. Using the Striker-Williams conjugacy result, they obtained a conceptual explanation for the periodicity of rowmotion on combinatorial models of bases for minuscule representations V_λ of a simple Lie algebra $\mathfrak{g}$, by connecting rowmotion to the action of a Coxeter element of the Weyl group W. Rush built on this machinery in [RW15, Rus16] to establish homomesy results, and very recently explained some generalizations in type A using canonical bases of quantum groups [Rus21] (see also [HLLY21]). Through the connection with Bender-Knuth involutions in type A, piecewise-linear rowmotion corresponds to the action of the cactus group on $V_{m\lambda}$ via Lusztig involutions, although this connection remains mysterious for general simple Lie algebras $\mathfrak{g}$. Using the reflection functors of quiver representation theory, Garver, Patrias, and Thomas gave a uniform proof of periodicity of piecewise-linear rowmotion on minuscule posets [GPT18]. Section 3 describes a proposal to explain these connections using Lusztig's action of the braid group $B(W)$ on the quantum group $U_q(\mathfrak{g})$.

1.2.4. *From Rowmotion on Root Posets to Cluster Algebras.* Panyushev conjectured that rowmotion on order ideals in the positive root poset of an irreducible crystallographic root system has order $2h$, where h is the Coxeter number. Using an equivariant bijection to noncrossing partitions under the Kreweras complement, Armstrong, Stump, and Thomas [AST13] resolved Panyushev's conjectures. Section 4 describes a proposal to study the PI's generalizations of the Panyushev conjectures, relating variants of rowmotion to the cluster isomorphisms induced by reflection functors on quivers and a conjectural relationship between the positive part of the associahedron and the canonical basis of the positive part $U_q^+(\mathfrak{g})$ of the quantum group.

2. Independence Posets and Polytopes

The most general combinatorial approach to date that encompasses the varied settings of the study of rowmotion is a novel notion of *independence polytope*, developed independently by the PI, which extends the earlier notion of *independence posets* introduced jointly by the PI and Hugh Thomas in [TW19]. Let G be a finite acyclic directed graph (without oriented cycles, loops, or multiple edges) so that the transitive closure of G admits a partial order on its vertices called *G-order*. An *independent set* $\mathcal{I} \subseteq G$ is a set of pairwise non-adjacent vertices of G. The PI's previous work with Thomas endows the independent sets of directed acyclic graphs with a partial order, from which rowmotion can be computed in several different ways [TW17, TW19]. The notion of "independence poset" is a natural generalization of that of "distributive lattice," but where the lattice requirement is eliminated: an independence poset that is a graded lattice is a distributive lattice (not every independence poset is a lattice). When $G = \text{Comp}(P)$ is the comparability graph of a poset P, the corresponding independence poset recovers the distributive lattice of order ideals $J(P)$ (where an order ideal corresponds to the antichain of its maximal elements, which is an independent set in $\text{Comp}(P)$). For example, Figure 1 is redrawn on the left in Figure 2 as an independence poset on the comparability graph. On the right of Figure 2 is the independence poset obtained by removing an edge from this comparability graph. The definition of independence posets was motivated by Coxeter–Catalan combinatorics, the representation theory of quivers, and lattice theory; many well-known posets (such as Cambrian lattices) are special cases.

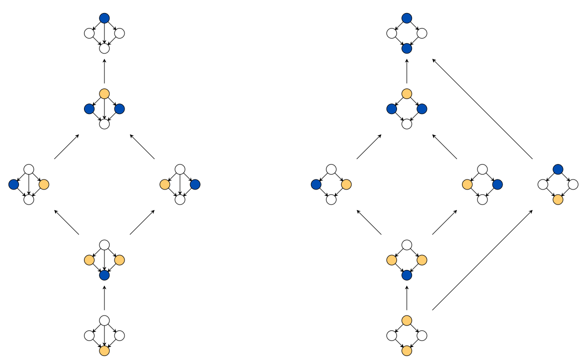


Figure 2. *Left:* independent sets in the comparability graph of the product of two chains arranged as an independence poset. *Right:* the independence poset obtained by removing a single edge from this comparability graph. Blue vertices correspond to elements of $\mathcal{D}$; yellow vertices to elements of $\mathcal{U}$.

The PI has recently defined a piecewise-linear generalization of independence posets to a theory of *independence polytopes*. Although the theoretical properties of the construction have not yet been proven, the PI has already implemented his definition as Sage code, and he proposes to further develop the theory as part of this proposal.

2.1. Independence Posets.

The definitions found below can be explored using the PI's FPSAC 2020 online interactive poster [TW20b].

Definition 1. *A pair $(\mathcal{D},\mathcal{U})$ of disjoint independent sets of G is called* orthogonal *if there is no edge in G from an element of $\mathcal{D}$ to an element of $\mathcal{U}$. An orthogonal pair of independent sets $(\mathcal{D},\mathcal{U})$ is called* tight *if whenever any element of $\mathcal{D}$ is increased (that is, removed and replaced by a larger element with respect to G-order) or any element of $\mathcal{U}$ is decreased, or* *a new element is added to either $\mathcal{D}$ or $\mathcal{U}$, then the result is no longer an orthogonal pair of independent sets.*

We write $\operatorname{top}(G)$ for the set of all **t***ight* **o***rthogonal* **p***airs* of G. One can show that for any independent set $\mathcal{I}$, there is a unique $(\mathcal{I},\mathcal{U}) \in \operatorname{top}(G)$ and a unique $(\mathcal{D},\mathcal{I}) \in \operatorname{top}(G)$. *Rowmotion*—now generalizing the definition given in Section 1.1 for distributive lattice—is defined as the map that sends an independent set $\mathcal{D}$ to $\mathcal{U}$, where $(\mathcal{D},\mathcal{U}) \in \operatorname{top}(G)$.

Problem 1. *Systematically study rowmotion for independent sets of various graphs arising in representation theory (see also Sections 3.2 and 7.4). Document those examples for which the rowmotion operator has interesting DAC properties.*

Tight orthogonal pairs allow us to define a non-local *flip* operation, which generate the cover relations of a partial order which we call the *independence poset*.

Definition 2. *The* flip *of* $(\mathcal{D},\mathcal{U}) \in \operatorname{top}(G)$ *at an element* $g \in G$ *is the tight orthogonal pair* $\operatorname{flip}_g(\mathcal{D},\mathcal{U})$ *defined as follows: if* $g \notin \mathcal{D}$ *and* $g \notin \mathcal{U}$*, the flip does nothing. Otherwise, preserve all elements of* $\mathcal{D}$ *that are not less than* g *and all elements of* $\mathcal{U}$ *that are not greater than* g *(and delete all other elements); after switching the set to which* g *belongs, then greedily add elements to* $\mathcal{D}$ *and* $\mathcal{U}$ *(respecting the conditions to form an orthogonal pair) in reverse* G*-order and* G*-order, respectively.*

Figure 3 illustrates a flip on a top in an orientation of $[7] \times [7]$. The *independence relations* on $\operatorname{top}(G)$ are the reflexive and transitive closure of the relations $(\mathcal{D},\mathcal{U}) < (\mathcal{D}',\mathcal{U}')$ if there is some $g \in \mathcal{U}$ such that $\operatorname{flip}_g(\mathcal{D},\mathcal{U}) = (\mathcal{D}',\mathcal{U}')$.

Theorem 3. *Independence relations are antisymmetric, and hence define an* independence poset*, denoted* $\operatorname{top}(G)$*. Flips and cover relations of* $\operatorname{top}(G)$ *coincide.*

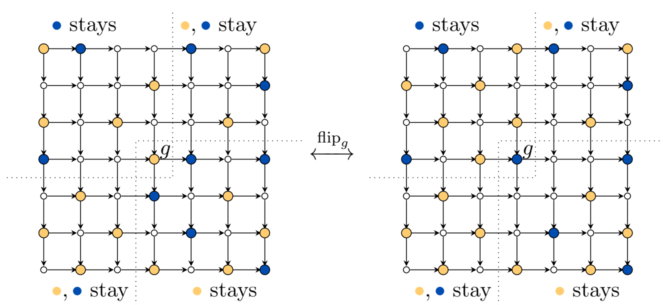


FIGURE 3. A flip on a top $(\mathcal{D},\mathcal{U})$ in the 7×7 grid oriented from top left to bottom right. Flipping at the vertex g changes its color, and divides the grid into 5 connected regions (delineated by the dotted lines): the blue vertices not less than g (i.e., not in the bottom right) and the orange vertices not greater than g (i.e., not in the top left) are preserved by the flip. The orange vertices in the top left are filled in greedily from bottom right to top left; the blue vertices in the bottom right are filled in greedily from top left to bottom right.

Birkhoff's *fundamental theorem of finite distributive lattices* proves that finite distributive lattices are parametrized by finite posets P (as the lattice $J(P)$ of order ideals under inclusion). Independence posets generalize Birkhoff's theorem: they are parametrized by acyclic directed graphs and their elements are the ubiquitous independent sets.

It is reasonable to suspect there is a simultaneos generalization of independence posets and semidistributive lattices—a motivating problem in this direction is that of reconstructing weak order from its Cambrian lattice: in type A, it is known how to reconstruct the collapsed intervals between sortable and anti-sortable elements [PP18]; more generally, for a class of independence posets called trim lattices it seems likely that one can replace vertices by another trim lattice to obtain a graded, semidistributive lattice. See also [RST19].

Problem 2. *Extend the theory of independence posets to general digraphs. In particular, the resulting construction should unify semidistributive and trim lattices.*

2.2. **Independence Polytopes.** In this section we propose a piecewise-linear generalization of independence posets. For G an acyclic directed graph, we conjecture in Problem 3 that a certain PL-generalization of flips given in Definition 5 defines the cover relations in a partial order $\mathrm{top}^{(m)}(G)$ on the integer points in the m-fold dilation of Chvátal's *independence polytope* $C(G)$.

Order ideals of a poset have a natural generalization to the theory of P-partitions [GHL$^+$16], which (after a piecewise-linear transfer map) can be interpreted as the lattice points inside of a certain polytope called the *chain polytope*. In this section we propose a piecewise-linear generalizations of independence posets, thereby providing a generalization of the construction of P-partitions.

Given a set $X = \{x_1, \ldots, x_n\}$, we write $\mathbb{R}^X$ for the set of functions $f : X \to \mathbb{R}$. For a poset P, the *chain polytope* in $\mathbb{R}^P$ is defined as the set of points $f \in \mathbb{R}^P$ satisfying the inequalities $0 \leq f(p)$ for all $p \in P$ and $\sum_{i=1}^{k} f(p_i) \leq 1$ for any chain $p_1 < \cdots < p_k$ in P. Stanley proved that the chain polytope is the convex hull of the characteristic functions of antichains of P [Sta86]. In fact (as Stanley remarks), this is a special case of a beautiful construction of Chvátal [Chv75]. Replacing order ideals of P by antichains in the comparability graph $G = \mathrm{Comp}(P)$ leads to the definition of the polytope $\mathcal{C}(G)$ as the set of points $f \in \mathbb{R}^G$ satisfying the inequalities

$$0 \leq f(g) \text{ for all } g \in G \text{ and } \sum_{g \in C} f(g) \leq 1 \text{ for any clique } C \subseteq G. \tag{1}$$

We call this the *independence polytope* of G. On the combinatorial side, the number of lattice points inside the m-fold dilation of the chain polytope is given by the number of multichains of order ideals $\emptyset = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_{m+1} = P$ in $J(P)$, or equivalently by $J(P \times [m])$. Since $\mathrm{top}(\mathrm{Comp}(P)) \simeq J(P)$, it is natural to search for a definition of the poset "$\mathrm{top}(G \times [m])$"—a partial order on the integer points in $m\mathcal{C}(G)$ that recovers $J(P \times [m])$ for $G = \mathrm{Comp}(P)$.

The PI has recently defined what appears to be the correct generalization—the PI has written Sage code to confirm this, and the task remains to prove these results. Rather than first defining the tight orthogonal pairs $\mathrm{top}(G)$ and then using these to define rowmotion, we *first* define rowmotion and then use rowmotion to give the correct generalization of $\mathrm{top}(G)$.

Definition 4. *Given $\mathcal{D} \in m\mathcal{C}(G)$, define* PL-toggle *operators $\mathrm{tog}_g^{(m)} : m\mathcal{C}(G) \to m\mathcal{C}(G)$ by*

$$\mathrm{tog}_g^{(m)}(\mathcal{D}(x)) = \begin{cases} \mathcal{D}(x) & \text{if } x \neq g \\ m - \max\limits_{\substack{C \text{ a clique} \\ g \in C}} \sum_{h \in C} \mathcal{D}(h) & \text{otherwise} \end{cases}. \tag{2}$$

Rowmotion *is the operator $\mathsf{row}^{(m)} : m\mathcal{C}(G) \to \mathcal{C}(G)$ given by $\mathsf{row}^{(m)}(\mathcal{D}) = \prod_{g \in G} \mathrm{tog}_g^{(m)}(\mathcal{D})$, where the product is in G-order.*

Definition 4 matches the combinatorial and piecewise-linear definitions of rowmotion for $G = \mathrm{Comp}(P)$ [SW12, EP13, Jos19, JR20]).

Definition 5. *The directed graph $\mathrm{top}^{(m)}(G)$ has vertices that are pairs $(\mathcal{D}, \mathsf{row}^{(m)}(\mathcal{D}))$ for an integer point $\mathcal{D} \in m\mathcal{C}(G) \cap \mathbb{Z}^G$. Its directed edges are defined using a PL-generalization of* flips*: for a vertex $g \in G$, subtract 1 from $\mathcal{U}(g)$ and add one to $\mathcal{D}(g)$ (if possible), and fill in the remainder of $\mathcal{U}$ and $\mathcal{D}$ above and below g in G-order using PL-toggles.*

An example is given in Figure 4; this definition has been coded in Sage although no theoretical properties have yet been proven.

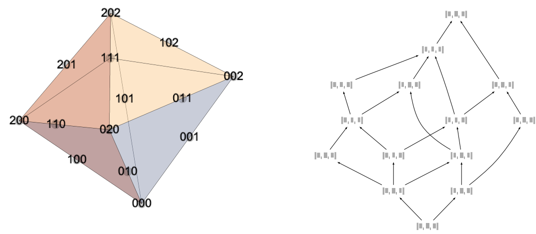


FIGURE 4. *Left:* the two-fold dilation of the independence polytope for the directed graph $G = 1 \to 2 \to 3$, with its 14 integer points labeled. *Right:* the same 14 lattice points in the generalized independence poset $\mathrm{top}^{(2)}(G)$.

Problem 3.

- *Prove that* $\mathrm{top}^{(m)}(G)$ *defines a partial order on the integer points in the polytope* $\mathcal{C}(G)$ *with cover relations given by PL-flips with unique minimal element* $(0, \mathsf{row}^{(m)}(0)$.
- *Prove that* $\mathrm{top}^{(m)}(\mathrm{comp}(P))$ *recovers the distributive lattice structure on the integer points on Stanley's chain polytope.*
- *Generalize central properties of P-partitions and Stanley's chain polytopes to* $\mathrm{top}^{(m)}(G)$ *and* $\mathcal{C}(G)$*. For example, a triangulation of* $\mathcal{C}(G)$ *should suggest a notion of "linear extensions" for independence posets, and then ought to give a formula for the Ehrhart generating function of* $\mathcal{C}(G)$.
- *Show that the polytopal interpretation of* $\mathrm{top}^{(m)}(G)$ *gives an efficient algorithm to generate the lattice points in* $m\mathcal{C}(G)$.
- *Extend existing problems (such as Problem 1) and theorems from* $\mathrm{top}(G)$ *to* $\mathrm{top}^{(m)}(G)$.

3. Symmetries of Quantum Groups

In this section we propose a connection between independence posets arising from the combinatorics of minuscule representations of Lie algebras and quantum groups.

3.1. **Definition of $U_q(\mathfrak{g})$, representations, and crystals.** Let $\mathfrak{g}$ be a simple Lie algebra with simple roots $\Delta = \{\alpha_1, \ldots, \alpha_n\}$, root lattice P and coroot lattice $P^\vee$. Take $(\cdot, \cdot)$ to be the bilinear form on P normalized so that short roots are of length 2; for $h \in P, \alpha \in \Delta$, we write $r_{h,\alpha} := -\alpha^\vee(h)$ and $r_{j,i} := r_{\alpha_j,\alpha_i}$. The *Weyl group* W is defined as the reflection group generated by *simple reflections* $\mathcal{S}$, defined by $s_i(h) := h + r_{h,\alpha_i}\alpha_i$ for $\alpha_i \in \Delta$. We write $w_\circ$ for the *longest element* of W. For q an indeterminate write $q_\alpha = q^{(\alpha,\alpha)/2}$ and $q_i = q_{\alpha_i}$. The *quantum group* $U_q(\mathfrak{g})$ is the unital associative algebra over $\mathbb{Q}(q)$ generated by e_i, f_i (for $i \in \{1, 2, \ldots, n\}$), and q^h for $h \in P$ subject to the relations

$$q^0 = 1 \text{ and } q^h q^{h'} = q^{h+h'} \text{ for } h, h' \in P,$$

$$q^h e_i = q^{(\alpha_i,h)} e_i q^h \text{ and } q^h f_i = q^{-(\alpha_i,h)} f_i q^h \text{ for } h \in P,$$

$$e_i f_j - f_j e_i = \delta_{i,j} \frac{q^{\alpha_i} - q^{-\alpha_i}}{q - q^{-1}},$$

$$e_j e_i^{(r_{j,i}+1)} = \sum_{k=0}^{r_{j,i}} (-1)^k e_i^{(k+1)} e_j e_i^{(r_{j,i}-k)} \text{ and } f_j f_i^{(r_{j,i}+1)} = \sum_{k=0}^{r_{j,i}} (-1)^k f_i^{(k+1)} f_j f_i^{(r_{j,i}-k)},$$

where $e_i^{(n)} = \frac{e_i^n}{[n]_{\alpha_i}!}$ and $f_i^{(n)} = \frac{f_i^n}{[n]_{\alpha_i}!}$, with $[n]_\alpha = \sum_{i=1}^n q_\alpha^{n-2i+1}$ for $n > 0$, $[n]_\alpha! = [n]_\alpha [n-1]_\alpha!$, and $[0]_\alpha! = 1$. Let λ be a dominant weight with V_λ an irreducible highest-weight representation with a *highest-weight vector* v_λ. For $a \in V_\lambda$, if $e_\alpha(a) = 0$ then for $k \geq 0$ we obtain basis vectors for the α-string through a by $\widetilde{f}_i^k(a) = f_i^{(k)}(a)$; these $\widetilde{f}_i$ are called *lowering operators*, with *raising operators* $\widetilde{e}_i$ defined analogously. The corresponding *crystal* C_λ is the directed graph with vertices given by the basis vectors of all such strings produced starting from the highest-weight vector v_λ, with labeled arrows $a \xrightarrow{i} b$ if $\widetilde{f}_i(a) = b$. The right side of Figure 1 gives an example of a crystal graph for $\mathfrak{sl}_3$.

3.2. **Weyl Group action on crystals.** There is a special class of representations for which the combinatorics is particularly simple; the connection between their combinatorics and representation theory has been exploited in these cases to explain good behavior of rowmotion [SW12, RS13]. A weight λ is called *minuscule* if each weight-space of V_λ is multiplicity-free and these weight spaces are spanned by the W-orbit of v_λ; any such λ is a fundamental weight, although not all fundamental weights are minuscule outside of type A (see Figure 5).

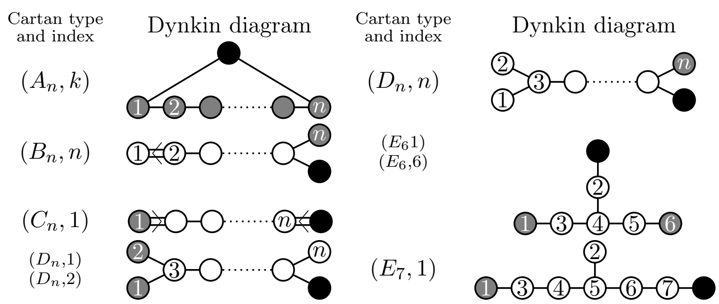
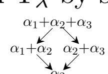
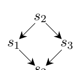


FIGURE 5. The roots α_i marked in gray have cominuscule fundamental weight λ_i; the affine simple root is marked in black.

The *root poset* Φ^+ is the partial order on the positive roots given by $\alpha < \beta \in \Phi^+$ iff $\beta - \alpha \in \mathrm{span}_{\mathbb{R}^+}\Delta$. Write Φ_λ for the restriction of the root poset to the inversion set of the longest element of the parabolic quotient of W by the stabilizer of λ; any path from v_λ to $w_\circ(v_\lambda)$ gives a natural labeling of the roots in Φ_λ by simple reflections; for example, when λ is the second fundamental weight in $\mathfrak{sl}_3$, $\Phi_J =$ 

with corresponding labeling by simple reflections 

. The vertices and edges of C_λ are then particularly easy to describe: any vertex is an order ideal in Φ_J and the raising operators act by *toggling* at all roots labeled by s_i (as in Section 2, adding or removing all such roots such that the result is again an order ideal). An example of this construction when λ is the second fundamental weight in $\mathfrak{sl}_3$ is given by comparing the left- and right-hand sides of Figure 1.

More generally, there is an action of the Weyl group W on C_λ where the simple reflection s_i reverses all α_i-chains. While combinatorially and geometrically natural, for $V_{m\lambda}$ (with λ minuscule) this action surprisingly does not correspond to the natural piecewise-linear toggles of Equation (2) on $\Phi_\lambda \times [m]$. Attempts to algebraically mirror the combinatorial toggles instead rely on a more subtle action, reversing not only α_i-chains but entire parabolic subcrystals via *Lusztig-Schützenberger involutions*.

3.3. **Cactus group action on crystals.** For W of rank n with $\mathcal{S} = \{s_1, s_2, \ldots, s_n\}$, write w_J for the longest element of the parabolic subgroup generated by $\{s_i\}_{i \in J}$; define an action of w_J on subsets $K \subset J$ by $w_J(K) = \{k' : w_J(s_k) = s_{k'}$ for $k \in K\}$. The (internal) *cactus* *group* of type W is the group $C(W)$ generated by symbols S_J for $J \subseteq \{1, 2, \ldots, n\}$ subject to the relations $S_J S_K = S_{w_J(K)} S_J$ when $K \subseteq J$, commutations $S_J S_K = S_K S_J$ if $K \cap J = \emptyset$ and $S_J^2 = 1$ (and no other relations). By construction, $C(W)$ acts on crystals as the usual Lusztig involution—for $j \in J$, S_J acts on vertices $a \in C_\lambda$ so that $S_J(f_j(a)) = e_{w_J(j)}(S_J(a))$. Note that when $J = \{j\}$ is a singleton, S_J recovers the previous action of W by reversing α_j-chains. In type A_{n-1}, vertices of C_λ are labeled by semistandard tableaux and c_J acts as a Schützenberger involution (partial evacuation).

The i-th Bender-Knuth involution B_i is a *different* lift of s_i to the cactus group:

$$B_i := S_{\{1,2,\ldots,i-1\}} S_{\{1,2,\ldots,i\}} S_{\{1,2,\ldots,i-1\}} S_{\{1,2,\ldots,i-2\}}, \tag{3}$$

with $B_1 := S_1$ and $B_2 := S_1 S_{1,2} S_1$. Write $W_\circ := S_{\{1,2,\ldots,n\}}$, $W_\circ^+ := S_{\{1,2,\ldots,n-1\}}$, and $w_\circ^+ := (w_\circ)_{\{1,2,\ldots,n-1\}}$. This construction leads to a trick (recently exploited in [Rus21, HLLY21]) to obtain the action of the long cycle $c := s_1 s_2 \cdots s_{n-1} = w_\circ w_\circ^+$ as promotion using the corresponding product in the cactus group $\mathsf{pro} := B_1 B_2 \cdots B_{n-1} = W_\circ W_\circ^+$. The trouble is that this trick does not appear to generalize to other types in order to obtain piecewise-linear toggles—in general type, a Coxeter element cannot be written as the product of two longest elements of parabolic subgroups. This trick, however, partially works in even orthogonal type $\mathfrak{so}(2n)$, and for multiples of both the vector and the spin representations, the PI has produced elements of the cactus group of $\mathfrak{so}(2n)$ that appear to act as piecewise-linear toggles on $\Phi_\lambda \times [m]$—but the construction is ad-hoc and only supported by many small computations.

We remark that it is plausible that the realization of promotion in type A is a special case of a different general phenomenon related to symmetries of the affine Dynkin diagram—in general type, generators of the quotient of the weight lattice by the root lattice Λ/P can be naturally lifted to $C(W)$ as the product of two longest elements of parabolic subgroups.

Problem 4. *Prove that the above lift of the group of symmetries of the affine Dynkin diagram to the cactus group $C(W)$ preserves the canonical basis of $C_{m\lambda}$ for λ minuscule, and corresponds to the corresponding PL-toggle order on $\Phi_\lambda \times [m]$.*

This statement can be checked in a type-by-type manner using well-developed combinatorial models for the canonical basis.

3.4. **Braid Group action on $U_q(\mathfrak{g})$.** We now begin a more systematic approach to encoding PL-toggles algebraically. For $\alpha_i \in \Delta$, there is an automorphism T_i of $U_q(\mathfrak{g})$ defined by

$$T_i(e_i) = -f_i q^{\alpha_i}, T_i(q^h) = q^{s_i(h)} \text{ and } T_i(f_i) = -q^{-\alpha_i} e_i$$

$$T_i(e_j) = \sum_{k=0}^{r_{j,i}} (-1)^k q_i^{-k} e_i^{(r_{j,i}-k)} e_j e_i^{(k)} \text{ and } T_i(f_j) = \sum_{k=0}^{r_{j,i}} (-1)^k q_i^k f_i^{(k)} e_j e_i^{(r_{j,i}-k)} \text{ if } i \neq j.$$

The T_i satisfy the braid relations and actually define an action of the braid group $B(W)$ by automorphisms of the quantum group $U_q(\mathfrak{g})$ [Jan96], and we write $T_w = T_{s_{i_1}} \cdots T_{s_{i_k}}$ if $\mathsf{s}_{i_1} \cdots \mathsf{s}_{i_k}$ is a reduced word for $w \in W$.

The T_i give a rigid lift of the simple reflection s_i on V_λ and it is exactly this property that we propose to exploit to algebraically recover PL-toggle actions. For example, the automorphism T_i on $U_q(\mathfrak{g})$ sends a highest-weight vector v_λ of weight λ to a vector of weight $s_i(\lambda)$. For minuscule λ, it follows from Section 3.2 and the fact that all weight spaces are one-dimensional that T_i can be interpreted as a toggle on the canonical basis of V_λ.

Fix a reduced word $\mathsf{s}_{i_1} \mathsf{s}_{i_2} \cdots \mathsf{s}_{i_N}$ in simple reflections for the long element $w_\circ \in W$ with corresponding total order on the positive roots given by $\beta_k = s_{i_1} s_{i_2} \cdots s_{i_{k-1}}(\alpha_{i_k})$, and define $f_{\beta_k} := T_{i_1} T_{i_2} \cdots T_{i_{k-1}}(f_{i_k})$. The *Poincaré-Birkhoff-Witt (PBW)* basis of $U_q(\mathfrak{g})$ corresponding to this choice of reduced word is given by elements of the form

$$f_{\beta_1}^{(k_1)} \cdots f_{\beta_N}^{(k_N)} q^h e_{\beta_1}^{(l_1)} \cdots e_{\beta_N}^{(l_N)} \text{ with } k_i, l_i \in \mathbb{Z}, h \in P.$$

The Lusztig-Schützenberger involutions also fit into this framework, since $T_{w_\circ}$ preserves the PBW basis while acting to interchange e_i and f_i. With a slight modification of the basis of $U_q^0(\mathfrak{g})$ (Lusztig's *integral form*), the product of two such basis elements is a linear combination of basis elements with coefficients in $\mathbb{Z}[q, q^{-1}]$.

Much of the combinatorics arising from $U_q(\mathfrak{g})$ has come from studying special choices of reduced words for $w_\circ$. For example, in $\mathfrak{sl}_n$ the choice of $w_\circ = \mathsf{s}_1\mathsf{s}_2\cdots\mathsf{s}_n\cdot\cdots\cdot\mathsf{s}_1\mathsf{s}_2\cdot\mathsf{s}_1$ leads to the standard theory of Gelfand-Tsetlin patterns and semistandard Young tableaux as a basis of C_λ. Furthermore, at $q=1$, we can check that $\mathsf{pro} = T_{w_\circ}T_{w_\circ^+}$.

We now explain a choice of a reduced word for $w_\circ$ that is ubiquitous in the combinatorics of quiver representations, cluster algebras, and Coxeter–Catalan combinatorics, and is therefore very natural from the Ringel-Hall algebra approach to quantum groups, which thus far has not been fully exploited by the combinatorics community.

3.5. **Coxeter-sorting words and PBW-preserving automorphisms of $U_q(\mathfrak{g})$.** A (standard) *Coxeter element* c for W is a product of all elements of $\mathcal{S}$ in some order.

Definition 6. *Let w be an element of W. The* c-sorting word $\mathsf{w}(\mathsf{c}) = \mathsf{c}\big|_{I_1}\cdots\mathsf{c}\big|_{I_k}$ *for w is the lexicographically first subword of $\mathsf{c}^\infty = (\mathsf{s}_1\cdots\mathsf{s}_n)^\infty$ that is a reduced expression for w.*

The connection between c-sorting words and the quantum group proceeds via the PBW basis corresponding to the c-sorting word $\mathsf{w}_\circ(\mathsf{c})$, which also naturally arises via Ringel's alternative quiver-theoretic construction of $U_q^+(\mathfrak{g})$. For u, w words in simple reflections, write $\bar{\mathsf{u}}\mathsf{u}\mathsf{w} = \mathsf{w}$, write $\mathsf{w} \equiv \mathsf{u}$ if both words are equal up to commutation of commuting simple reflections, and write $\psi(\mathsf{w})$ for the action conjugating all letters of w by $w_\circ$. Write $s = s_1$ for the initial simple reflection of c; repeatedly using the fact that $\mathsf{w}_\circ(\bar{\mathsf{s}}\mathsf{c}\mathsf{s}) \equiv \bar{\mathsf{s}}\mathsf{w}_\circ(\mathsf{c})\psi(\mathsf{s})$ shows that $\mathsf{w}_\circ(\mathsf{c}) \equiv \bar{\mathsf{c}}\mathsf{w}_\circ(\mathsf{c})\psi(\mathsf{c})$ so that the composition of $U_q(\mathfrak{g})$ automorphisms $T_1\cdots T_n$ is an automorphism preserving the PBW basis coming from $\mathsf{w}_\circ(\mathsf{c})$.

Unfortunately, outside of the usual linear ordering on the simple roots of $\mathfrak{sl}_{n-1}$, one can check that Lusztig's canonical basis for C_λ is *not* preserved (even at $q=1$) by this composition of automorphisms. This is not particularly surprising: unlike the T_i, the Bender-Knuth lifts S_i of the simple reflections do not satisfy the braid relations. However, for λ minuscule, Garver, Patrias, and Thomas [GPT18] showed using representations of quivers that PL-promotion on $\Phi_\lambda \times [m]$ can be computed using a composition of reflection functors in an order compatible with the initial orientation of the Dynkin quiver. Since this quiver-theoretic perspective is only sensitive to the leading monomial of the canonical basis, this work strongly suggests that there is a deformation of the automorphism T_i of $U_q(\mathfrak{g})$ that acts by PL-toggles on the canonical basis of $C_{m\lambda}$ for λ minuscule. From the point of view of the dual braid group and Catalan combinatorics, there is a natural choice generalizing the Bender-Knuth lifts B_i. If we write $T_{\beta_k} = T_{i_1}\cdots T_{i_k}\cdots T_{i_1}$ when $\beta_k = s_{i_1}s_{i_2}\cdots s_{i_{k-1}}(\alpha_{i_k})$ with $\mathsf{w}_\circ(\mathsf{c}) = \mathsf{s}_{i_1}\cdots\mathsf{s}_{i_N}$, then for $1 \le k \le n$, $T'_k = T_{\beta_1}\cdots T_{\beta_{k-1}}T_{\beta_k}T_{\beta_{k-1}}\cdots T_{\beta_1}$ is a lift of s_k.

To show how this construction recovers the Bender-Knuth for $\mathfrak{sl}_4$ with the linear ordering, we compute that $T_{\alpha_1} = T_1$, $T_{\alpha_2} = T_1T_2T_1$, and $T_{\alpha_3} = T_1T_2T_3T_2T_1$. Then we obtain the following very interesting factorization—which we note is different from Equation (3) (cf. Problem 4):

$$\begin{aligned} B_3 = S_{1,2}\cdot S_{1,2,3}\cdot S_1\cdot S_{1,2,1} &= T_1T_2T_1\cdot T_1T_2T_3T_1T_2T_1\cdot T_1\cdot T_1T_2T_1 \\ &= T_{\alpha_1}\cdot T_{\alpha_2}\cdot T_{\alpha_3}\cdot T_{\alpha_2}\cdot T_{\alpha_1} = T_1\cdot T_1T_2T_1\cdot T_1T_2T_3T_2T_1\cdot T_1T_2T_1\cdot T_1. \end{aligned}$$

Problem 5. *Using c-sorting words and the automorphisms T_i of $U_q(\mathfrak{g})$, uniformly define a lift of the simple reflections s_i that acts by PL-toggles on combinatorial models for the canonical basis of $C_{m\lambda}$ for λ minuscule. Does the above factorization satisfy these requirements?*

As in [KB96], a helpful step will be to perform calculations within the piecewise-linear crystal model $B(\infty)$ of $U_q^+(\mathfrak{g})$ before passing to the entire quantum group.

4. Symmetries of Cluster Algebras

In this section, we study independence posets arising from the combinatorics of quiver representations, intimately related to cluster algebras. For Φ an irreducible root system with positive roots Φ^+ and simple roots Δ, let $\Phi_{\geq -1} = \Phi^+ \cup -\Delta$ be the set of *almost positive roots*. For $s \in \mathcal{S}$ with corresponding $\alpha_s \in \Delta$, define a bijection $\tau_s : \Phi_{\geq -1} \longrightarrow \Phi_{\geq -1}$ by

$$\beta \longmapsto \begin{cases} \beta & \text{if } \beta \in -(\Delta \setminus \alpha_s) \\ s(\beta) & \text{otherwise} \end{cases}. \tag{4}$$

In their study of finite type cluster algebras, Fomin and Zelevinsky used τ_s to define a binary relation on $\Phi_{\geq -1}$ [FZ02, FZ03]. Marsh, Reineke, and Zelevinsky [MRZ03] and, independently, Reading [Rea07] interpreted this as the bipartite case of a more general family of relations, depending on a Coxeter element c.

Definition 7. *The c-compatibility relations are the unique family of relations $\|_c$ on $\Phi_{\geq -1}$ characterized by: for $\alpha \in \Delta$, $-\alpha \|_c \beta \Leftrightarrow \beta \in \Phi_{\langle s_\alpha \rangle}$, and for s final in c, $\beta_1 \|_c \beta_2 \Leftrightarrow \tau_s(\beta_1) \|_{sc\bar{s}} \tau_s(\beta_2)$. The c-cluster complex $\mathsf{Asso}(W, c)$ is the simplicial complex given by all collections of pairwise c-compatible almost positive roots.*

Particular orientations of cluster exchange graphs are known to be independence posets, as they coincide with Cambrian lattices. In crystallographic type, the c-cluster complexes are isomorphic to the cluster complex defined in [FZ02]. A *c-cluster* is a facet of the c-cluster complex $\mathsf{Asso}(W, c)$—that is, a maximal subset of almost positive roots which are pairwise c-compatible. Igusa and Schiffler found an explicit rule for the compatibility of two roots under $\|_c$ in [IS10, Theorem 2.5].

Theorem 8. *Fix W a finite Coxeter group and c a Coxeter element. Let $t_1 <_c \cdots <_c t_N$ be the total order on all reflections $\mathcal{R}$ given by the c-sorting word $\mathsf{w}_\circ(\mathsf{c})$, write $\mathsf{Q} = \mathsf{t}_1\mathsf{t}_2 \cdots \mathsf{t}_N \mathsf{t}_1 \mathsf{t}_2 \cdots \mathsf{t}_n$, and $[N] = \{1, \ldots, N\}$.*

- *The c-cluster complex $\mathrm{Asso}_c(W, c)$ is isomorphic to the flag simplicial complex on $[N + n]$ such that $(i_1 < i_2 < \cdots < i_k)$ is a face iff $\mathsf{Q}_{i_1}\mathsf{Q}_{i_2} \cdots \mathsf{Q}_{i_k}$ is a reduced $\mathcal{R}$-word for an element less than c^{-1} in absolute order.*
- *The positive c-cluster complex $\mathrm{Asso}_c^+(W)$ is the flag simplicial complex obtained by intersecting the simplices of $\mathrm{Asso}_c(W, c)$ with $[N]$.*

4.1. Quantum groups and positive cluster complexes.

We first state a conjectural relationship between $U_q(\mathfrak{g})$ and $\mathrm{Asso}_c^+(W)$ that is supported by many computations using the standard piecewise-linear operations on the $B(\infty)$-crystal tracking the leading coefficients of the canonical basis of $U_q^+(\mathfrak{g})$ as the choice of reduced word for $w_\circ$ varies, resulting in a change of PBW basis. This conjectural relationship exploits the comparison of the choice $\mathsf{w}_\circ(\mathsf{c})$ with the choice $\mathsf{w}_\circ(\mathsf{c}^{-1})$, and is suggested both by the conjecture in Section 4.2 as well as the discussion in Section 3.5 of the symmetries of the quantum group.

Problem 6. *Fix $U_q(\mathfrak{g})$ with PBW basis given by $\mathsf{w}_\circ(\mathsf{c})$ with $c = s_1 \cdots s_n$ and root order $\beta_1 <_c \cdots <_c \beta_N$. Define a map $\phi : \mathbb{N}^n \to \mathbb{N}^N$ by*

$$\phi(a_1, a_2, \cdots, a_n) = ((i_1 < i_2 < \cdots < i_k), (b_{i_1}, b_{i_2}, \ldots, b_{i_k}))$$

if $f_n^{(a_n)} \cdots f_1^{(a_1)} \in f_{\beta_1}^{(b_1)} f_{\beta_2}^{(b_2)} \cdots f_{\beta_N}^{(b_N)} + qU_q^+(\mathfrak{g})$ with $b_l \neq 0$ iff $l \in \{i_1, \ldots, i_k\}$.

Then ϕ is a bijection from $\mathbb{N}^n$ to $\mathrm{Asso}_c^+(W) \times \mathbb{N}^n$.

Example 9. *Using the piecewise-linear rules for converting between $B(\infty)$ bases depending on the choice of reduced word for $w_\circ$, for $\mathfrak{g} = \mathfrak{sl}_3$ and $c = s_1 s_2$, we have*

$$f_2^x f_1^y = \begin{cases} f_{\alpha_1}^{x-y} f_{\alpha_1+\alpha_2}^{y} + qU_q^+(\mathfrak{sl}_3) & \text{if } x \geq y \\ f_{\alpha_1+\alpha_2}^{x} f_{\alpha_2}^{y-x} + qU_q^+(\mathfrak{sl}_3) & \text{otherwise.} \end{cases}$$

Observe that $c^{-1} = (12)(13) = (13)(23)$.

This suggests that the generalized independence poset $\mathrm{top}^{(m)}(G)$ for G a certain refinement of the Auslander-Reiten quiver can be embedded in the positive part of the quantum group.

4.2. **Nonnesting partitions and cluster complexes.** Although no uniform proof is currently known, the facets of the (positive) cluster complex are counted by the *Catalan number of type* W,

$$\mathsf{Cat}(W) := \prod_{i=1}^{n} \frac{h+1+e_i}{d_i} \text{ and } \mathsf{Cat}^+(W) := \prod_{i=1}^{n} \frac{h-1+e_i}{d_i}$$

where $d_1 \leq d_2 \leq \ldots \leq d_n$ are the degrees of the fundamental invariants of W, $e_i = d_i - 1$ and $h := d_n$ is the Coxeter number. For the symmetric group $\mathfrak{S}_n$, this definition recovers the classical Catalan numbers.

Problem 7. *Uniformly prove that facets of the (positive) cluster complex are counted by* $\mathsf{Cat}(W)$ *(resp.* $\mathsf{Cat}^+(W)$*).*

We propose a strategy for Problem 7. Catalan numbers naturally appear in a markedly different context—in the study of affine Weyl groups and rational Cherednik algebras (this is related to the Macdonald theory of the previous section). $\mathsf{Cat}(W)$ (uniformly) counts the number of coroot points inside an $h+1$-fold dilation of the fundamental alcove in the corresponding affine Weyl group [Hai94, Sut98]. These coroot points are called *nonnesting partitions*, and are in bijection with order ideals in the root poset (or, equivalently, ad-nilpotent ideals in a Borel subalgebra of the corresponding complex simple Lie algebra). Although nonnesting and noncrossing partitions have many similarities, finding a uniform bijection between the two sets has been an active and motivating area of research since the late 1990s [Rei97, Ath98]. Since nonnesting partitions are uniformly enumerated, such a bijection would answer Problem 7.

In [Wil13b], the PI conjectured exactly such a bijection between nonnesting and noncrossing objects for any Coxeter element and any finite Weyl group, suggesting that the root poset encodes a remarkable amount of information related to the corresponding Weyl group (compare with the duality between the heights of roots and the degrees). In more detail, the PI's method is based on an original analogy between clusters and nonnesting partitions: clusters have a natural order $h+2$ cyclic action called *Cambrian rotation* Camb_c, obtained by sending the initial copy of c in the initial $\mathsf{w}_\circ(\mathsf{c})$ of Q to the final copy of $\psi(c)$ in Q.

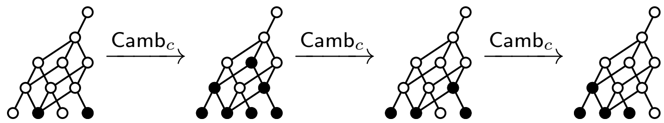
Definition 10. *Define* Krew *to be the composition of toggles in* $\mathsf{inv}(\mathsf{w}_\circ(\mathsf{c}))$ *root order and* Krew^+ *to be the same composition without simple roots. Define* $\mathsf{Camb}_c := \mathsf{Krew} \circ \mathsf{Krew}^+$.

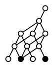
We conjecture that Camb_c on $\mathsf{NN}(W)$ has the same orbit structure as Camb_c on $\mathsf{NC}(W, c)$.

Problem 8. *For any irreducible Weyl group* W *and Coxeter element* c*, there is a unique bijection* $C_c : \mathsf{NN}(W) \to \mathsf{Asso}(W, c)$ *satisfying* • $C_c(\emptyset) = 1$*,* • $C_c \circ \mathsf{Camb}_c = \mathsf{Camb}_c \circ C_c$*, and* • $\mathsf{supp}(\mathcal{I}) = \mathsf{supp}(C_c(\mathcal{I}))$.

In fact, the PI has found an explicit way to conjecturally compute C_c using support. Given a nonnesting partition $\mathcal{I}$, form the orbit $\left(\mathcal{I}, \mathsf{Camb}_c(\mathcal{I}), \mathsf{Camb}_c^2(\mathcal{I}), \ldots, \mathsf{Camb}_c^{h+1}(\mathcal{I})\right)$. For each order ideal $\mathsf{Camb}_c^j(\mathcal{I})$ in the orbit, record only the simple roots that are *not* in $\mathsf{Camb}_c^j(\mathcal{I})$. These simple roots then spell out the corresponding facet in $\mathsf{Asso}(W, c)$. Some care is required when $\overline{s} \neq s$. Example 11 illustrates this computation for an orbit in type D_4.

Example 11. *In type* D_4 *with* $c = s_1s_2s_3s_4 = (123\overline{1}\overline{2}\overline{3})(4\overline{4})$, $\mathsf{w}_\circ(\mathsf{c}) = \mathsf{c}^3$*, we have the root order is* $(12), (13), (14), (1\overline{4}), (14), (23), (2\overline{3}), (1\overline{2}), (2\overline{4}), (24)(1\overline{3}), (2\overline{3}), (34), (3\overline{4})$. *We compute*



so that  *maps to the facet* $\{1, 3, 11, 16\}$*, which corresponds to* $c^{-1} = (12)(14)(34)(1\bar{4})$.

The conjectural bijection stated in Problem 8 has been exhaustively checked up to rank eight [Wil13a, Wil14, STW17], and there are further conjectural relations and compatibility with the Kreweras complement that we omit here.

5. Prior Support

The PI has not held a standard NSF grant before; he applied for and received the NSF grant for the 2018 Graduate Student Combinatorics Conference (with over 70 outside graduate student participants), and he is currently the US grant coordinator for the international conference Formal Power Series and Algebraic Combinatorics.

6. Intellectual Merit

The PI's research is in algebraic combinatorics, with a broad interest in motivation from other areas of mathematics such as Lie theory, geometric group theory, and reflection groups. The PI has a strong record of solving long-standing problems using an original toolkit and perspective: he has been selected to give six talks at FPSAC and will be an invited speaker at the 2020 Triangle Lectures in Combinatorics as well as Open Problems in Algebraic Combinatorics 2021 at the University of Minnesota.

The PI's related work in [SW12] has served as a catalyst for the involvement of undergraduate and beginning graduate students in cutting-edge research at REUs and doctoral programs. There have been many developments motivated by the appearance of [SW12]—to name a few: [CHHM15, EP13, EFG+15, Had14, Hop16, GR14, GR15, GR16, PR15, Rob16, RS13, RW15, Rus16, DPS17, Str15, Str16, JR18, MR19, DSV19, Jos19, JR20, Hop20, JR20]. In 2015, the PI, Striker, Propp, and Roby organized an AIM workshop that launched a new field of combinatorics now termed "Dynamical Algebraic Combinatorics." This same group organized a follow-up BIRS online workshop in the Fall 2020 (originally accepted in-person, but held online due to COVID-19; the PI is took advantage of this to arrange for his undergraduate honors reading class to attend the workshop). The PI has additionally organized several successful AMS and JMM special sessions in this field. An integral part of this proposal is to continue supporting the PI's ongoing and future efforts to involve students in cutting-edge research in algebraic combinatorics and related areas.

The PI has already laid some of the theoretical groundwork underpinning this proposal in the two recent publications [TW17, TW19]. Over the course this previous research, the PI has developed an original toolkit and perspective that has yielded substantial new progress in related fields. Based on this new perspective, the PI has created an interconnected library of concrete combinatorial problems especially suitable for early-stage students.

7. Broader Impacts

The PI has substantial past experience in involving students and underrepresented students in research: he has mentored undergraduate research over six different summers (at UTD, LaCIM, and UMN), supervised three honors theses at UTD, and he currently has two Ph.D. students pursuing thesis research in areas related to this proposal. The PI's related work in [SW12] has served as a catalyst for the involvement of undergraduate and beginning graduate students in cutting-edge research at REUs and doctoral programs. There have been many developments motivated by the appearance of [SW12]—to name a few: [CHHM15, EP13, EFG+15, Had14, Hop16, GR14, GR15, GR16, PR15, Rob16, RS13, RW15, Rus16, DPS17, Str15, Str16, JR18, MR19, DSV19, Jos19, JR20, Hop20, JR20]. The PI has a record of producing problems and research areas accessible to beginning researchers, including the now-active area of dynamical algebraic combinatorics. At least four of the PI's papers have independently led to Research Experience for Undergraduates (REU) projects at four different institutions. In 2015, the PI, Striker, Propp, and Roby organized an AIM workshop that launched a new field of combinatorics now termed "Dynamical Algebraic Combinatorics." This same group organized follow-up BIRS workshops in Fall 2020 and 2021. The PI has additionally organized several successful AMS and JMM special sessions in this field.

An integral part of this proposal is to continue supporting the PI's ongoing and future efforts to involve students in cutting-edge research in algebraic combinatorics and related areas. As the only combinatorialist at UTD, the PI has designed new undergraduate and graduate courses in combinatorics; due to the success of his undergraduate Discrete Math and Combinatorics class, the PI was asked by the honors college to teach honors reading courses in Fall 2019, 2020, and 2021. The PI has a history of service to the combinatorial community: he has refereed for over twenty journals, became an editor for Annals of Combinatorics in 2019, served on the program committee of FPSAC in 2019, serves currently on the organizing committee as the US funding coordinator, and has organized many conferences, workshops, and special sessions. He has represented the larger mathematical community to the public by appearing as a mathematical consultant in a 2018 nationally televised report (WFAA) regarding the NCAA basketball bracket, and hosting mathematical events at UTD (annual freshman orientation tables for potential math majors, faculty speaker at a MATHCOUNTS competition, π-day events for the local Dallas International School, etc.).

7.1. **Interactive JavaScript textbook.** During the COVID-19 pandemic, the PI experimented with novel methods to disseminate his research. The 2020 summer international conference Formal Power Series and Algebraic Combinatorics (FPSAC) was held online, and the PI used the opportunity of the remote poster session to develop a JavaScript browser-based interactive poster (see Figure 6 and [TW20b]). This poster was a highly successful experiment: the conference organizers selected it as an example for other presenters of how the online format could be harnessed to be even more engaging than a static in-person poster, and asked the PI for advice on how other presenters could develop a similar poster. The PI is currently developing a new interactive poster for his FPSAC 2021 submission.

The PI would like to build on this success by extending such interactive materials from his research to his undergraduate teaching by creating a **browser-based interactive discrete math textbook**. The PI designed a discrete math and combinatorics course as part of the new data science program at UTD. He has currently taught the course five times, and he would like to use the expertise he developed while creating interactive posters to render his notes of course content and classroom activities more engaging by using JavaScript to both animate concepts and allow students to interact with new definitions and proofs. Materials include introduction to proof, naive set theory, relations, introduction to algorithms, modular arithmetic, basic combinatorial objects (combinations and permutations), recurrences, inclusion-exclusion, the cycle lemma, and trees. The PI has notes in TeX for this course, as well as lecture recordings and handwritten notes from the past two online semesters.

An example of how interactive content could be modified from the classroom to the textbook is the following two-player game the PI uses to introduce inductive reasoning, typically played by students by crossing out circles on paper: "There are nine coins. The players take turns, each of which consists of taking either one or two coins. A player loses if they can't take a coin. Do you want to go first or second, and why?" Making this a virtual exercise will

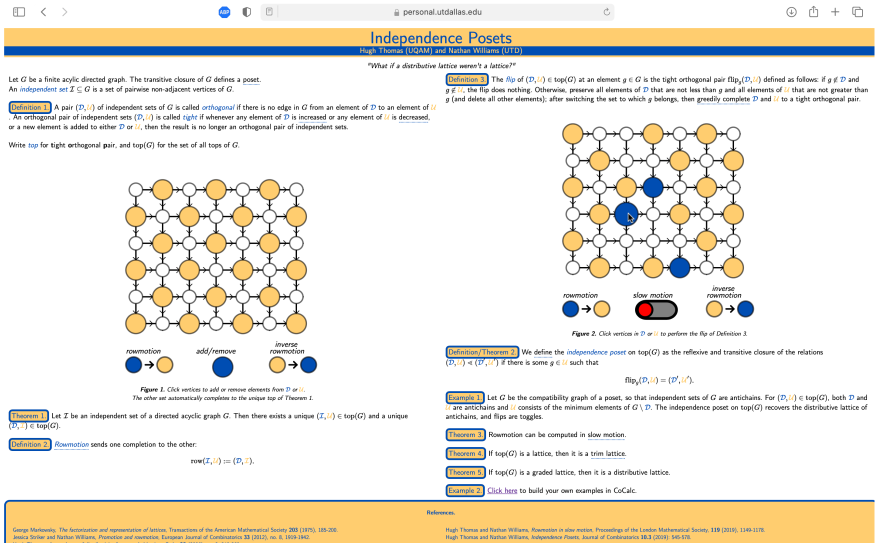


FIGURE 6. A screen shot of the PI's interactive poster presented at FPSAC 2020. Each of the grids is a JavaScript applet that allows the participant to experiment with various notions from Section 2, including Figure 3.

allow students to more easily play and experiment (and allow them to more easily change the number of coins), which will allow students to more productively engage with the problem.

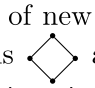
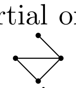
7.2. **Conferences and Workshops Organized.** The PI has been very active in **organizing conferences and workshops**: • 2015 - week-long workshop at the American Institute of Mathematics; • 2018 - Graduate Student Combinatorics Conference at UTD, with over 75 attendees (also obtaining $20,000 of NSF funding); • 2019 - FPSAC program committee.• 2017–2021 - Organized four AMS special sessions. With UTD colleague M. Arnold, organized a special session in Hawaii in 2019, and another special session at the 2020 Joint Mathematical Meetings in Denver, both relating to the interactions between dynamical systems and combinatorics; • 2018 - two-week "research-in-pairs" program at Oberwolfach, resulting in a 132-page preprint accepted to Memoirs of the AMS; • 2019 - two minisymposia on "Coinvariant Spaces and Parking Functions" at the SIAM Texas Louisiana Section at Southern Methodist University under the meta-organization of Sottile; • 2020 and 2021 - BIRS workshops with Propp, Roby, and Striker on "Dynamical Algebraic Combinatorics"; and • 2021–2023 - Member of the FPSAC organizing committee as US funding coordinator.

7.3. **Mentoring - Online Workshop.** The PI has substantial past experience in involving students and underrepresented students in research: this past Spring 2020, the PI supervised two undergraduate honors theses (both summitted for publication, one already accepted), and this past Summer 2021, the PI supervised two graduate students on a research project. He currently has two Ph.D. students (A. Kaushal and P. Palit) pursuing their thesis research in areas related to this proposal. The PI will continue to seek out such opportunities with the goal to eventually build a strong combinatorics program at UT Dallas, include applying for REU funding. The PI intends to use his past experience in conference organization and research mentoring to set up a **yearly online workshop** with the goal of bringing together **early graduate and undergraduate students** (including the honors students in his honors reading courses, as well as Ph.D. students of the PI's collaborators).

While at UT Dallas the PI has worked with **graduate students** in the following ways: • currently the **thesis advisor** of A. Kaushal (since Fall 2018); • currently the **thesis advisor** of P. Palit (since Spring 2019); • organized the 2018 **Graduate Student Combinatorics**

Conference; • supervised several **independent study/research** courses with graduate students (Fall 2017, Spring 2019, Summer 2020, Summer 2021).

While at UT Dallas the PI has worked with **undergraduates** in the following ways: • Spring 2018 - Supervised K. Zimmer's **senior honors thesis**; • Summer 2018 - Mentored rising senior R. Hubbard for eight weeks as part of the **Pioneer REU program** (now pursuing his Ph.D. at UNC Chapell Hill); • Spring 2019 - Supervised **independent research** with junior J. Marsh; • Spring/Summer 2019 - Supervised **independent research** with undergraduates C. Kondor and M. Patten; • Due to the success of the Discrete Math and Combinatorics course the PI designed for the new BS in Data Science program, the PI was asked by the honors college to teach an **honors reading course** in Fall 2019, 2020, and 2021. In Fall 2020, this reading class took part in the BIRS Dynamical Algebraic Combinatorics conference (held online due to COVID-19).• Spring 2020 - Supervised J. Marsh's **senior honors thesis** (now pursuing his Ph.D. studies at GA tech; submitted for publication); • Spring 2020 - supervised B. Cotton's **senior honors thesis** (accepted for publication). Further past experience involving undergraduate students in research includes **two summers as an REU mentor at the University of Minnesota** and **two summers mentoring undergraduate students at LaCIM**.

7.4. **Digital database of independence posets.** Since distributive lattices $J(P)$ are recovered by independence posets when $G = \mathrm{Comp}(P)$ is the comparability graph of the poset P (antichains in P become independent sets of $\mathrm{Comp}(P)$), many classical combinatorial objects (to name a few: integer partitions in a box, various classes of plane partitions, domino tilings, stable marriages, alternating sign matrices, and minuscule lattices) can be encoded as independent sets of particular graphs, and these objects can now be endowed with a wide variety of new partial orders using the framework of independence posets. For example, the graphs  and  each have 7 independent sets (one graph encodes the seven 3×3 alternating sign matrices, the other the corresponding set of totally symmetric self-complementary plane partitions). A broader impact of this proposal is to **compile a library of combinatorially-relevant graphs and their independent sets, and integrate them into Sage for widespread use**. The PI has already written and made publicly available some code for dealing with independence posets [TW20a].

7.5. **Math circles.** UTD has recently hired several assistant professors in mathematics, and the PI proposes to start **math circles** with these colleagues (in particular Arnold and Arreche, both of whom have had substantial past experience with and interest in such activities through the Russian School of Mathematics and MathCounts). The PI has already engaged with Mark Smith, the Campus Director of the Art of Problem Solving Academy in Frisco, and the UTD Department of Mathematical Sciences to gauge student interest and make sure that local funding is available.