Strange Ěxpectations

Nathan Williams (UTD)
Joint with Marko Thiel (Marshall Wace LLP) *and* Eric Stucky (UMN)

UCLA Combinatorics Seminar

December 3, 2020

Blue checkmark:

$\alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha \rangle}$

$\widetilde{\alpha} = \sum c_i \alpha_i$ $\qquad$ $h = 1 + \sum c_i$

$\widetilde{\alpha}^\vee = \sum c_i^\vee \alpha_i^\vee$ $\qquad$ $g = 1 + \sum c_i^\vee$

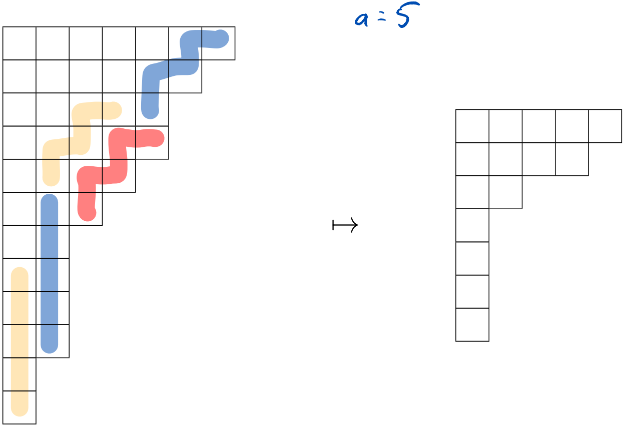
Φ	root system
Φ^+	positive roots
n	rank
W	Weyl group
h	Coxeter number
g	dual Coxeter number
r	ratio of length of a long to short root
$\widetilde{\alpha}$	highest root
$\mathcal{Q}$	root lattice
ρ	half-sum of the positive roots
$\widetilde{\Phi}$	affine root system
$\widetilde{W}$	affine Weyl group
$\Phi^\vee$	dual root system
$g^\vee$	dual Coxeter number of dual root system

1

Cores and the coroot lattice

Cores

An *a-rim hook* of λ is a connected boundary strip of a boxes.

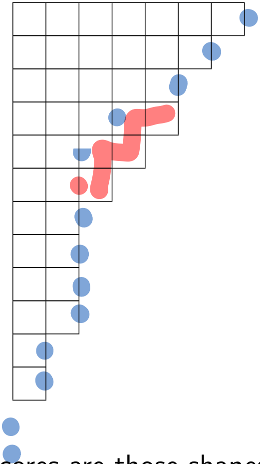


- For a fixed $a \in \mathbb{N}$, we can try to remove all a-rim hooks.
- *Order doesn't matter!?!*
- Partitions with no a-rim hooks are called *a-cores*.

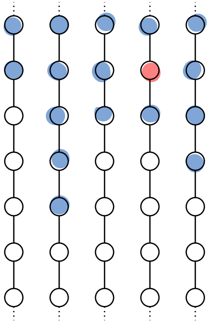
Abaci

The *a-abacus* records the boundary of λ on a runners. $a = 5$

Removing an a-rim hook pushes an • up a runner.



$\mapsto$

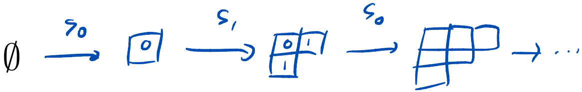


$\therefore$ a-cores are those shapes that are "flush" on the a-abacus.

Generating 2-cores

Label points (i, j) in $\mathbb{N} \times \mathbb{N}$ by content $(i - j) \bmod 2$.

0	1	0	1	0	1	0
1	0	1	0	1	0	1
0	1	0	1	0	1	0
1	0	1	0	1	0	1
0	1	0	1	0	1	0
1	0	1	0	1	0	1
0	1	0	1	0	1	0

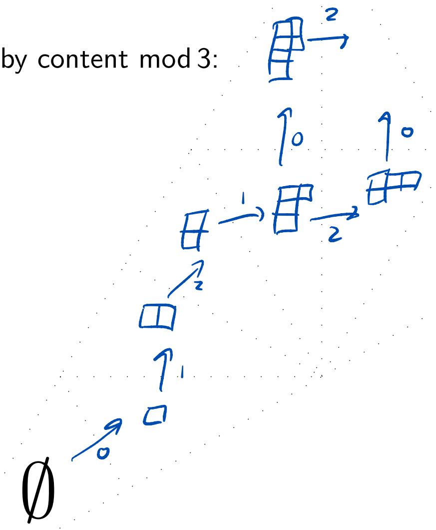


- s_0 adds or removes all boxes with content 0
- s_1 adds or removes all boxes with content 1.

Generating 3-cores

"Same thing" for $a = 3$: label by content mod 3:

0	1	2	0	1	2
2	0	1	2	0	1
1	2	0	1	2	0
0	1	2	0	1	2
2	0	1	2	0	1
1	2	0	1	2	0



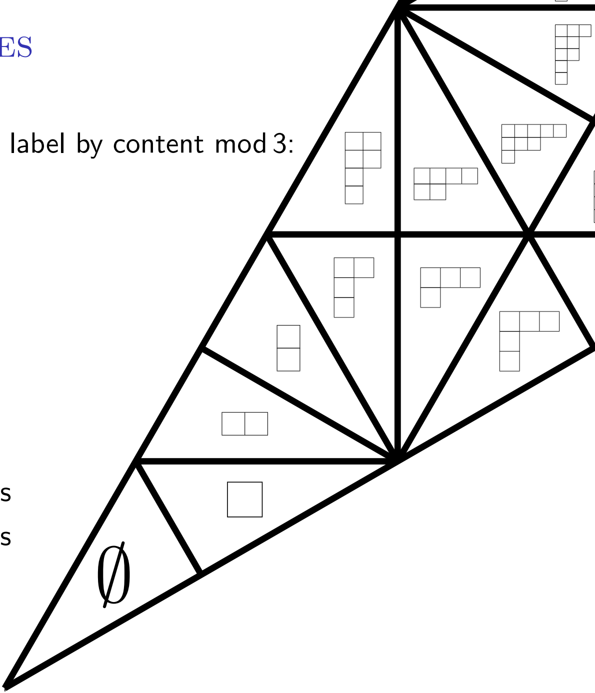
- s_0 adds/dels 0 boxes
- s_1 adds/dels 1 boxes
- s_2 *conjugates* (?!?)

Generating 3-cores

"Same thing" for $a = 3$: label by content mod 3:

0	1	2	0	1	2
2	0	1	2	0	1
1	2	0	1	2	0
0	1	2	0	1	2
2	0	1	2	0	1
1	2	0	1	2	0

- s_0 adds/dels 0 boxes
- s_1 adds/dels 1 boxes
- s_2 *conjugates* (?!?)

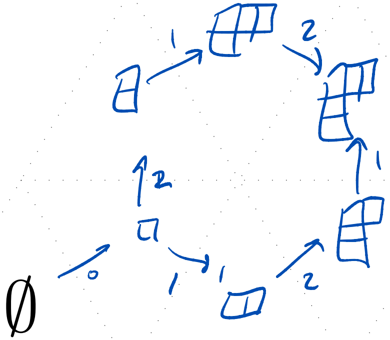


Generating a-cores

Same thing for higher a:

- label by content mod a
- s_i adds/removes all boxes with content i

0	1	2	0	1	2
2	0	1	2	0	1
1	2	0	1	2	0
0	1	2	0	1	2
2	0	1	2	0	1
1	2	0	1	2	0

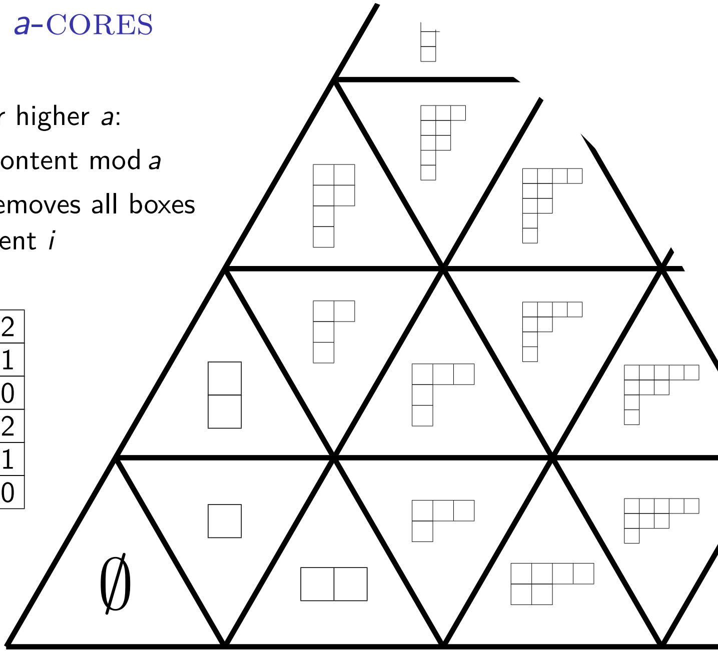


Generating a-cores

Same thing for higher a:

- label by content mod a
- s_i adds/removes all boxes with content i

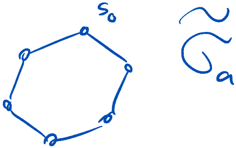
0	1	2	0	1	2
2	0	1	2	0	1
1	2	0	1	2	0
0	1	2	0	1	2
2	0	1	2	0	1
1	2	0	1	2	0

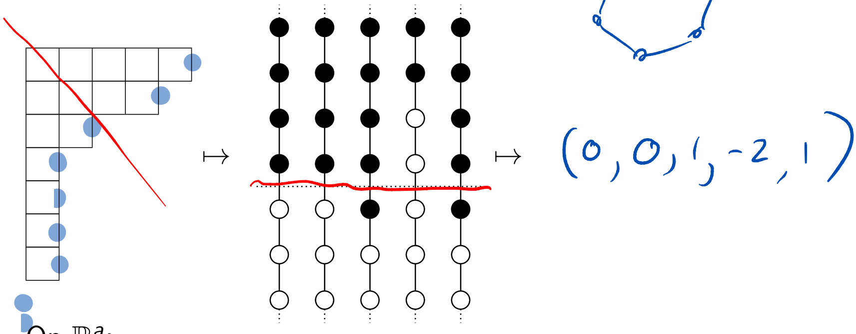


Lattice Points

a-cores are really integer points in $\mathbb{R}^a$ with zero sum ($\mathcal{Q}_a$):

- "balance" the abacus and
- record the heights of the runners.





On $\mathbb{R}^a$:

- s_i swaps the i and $(i+1)$st coordinates
- s_0 swaps the first and last coordinates (and adds $e_1 - e_a$).

Generalizing

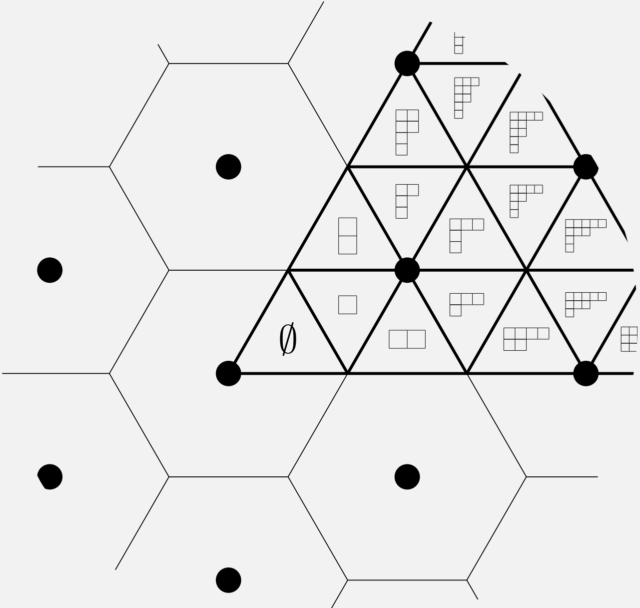
This set $\mathcal{Q}_a$ is a (co)root lattice of type A... so

- $\mathcal{Q}_a \mapsto$ coroot lattice $\mathcal{Q}^\vee$
- $\mathfrak{S}_a \mapsto$ Weyl group W
- $\widetilde{\mathfrak{S}}_a \mapsto$ affine Weyl group $\widetilde{W} = W \ltimes \mathcal{Q}^\vee = W \ltimes (\widetilde{W}/W)$.

Exercise: find combinatorial models for the action of classical $\widetilde{W}$ on $\mathcal{Q}^\vee$. (*Hint:* embed $\widetilde{W}$ into $\widetilde{\mathfrak{S}}_a$ and $\mathcal{Q}^\vee$ into $\mathcal{Q}_a$).

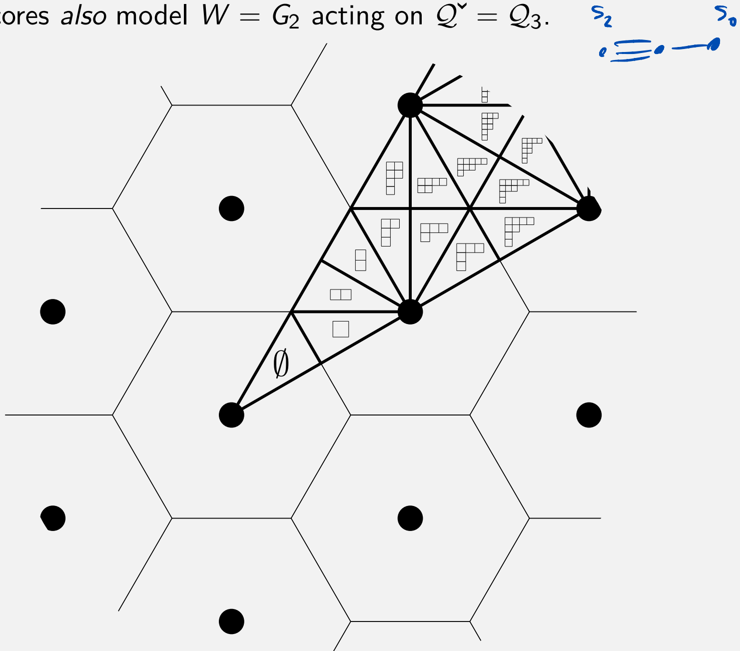
NICE CASE: TYPE A

- a-cores model $\widetilde{W} = \widetilde{\mathfrak{S}}_a$ acting on $\mathcal{Q}_a = \{q \in \mathbb{Z}^a : \sum_i q_i = 0\}$.



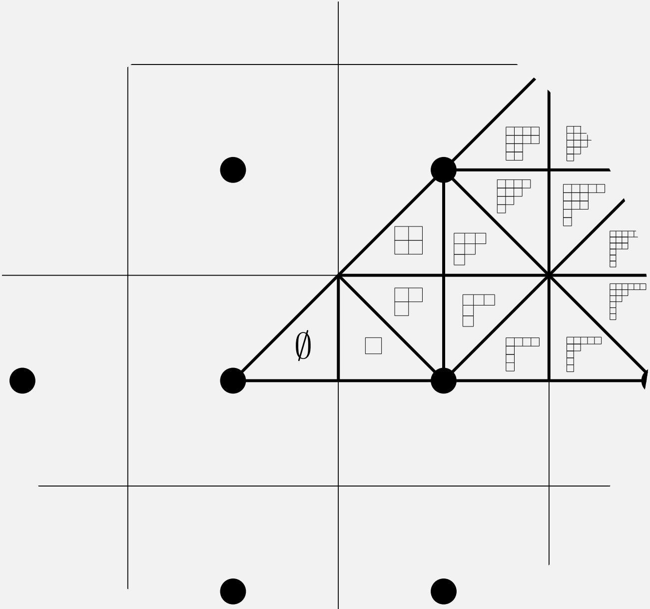
Nice case: type G_2

- 3-cores *also* model $\widetilde{W} = \widetilde{G}_2$ acting on $\mathcal{Q}^\vee = \mathcal{Q}_3$.



Nice case: type C

- Self-conjugate $2a$-cores model $\widetilde{W} = \widetilde{C}_a$ acting on $\mathcal{Q}^\vee = \mathbb{Z}^a$.



1. (Co)root lattices $\mathcal{Q}^\vee$ generalize a-cores.

2

Macdonald's identities and the size statistic

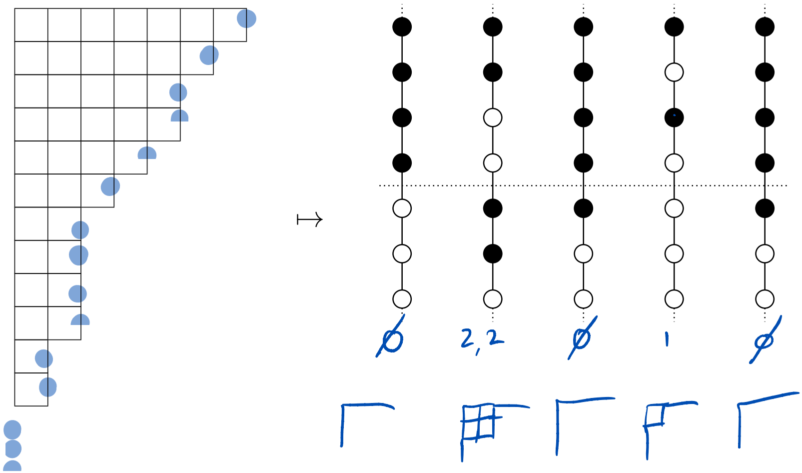
Theorem

gen. func for partitions by # boxes

g.f. a-tuple of partitions a · #boxes

g.f for a-cores by # boxes

$$\prod_{i=1}^{\infty} \frac{1}{1-x^i} = \left(\prod_{i=1}^{\infty} \frac{1}{1-x^{ai}} \right)^a \sum_{q \in \mathrm{core}(a)} x^{\mathrm{size}(q)}.$$



Macdonald's affine *denominator formula*

Theorem (I. G. Macdonald 1971, Kac and Moody)

$$\prod_{\alpha \in \widetilde{\Phi}^+} (1 - e^{-\alpha})^{mult(\alpha)} = \sum_{w \in \widetilde{W}} (-1)^{\ell(w)} e^{w(\rho)-\rho}.$$

- generalizes Weyl's denominator formula for simple Lie algebras
- explicit: imaginary roots indexed by $\mathbb{Z}$ with multiplicity n

Famous Specializations

Specializations of

$$\prod_{\alpha\in\widetilde{\Phi}^+}(1-e^{-\alpha})^{m_\alpha} = \sum_{w\in\widetilde{W}}(-1)^{\ell(w)}e^{w(\rho)-\rho}$$

for various root systems give many famous partition identities:

- Euler's pentagonal number theorem
  $(q)_\infty = \sum_{i=-\infty}^{\infty}(-1)^i q^{i(3i-1)/2}$
- $(q)_\infty^3 = \sum_{i=0}^{\infty}(-1)^i(2i+1)q^{i(i+1)/2}$
- Jacobi's triple product identity
- Dyson's identity for Ramanjuan's τ-function
  $\tau(n) = \sum \frac{(a-b)(a-c)(a-d)(a-e)(b-c)(b-d)(b-e)(c-d)(c-e)(d-e)}{1!2!3!4!}$
- $(q)_\infty^{\dim\mathfrak{g}}$ for any simple Lie algebra $\mathfrak{g}$
  (adjoint or *short adjoint*)
- . . . many more

Dyson's "Missed Opportunities"

Pursing these identities further by my pedestrian methods, I found that there exists a formula of the same degree of elegance as [Dyson's formula for Ramanujan's τ function] for the dth power η whenever d belong to the following sequence of integers:

$$d = 3, 8, 10, 14, 15, 21, 24, 26, 28, 35, 36, \ldots$$

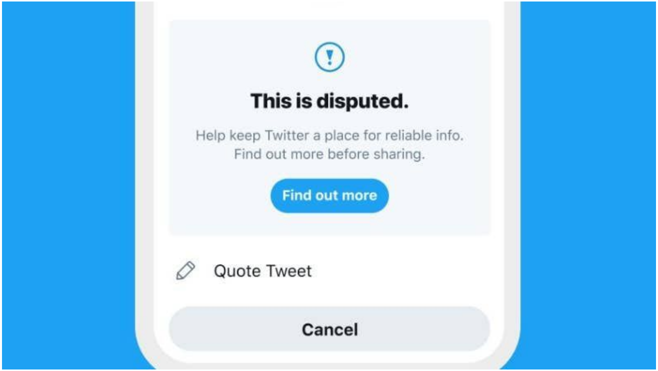
If the numbers had appeared in the context of a problem in physics, I would certainly have recognized them as the dimensions of the finite-dimenstional simple Lie algebras. Except for 26. Why 26 is there I still do not know.

— *F. Dyson "Missed Opportunities"*

Dyson's "Missed Opportunities"

This was another missed opportunity, but not a tragic one, since MacDonald cleaned up the whole subject very happily without any help from me. The only thing he did not clean up is the case $d = 26$, which remains a tantalizing mystery.

— F. Dyson "Missed Opportunities"



Dyson's "Missed Opportunities"

A more careful study of Macdonald's article reveals that the identity for the 26th power of $\eta(x)$ is not really such a mystery. It is related to the exceptional group F_4 of dimension 52, where the space of dual roots $F_4^\vee$ and the space of roots F_4 are not the same... A similar situation prevails in the case of the algebra G_2 of dimension 14... The identities for $\eta^{26}(x)$ and $\eta^7(x)$ are considerably more complicated.

— *M. Monastyrskii "Appendix to F. J. Dyson's paper 'Missed Opportunities"'*

Specializations for simply-laced type

Theorem (Macdonald) *In* **simply-laced** *type,*

$$\prod_{i=1}^{\infty} c(x^i) = \left(\prod_{i=1}^{\infty} \frac{1}{1-x^{hi}}\right)^n \sum_{q\in Q} x^{\langle \frac{h}{2}q-\rho,q\rangle}, \text{ where}$$

h is the Coxeter number,
c(x) is the characteristic polynomial of a Coxeter element.

$e_\alpha \longmapsto \omega_h^{ht(\alpha)}$

$(1,2,\dots,a)$

S_a, $h=a$, $n=a-1$, $c(x) = \frac{1-x^a}{1-x}$

$$\prod_{i=1}^{\infty} \frac{1}{1-x^i} = \left(\prod_{i=1}^{\infty} \frac{1}{1-x^{ai}}\right)^{a-1} \sum_{q\in Q} x^{\langle \frac{a}{2}q-\rho,q\rangle}$$

Non-simply-laced type

(8.16) **Theorem.** *Let R be a reduced irreducible finite root system such that $\|\alpha\| = 1$ for all $\alpha \in R$. Then*

$$\sum_{\lambda \in L(R)} X^{h^{-1}\|h\lambda+\rho\|^2} = \eta(X^h)^l \cdot X^{l/24} \prod_{n=1}^{\infty} c(X^n)$$

$$= \eta(X^h)^l \prod_{i=1}^{l} \eta(\omega_i X)$$

where $c(X)$ is the characteristic polynomial and $\omega_1, \ldots, \omega_l$ the eigenvalues of a Coxeter element of $W(R)$.

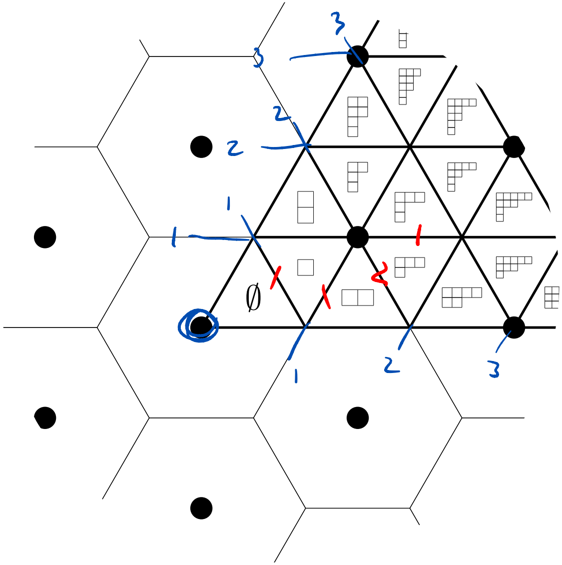
When R contains roots of different lengths, the formula corresponding to (8.16) is more complicated, and we shall not reproduce it here.

Non-simply-laced type

Theorem (Macdonald)

$$\sum_{q\in Q} x^{\left\langle \frac{h}{2}q-\rho,q\right\rangle} = \prod_{i=1}^{\infty}\left[(1-x^i)^{n_s}(1-x^{ri})^{n_\ell}\left(\prod_{\alpha\in\Phi_s}(1-x^i\omega^{\mathrm{ht}(\alpha)})\right)\left(\prod_{\alpha\in\Phi_\ell}(1-x^{ri}\omega^{\mathrm{ht}(\alpha)})\right)\right],\ \textit{where}$$

n_s/n_ℓ count the number of short/long roots,
ω is a primitive hth root of unity,
r is the ratio of the length of a long to short root,
Φ_s/Φ_ℓ are the sets of short/long roots,
$ht(\alpha)$ is the height of the root α.



For several reasons, Marko and I missed the correct definition for the statistic size in the non-simply-laced types.

1. (Co)root lattices $\mathcal{Q}^\vee$ generalize a-cores.
2. The quadratic form

$$\mathsf{size}(q) = \left\langle \frac{h}{2}q - \rho^\vee, q \right\rangle$$

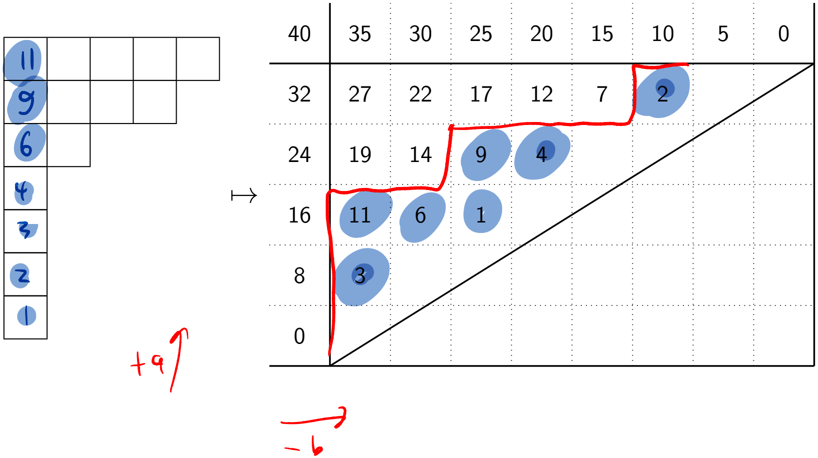
generalizes the statistic "number of boxes" on cores.

3

Simultaneous cores and the Sommers Region

Theorem (Anderson 2002)

For a, b coprime, there are $\frac{1}{a+b}\binom{a+b}{b}$ partitions that are simultaneously a-cores and b-cores.



Sommers regions

By division, write $b = t_b a + r_b$ with $0 < r < a$. The condition for $q = (q_1, \ldots, q_a) \in \mathcal{Q}_a$ to also be a b-core is

$$q_i - q_{i+r_b} \geq -t_b \text{ and } q_i - q_{i+a-r_b} \leq t_b + 1.$$

More generally, write $b = t_b h + r_b$ with $0 < r < h$.

Definition

For a root system Φ and b coprime to h, the *Sommers region* is

$$\mathcal{S}_b = \left\{ x \in V : \begin{array}{c} \langle x, \alpha \rangle \geq -t_b \text{ for } \alpha \in \Phi_{r_b}, \\ \langle x, \alpha \rangle \leq t_b + 1 \text{ for } \alpha \in \Phi_{h-r_b} \end{array} \right\}.$$

So coroot points in $\mathcal{S}_b$ are "simultaneous cores" in other types. Enumeration?

The fundamental alcove A_0

Write $\widetilde{\alpha}$ for the *highest root* of Φ. We can express $\widetilde{\alpha} = \sum_{i=1}^{n} c_i \alpha_i$.

Definition

The *fundamental alcove* has vertices $0, \frac{\omega_1^\vee}{c_1}, \cdots, \frac{\omega_n^\vee}{c_n}$.

Theorem

For b coprime to h, there is an element $w_b \in \widetilde{W}$ such that $w_b(\mathcal{S}_b) = bA_0$. In particular, $|\mathcal{Q}^\vee \cap \mathcal{S}_b| = |\mathcal{Q}^\vee \cap bA_0|$.

Counting lattice points in bA_0

Write $c = \mathrm{lcm}(c_1, \ldots, c_n)$ with $\widetilde{\alpha} = \sum_{i=1}^{n} c_i \alpha_i$.

Theorem (R. Suter 1998)

For b coprime to c,

$$|\mathcal{Q}^\vee \cap bA_0| = \frac{1}{|W|} \prod_{i=1}^{n} (b + e_i).$$

Proof.

The generating function

$$\prod_{i=0}^{n} \frac{1}{1 - x^{c_i}} = \sum_{b \in \mathbb{N}} |\Lambda^\vee \cap bA_0| \, x^b$$

counts coweights inside of bA_0. Expand case-by-case and (by coprimality) divide by the index of connection $f = |\Lambda/\mathcal{Q}|$. □

Counting lattice points in bA_0

Theorem (R. Suter 1998)

For b coprime to c,

$$|\mathcal{Q}^\vee \cap bA_0| = \frac{1}{|W|} \prod_{i=1}^{n} (b + e_i).$$

Theorem (M. Haiman 1994)

For b coprime to c,

$$|\mathcal{Q}^\vee \cap bA_0| = \frac{1}{|W|} \prod_{i=1}^{n} (b + e_i).$$

Ehrhart I

Generalizing Pick's theorem for lattice points in lattice polygons...

Theorem (E. Ehrhart 1962) *Fix*

- *A lattice* $L \simeq \mathbb{R}^n$
- *a convex polytope* $\mathcal{P}$ *with* $r\mathcal{P}$ *having vertices in* L *(*$r \in \mathbb{N}$*).*

Then the lattice point enumerator enumerator

$$\mathcal{P}^L(b) = |b\mathcal{P} \cap L|$$

is a quasipolynomial of degree n *in* b *with period dividing* r*.*

Theorem (M. Haiman 1994)

For b coprime to c,

$$\frac{1}{a}\binom{a+b-1}{b}$$

$$|\mathcal{Q}^\vee \cap bA_0| = \frac{1}{|W|}\prod_{i=1}^{n}(b+e_i).$$

$$\frac{1}{a+b}\binom{a+b}{b}$$

Proof.

(A) By Ehrhart theory, $\mathcal{Q} \cap pA_0$ is a quasipolynomial of period fa, since aA_0 has integral vertices in the coweight lattice so that faA_0 is integral in the lattice $\mathcal{Q}$.

(B) By Dirichlet's theorem on primes in arithmetic progressions, there are infinitely many primes p in any residue class $b \bmod fa$.

(C) The lattice points $\mathcal{Q} \cap pA_0$ are in bijection with W-orbits on $\mathcal{Q}/p\mathcal{Q}$. By the lemma that is not Burnside's, this can be computed as $\frac{1}{|W|}\sum_{w\in W}|\mathrm{Fix}(w|_{\mathcal{Q}/p\mathcal{Q}})|$.

□

Proof.

(D) The matrix for the reflection representation V of w in the root basis has integral coefficients and for p a sufficiently large prime has the same rank as over $\mathbb{R}$ and so $|\mathrm{Fix}(w)| = p^{\dim \mathrm{Fix}(w|_V)}$.

(E) By Shephard-Todd,

$$\frac{1}{|W|}\sum_{w\in W}|\mathrm{Fix}(w|_{\mathcal{Q}/p\mathcal{Q}})| = \frac{1}{|W|}\sum_{w\in W}p^{\dim \mathrm{Fix}(w|_V)}$$

$$= \frac{1}{|W|}\prod_{i=1}^{n}(p+e_i).$$

□

1. (Co)root lattices $\mathcal{Q}^\vee$ generalize a-cores.
2. The quadratic form

$$\text{size}(q) = \left\langle \frac{h}{2} q - \rho^\vee, q \right\rangle$$

   generalizes the statistic "number of boxes" on cores.
3. Lattice points $\mathcal{Q}^\vee \cap \mathcal{S}_b$ generalize simultaneous cores.

4

Armstrong's Conjecture and our generalization

Around 2011, D. Armstrong conjectured the following theorem.

THEOREM (P. JOHNSON 2015) *For* $\gcd(a,b)=1$,

$$\mathop{\mathbb{E}}_{\lambda\in\mathrm{core}(a,b)}(\mathrm{size}(\lambda)) = \frac{(a-1)(b-1)(a+b+1)}{24} = \mathop{\mathbb{E}}_{\substack{\lambda\in\mathrm{core}(a,b)\\ \lambda=\lambda^{\mathsf{T}}}}(\mathrm{size}(\lambda)).$$

P. Johnson gave a beautiful proof of this conjecture using a generalization of Ehrhart theory (the "Paul-ynomial" method).

Ehrhart II

Theorem *Fix*

- *A lattice* $L \simeq \mathbb{R}^n$
- *a convex polytope* $\mathcal{P}$
  with $r\mathcal{P}$ *having vertices in* L*, and*
- *a polynomial* $h : \mathbb{R}^n \to \mathbb{R}$ *of degree* d.

Then the weighted lattice point enumerator enumerator

$$\mathcal{P}_h^L(b) = \sum_{q \in b\mathcal{P} \cap L} h(x)$$

is a quasipolynomial of degree $n + d$ *in* b *with period dividing* r.

Z1

Theorem (Ekhad, Zeilberger, Johnson)

For $\gcd(a,b)=1$*, the sixth moment of size on* $\text{core}(a,b)$ *is*

$$\frac{1}{4184557977600}\,ab(b-1)(a-1)(a+b+1)(a+b)(307561a^8b^4+1230244a^7b^5+1845366a^6b^6+1230244a^5b^7$$
$$+307561a^4b^8-2056306a^8b^3-8225224a^7b^4-14394142a^6b^5-14394142a^5b^6-8225224a^4b^7-2056306a^3b^8$$
$$+5372061a^8b^2+21488244a^7b^3+42976488a^6b^4+53720610a^5b^5+42976488a^4b^6+21488244a^3b^7+5372061a^2b^8$$
$$-6453396a^8b-25813584a^7b^2-60704054a^6b^3-91764618a^5b^4-91764618a^4b^5-60704054a^3b^6-25813584a^2b^7$$
$$-6453396ab^8+2985120a^8+11940480a^7b+39743142a^6b^2+77437746a^5b^3+96285048a^4b^4+77437746a^3b^5$$
$$+39743142a^2b^6+11940480ab^7+2985120b^8-11104272a^6b-33312816a^5b^2-55521360a^4b^3-55521360a^3b^4$$
$$-33312816a^2b^5-11104272ab^6+2985120a^6+8955360a^5b+23840061a^4b^2+32754522a^3b^3+23840061a^2b^4$$
$$+8955360ab^5+2985120b^6-9109476a^4b-18218952a^3b^2-18218952a^2b^3-9109476ab^4+2985120a^4+5970240a^3b$$
$$+8955360a^2b^2+5970240ab^3+2985120b^4+8664840a^2b+8664840ab^2-62687520a^2-62687520ab-62687520b^2$$
$$+626875200).$$

Armstronger

Theorem (E. Stucky, M. Thiel, W.)

For X_n an irreducible rank n Cartan type with root system Φ, and b coprime to h

$$\mathop{\mathbb{E}}_{\lambda \in \mathrm{core}(X_n,b)}(\mathrm{size}(\lambda)) = \frac{rg^\vee}{h} \frac{n(b-1)(h+b+1)}{24}, \text{ where}$$

h is the Coxeter number of X,
$g^\vee$ is the dual Coxeter number for $\Phi^\vee$,
r is the ratio of the length of a long to short root.

The factor $\frac{rg^\vee}{h}$ is 1 in simply-laced type: $g^\vee = h$ and $r = 1$.

Special cases

$\mathfrak{S}_a$: a-cores, $n = a - 1$, $h = g^\vee = a$, $r = 1$.

$$\frac{rg^\vee}{h}\,\frac{n(b-1)(h+b+1)}{24} = \frac{(a-1)(b-1)(a+b+1)}{24}$$

For a even, $C_{a/2}$: self-conjugate a-cores, $n = a/2$, $h = a$, $g^\vee = a - 1$, $r = 2$.

$$\frac{rg^\vee}{h}\,\frac{n(b-1)(h+b+1)}{24} = \frac{2(a-1)}{a}\,\frac{\frac{a}{2}(b-1)(a+b+1)}{24}$$

Proof strategy

1. Work with coweights $\Lambda^\vee$ rather than coroots $\mathcal{Q}^\vee$:
   quadratic forms invariant under $W \subset O(V)$
   all $\Lambda^\vee/\mathcal{Q}^\vee$-orbits are free since b coprime to h
   divide at the end by $f = |\Lambda^\vee/\mathcal{Q}^\vee|$
2. Reduce problem from $\mathcal{S}_b$ to bA_0:
   multiplication by a particular element of $\widetilde{W}$
   translate size statistic ("remove" dependence on b!)
3. Conclude quasipolynomiality by Ehrhart theory II.
4. Find zeros!
   use Ehrhart reciprocity: "small" dilations
   of the fundamental alcove contain no interior lattice points.

1. (Co)root lattices $\mathcal{Q}^\vee$ generalize a-cores.
2. The quadratic form

$$\mathrm{size}(q) = \left\langle \frac{h}{2} q - \rho^\vee, q \right\rangle$$

   generalizes the statistic "number of boxes" on cores.
3. Lattice points $\mathcal{Q}^\vee \cap \mathcal{S}_b$ generalize simultaneous cores.
4.
$$\underset{\lambda \in \mathrm{core}(X_n, b)}{\mathbb{E}} (\mathrm{size}(\lambda)) = \frac{rg^\vee}{h} \frac{n(b-1)(h+b+1)}{24}$$

   generalizes Armstrong's conjecture for expected size.

One more thing: "strange"

$$\mathop{\mathbb{E}}_{\lambda \in \mathrm{core}(X_n, b)}(\mathrm{size}(\lambda)) = \frac{rg^\vee}{h} \frac{n(b-1)(h+b+1)}{24}$$

With translations $\mathcal{S}_b \leftrightarrow bA_0$, the value of size at 0 is given by

$$-\frac{1}{2h}\langle \rho^\vee, \rho^\vee \rangle = -\frac{rg^\vee}{h} \cdot \frac{n(h+1)}{24},$$

equivalent to the *strange formula* of Freudenthal and de Vries.

Thank you!

Future work

- finite type.
- twisted affine type.
- ...