Optimizing Service Operations with Price- and Density-Dependent Demand: A Copula-Based Approach

Andrew E. Frazelle, Toghrul Rasulov, Shouqiang Wang
Jindal School of Management, University of Texas at Dallas, Richardson, TX 75080
andrew.frazelle@utdallas.edu, toghrul.rasulov@utdallas.edu, shouqiang.wang@utdallas.edu

We study a service provider's pricing and density decisions when customers are heterogeneous both in their valuation and in their sensitivity to crowd *density*, the latter originating from, e.g., safety concerns in a pandemic or a desire for privacy and exclusivity. We develop a novel *copula*-based framework to model such multidimensional preferences and their dependence structure. We analytically characterize the provider's optimal price and density limit. For all but severely density-sensitive customer populations, (i) the provider optimally serves all segments of the market and activates the price (rather than the density) to regulate the demand and (ii) as customers' valuations and density tolerances become more positively dependent, the provider earns higher revenue and optimally increases its service density. By contrast, the optimal price may be decreasing, increasing, or decreasing followed by increasing. On the other hand, with severely density-sensitive customers, it may be optimal to partially cover the market and to activate density to regulate the demand. Finally, a calibrated numerical study based on a choice experiment on train travel preferences during the COVID-19 pandemic validates our model assumption and results. Our findings offer prescriptive guidelines for service providers in the presence of density-sensitive demand (especially in the aftermath of a pandemic). We also put forth an explanation for the stringent gatekeeping practices of exclusive private clubs, and suggest a new lever to fight the ongoing inflation in the service industry, namely by reshaping the dependence between density sensitivity and valuation. Notably, service providers' strategy to regain customers after the pandemic could conflict with government efforts to fight inflation.

Key words: service operations, revenue management, copulas, pandemic, exclusivity/privacy
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1. Introduction

When the onset of the COVID-19 pandemic ground the service industry in the United States (and around the world) to a nearly complete halt in early 2020, movie theaters were shuttered and dine-in service at restaurants was banned (Rubin and Maddaus 2020), passenger volume on U.S. commercial airlines dropped by more than 95% (BTS 2020), and ridership on public transport reduced to near zero (Kaufman et al. 2020). Upon reopening, service providers did and continue to wrestle with a vexing problem: how to successfully operate a service business with customers who become notably sensitive to crowd *density*, i.e., the maximal number of customers allowable in a service setting. Even though dine-in service in most U.S. states returned by summer 2020 (Sontag 2020), aggregate restaurant sales still decreased from 2019 to 2020 by an estimated $240 billion (Daniels 2021). Similarly, a year later in 2021, box office revenues for U.S. movie theaters were still less than half of 2019 levels (McClintock

2022). Even in 2022, two years after the COVID-19 outbreak, the Chinese government had been capping the capacity of all inbound flights at 75% (Koenig 2022).

Different businesses have taken very different approaches. For instance, of the major U.S. air carriers, only Delta Air Lines continued to block middle seats throughout its planes after January 2021 (Honig 2021), continuing the policy until May 1, 2021 (Pallini 2021). In announcing one of the extensions to the policy, a spokesperson for the company said "We want our customers to have complete confidence when traveling with Delta, and they continue to tell us that more space provides more peace of mind" (Delta 2021). While by January 2021, other carriers preferred to fly full planes when possible, Delta judged that its customers continued to be sensitive enough to safety and social distancing that it was in the airline's best interest to forgo revenue from a third of the seats in economy class, despite incurring almost the same operating cost as it would for a full plane.[1]

Service providers like Delta Air Lines must now assess their customers' sensitivity to the crowd density, namely their willingness to *accept* the provider's announced density cap, in addition to their willingness to *pay* (WTP) for the service. That is, the service provider must choose at what density cap to operate in addition to the price decision. For instance, movie theaters upon reopening had to carefully consider customers' density preferences. Fandango, a movie-ticket retailer, has informed its customers that when choosing seats, "You can see which seats are being blocked out for social-distancing purposes," and it also reminded them that "Each theater may have different policies." Maya Cinemas assured customers that that not only would it follow government guidelines, but it would also "[take] many additional steps to ensure your safety." Describing seating density, they stated that "Maya Cinemas utilizes dynamic social distancing automatically applying seat gaps between parties, as well as row spacing to ensure front and back spacing at locations that are not already spaced by design" (Fandango 2022).

Moreover, different customer groups likely also differ in their WTP, suggesting a non-trivial relation between the WTP and the sensitivity to crowd density. For example, "younger moviegoers between ages 18 and 34—and especially younger males—have fueled the recovery," but "families and older moviegoers have been more reluctant to return to cinemas" (McClintock 2022). Such correlation between the WTP and the sensitivity to the provider's announced density limit can be a major consideration for service providers when making their pricing and other operational decisions. To illustrate, consider Alaska Airlines, which adopted an unusual policy during the COVID-19 pandemic. It blocked middle seats in Premium Class—which has extra legroom and can be purchased for an extra fee—but not in the rest of the plane (Griff 2021). Alaska Airlines seems to have judged that

[1] Passenger service cost and reservations/sales costs represent only a combined 13% of the operating cost of a flight, so even a lofty estimate of the savings from serving 1/3 fewer passengers on each plane would be less than 5%; see FAA (2022, Table 4-1).

customers in different cabins have different density preferences, on top of their differences in WTP. Specifically, since it blocked middle seats only in Premium Class, Alaska may have determined that the passengers willing to pay for Premium Class tended to be more sensitive to the crowd density limit and that their demand would be more elastic to a change in the middle-seat-blocking policy than would demand for Main Cabin seats.

Customer density sensitivity arises beyond just safety concerns. Hillstone Restaurant Group, which operates upscale eateries around the U.S., caps the crowd density in its restaurants even in 2023 after all COVID restrictions have been lifted. Although its tables can fit as many as four diners (Hillstone Restaurant Group 2023), Hillstone avoids serving such parties, informing customers that "We prefer to host experiences based on smaller, more intimate parties of two. If a reservation is not available for three or more guests, we may not be able to accommodate you. Walk-ins are welcome for parties of two" (see, e.g., Hillstone 2023a,b). In this case, Hillstone aims to provide an "intimate" experience for discerning diners by strategically imposing a density limit. Furthermore, numerous service operations exist for which privacy and exclusivity are key selling points of the offering, some even more so than for Hillstone Restaurants. To create an image of unique exclusivity, such service providers must consider capping the crowd density to limit the general public's access to their service. Prominent examples abound and include private social clubs like Soho House (Burton 2017) and member-exclusive country clubs like the (private ski resort) Yellowstone Club (Tong 2022) and Augusta National Golf Club (Buteau and Paskin 2015). With high-profile members such as Bill Gates and Warren Buffett at Yellowstone Club and Augusta National, and celebrities such as Jessica Biel and Leonardo DiCaprio at Soho House (Ginsberg 2010), privacy and exclusive access to service are naturally of paramount importance for these clubs to maintain. Accordingly, the membership in such clubs is extremely restrictive. For instance, Yellowstone Club states that it is "limited to 864 residential properties to protect exclusivity and exceptional, highly personalized service" (Yellowstone Club 2016). Based on the density cap set by the service provider, prospective members form their perception about the service's exclusivity and decide whether they wish to join (if slots are still available, that is).

Motivated by these practical settings, our main research question asks how a profit-maximizing service provider should price and set its density cap in the presence of customers' heterogenous density *tolerance* (i.e., their maximum acceptable density) and, in particular, its *dependence* with their WTP. In the above examples, the density tolerance is likely to be "rigid" for each customer in that it is a hard threshold above which she is unwilling to accept the service. If an individual would not feel safe boarding a plane, then she is unlikely to become convinced to buy a ticket merely because the price is reduced without reducing the density cap. Indeed, despite inflation-adjusted U.S. domestic airfares being the lowest ever recorded in 2020 (almost 20% lower than the closest other year), and even though the numbers include January and February 2020 (which operated relatively normally before the outbreak began affecting the U.S.), total U.S. air passenger volume still plummeted from 331 million in 2019 to only 131 million in 2020 (BTS 2021). For train travel, Shelat et al. (2022b) finds that "travellers are significantly more likely to choose routes with less COVID-19 risk (e.g., due to crowding)," underscoring the critical role of density in their preferences. Similarly, extremely low density is the ultimate determinant for celebrities or jet-setters, who value privacy and exclusivity the most, to join elite private social or country clubs. Referring in a New York Times interview to the celebrity membership of his ultra-exclusive San Vicente Bungalows (SVB) in Los Angeles—which includes, e.g., Taylor Swift and Elon Musk (Kay 2023)—Jeffrey Klein notes that "Privacy is the new luxury" and describes the club as "an oasis" (Trebay 2019). The article observes that "Mr. Klein understood better than most that the one craving that famous people cannot satisfy is to get their anonymity back," and SVB allows them to do so. Maintaining this aura of pampered privacy is critical to this and similar clubs' success, and if a privacy-seeking celebrity member resigns from SVB because management increases the density cap, then a reduced price will likely be completely ineffective at wooing her back. In other words, an individual customer deems WTP and density tolerance as non-substitutable: each customer will make the purchase if and only if both (i) the price is below her valuation *and* (ii) the density cap is below her tolerance. Demand in this context may not be adequately modeled in a conventional way, whereby the service provider can lower the price to induce customers to accept the service at a higher density.

To model such preference rigidity and its dependence with WTP, we leverage the notion of the *copula* to develop a novel framework for such multidimensional preferences. The copula is a useful tool to capture a wide variety of dependence structures by linking the marginal distributions for customer valuation and density tolerance to their joint distribution without needing to explicitly specify conditional distributions. The marginal distributions capture the customer heterogeneity along these two dimensions. To permit a parsimonious and tractable model of dependence, we use the two-parameter Fréchet family of copulas that decompose the demand into a mixture of three polar dependence structures: both perfect positive and negative dependence as well as independence. As the proportions of these three clusters change, a continuum of dependence structures is covered. In this context, with multidimensional preferences and dependent valuation and density tolerance, we formulate and solve the provider's revenue maximization problem over price and density.

For a wide range of distributional parameters that excludes only the most severely density-sensitive customer populations, we provide the complete characterization of the provider's optimal strategy, which always takes a "full-coverage, price-active" structure. A "full-coverage" strategy induces positive demand from all three clusters of polar dependence structure (in particular, from the cluster of perfect negative dependence); under a "price-active" strategy, the demand from the cluster of perfect positive dependence is determined only by the price and not by the density. In other words, for those lucrative customers who have both high valuation and high density tolerance, it is better for the provider to price some of them out (with a high price) rather than to crowd some out (with a high density).

Furthermore, we find that as the two preference dimensions become more positively dependent, both the optimal density and revenue increase, but not necessarily the optimal price, which can be increasing, decreasing, or first decreasing followed by increasing. We characterize the parametric condition for these distinct price movements, which we show is driven by the provider's intention to maintain the *balance* between the demand and the density. In particular, the optimal price is shown to be higher than the price that the provider should charge when customers are insensitive to the service density (i.e., have only one-dimensional preference of WTP as in the classical model). The positive dependence between the customer valuation and density tolerance exerts a *direct positive* and an *indirect negative* effect on the optimal price. For a given price-density pair, more positive dependence between these two dimensions enhances the demand, calling for a *higher* price to rebalance the demand with the density if the latter is fixed. Besides this direct positive effect, however, more positive dependence also incentivizes the provider to increase the density cap, lowering the demand if the price remains unchanged. Again, to rebalance the demand with the density, the provider now has the incentive to *lower* the price, resulting in an indirect negative effect. When customers' density tolerance is more evenly distributed (e.g., follows a uniform distribution), the direct positive effect dominates the indirect negative effect, leading to an increasing optimal price as the two preference dimensions become more positively dependent. On the other hand, when customers' density tolerance is sufficiently skewed towards the higher level, the indirect negative effect is dominant and hence the monotonicity of the optimal price is reversed. Naturally, the optimal price exhibits non-monotone behavior in between these two regimes.

Additionally, we characterize the optimal policy for the remaining range of distributional parameters, which reflects a severely density-sensitive population. In this regime, the full-coverage, price-active strategy is not always optimal. Instead, it can be optimal to only partially cover the market without serving the cluster with negative dependence, and/or to use density instead of price to regulate the demand from the cluster with positive dependence. In particular, partial coverage is optimal only when customers are extremely density-sensitive. Under either full or partial coverage, density (resp., price) should be the active lever when customers are highly (resp., relatively less) sensitive to the density cap. Finally, we leverage a real dataset to numerically validate our distributional assumptions, calibrate the model parameters, and subsequently evaluate the provider's optimal strategy.

Managerially, we provide guidelines for service providers on how to set the price and density cap when density-sensitive customers have dependent valuations and density tolerances (e.g., in the aftermath of a pandemic). Our results suggest that the more positively dependent the valuation and density tolerance are, the better for service providers in that their revenues will be higher. They can also afford to operate at higher densities in this case as there are more customers with both high willingness to pay and high density tolerance. On the other hand, for exclusivity or privacy-based service businesses, we conjecture that customers like celebrities or other wealthy individuals with high willingness to pay for a service are likely to have especially low density tolerances. This would imply a strong negative dependence between valuation and density tolerance. Accordingly, our results suggest that the operator of an exclusive country club or social club will optimally operate at a very low density. This is consistent with the well-known gatekeeping strategy of private clubs in practice. Yellowstone Club has an explicit cap of 864 resident members, as mentioned above; Augusta National, despite boasting a world-class golf course and massive financial resources earned from the annual Masters Tournament, reportedly has only around 300 members (Buteau and Paskin 2015); and similarly, Soho House carefully manages and, when necessary, even purges its membership (Burton 2017), resulting in a list of more than 30,000 outstanding applicants at one point (Smith 2015).

In contrast with the classical pricing theory of a single-dimensional preference, our results call for attention to the customers' density tolerance distribution and its dependence with valuation when optimizing the price. In the pandemic's aftermath, rampant inflation has become the most urgent economic issue for central banks in the U.S. and worldwide to combat (Wallace 2022, Derby 2023). Monetary policy tools wielded by central banks, in particular interest rate hikes, seem to be less effective in containing the inflation in the service sector even as inflation for goods prices has tapered off (Mutikani and Nomiyama 2023, Anstey 2023). From an operational perspective, our findings identify other potential levers to address this issue. As discussed above, density-sensitive customers tend to drive prices higher than when customers are insensitive, and the correlation of customers' density sensitivity with their service valuation may even further exacerbate such inflation. Thus, a new angle to fight price inflation in the service sector is to reshape customers' density sensitivity. For policymakers, this can be implemented in a variety of ways, perhaps most simply by communicating with the population to ease the public fear. Indeed, the U.S. government ended the national state of emergency on April 11, 2023, earlier than scheduled (Associated Press 2023). Furthermore, our findings also advocate exploring and influencing the dependence between WTP and density tolerance, for example, by tailoring messages and mitigation measures to target different sub-populations. For instance, "families and older moviegoers have been more reluctant to return to cinemas" (McClintock 2022); these groups tend to be more economically affluent and hence may have higher WTP (Pino 2022). Cinemas would then have incentives to aim their messages and safety measures at these demographics to increase the positive dependence between density tolerance and WTP, because doing so always helps boost their revenue. However, our results caution that such strategies may not necessarily lead to lower prices. Thus, to ease the price pressure, public agencies have to fine-tune the positive dependence between density tolerance and WTP, as suggested by our results.

2. Literature Review

As elaborated below, our work speaks and contributes to multiple research streams ranging from the pandemic-motivated operations to the applications of copulas in operations management.

Pandemic Operations. The recent atrocity of the Covid-19 outbreak has spurred fast-growing research efforts among the operations management community to study policy issues in containing the pandemic spread (see Gupta et al. 2022, for a comprehensive review). These efforts range from evaluating different social distancing measures (Bai et al. 2022), testing optimization (Yang et al. 2022), vaccination policies (Nageswaran 2022), capacity management of medical equipment (Jain and Rayal 2022), to the design of public messages (de Véricourt et al. 2021). In a spirit similar to ours, Hassin et al. (2023) derives equilibrium queue-joining strategies in a queueing system where infection-averse customers are sensitive to their waiting time as well as the number of other customers encountered and cumulative exposure to them. However, they do not consider the service provider's decision problem nor the consumers' rigid preferences. Zhong et al. (2021) studies the effect of COVID-19 vaccination progress on the demand for public transportation, finding that increases in vaccination rates lead to higher demand for public transportation. Their finding offers an empirical ground for a key premise of our study, namely customers' sensitivity to service density in situations such the pandemic. Such an empirical phenomenon is also observed in Shelat et al. (2022b), which we use for our calibrated numerical study in Section 6. However, optimization of private-sector operations—especially those that are not directly healthcare-related—in response to pandemic conditions has received minimal attention. One exception is the recent work by Tang et al. (2023), which investigates retailers' (time-based) pricing decisions to smooth demand in a duopoly setting with homogenous congestion-averse consumers. We differ from their study by also including the density decision and specifically considering consumers' heterogeneity and correlation in both WTP and sensitivity to the crowdedness, albeit in a monopolistic setting.

Service Management. As elaborated in Section 1, density sensitivity can also arise beyond pandemic-induced safety concerns, as exemplified by luxury services. In economics, there is a small stream of literature on the theory of clubs (e.g., swimming pools) initiated by Buchanan (1965) and continued by Scotchmer (1985) and others (see Sandler and Tschirhart 1997 for a survey). This literature examines a competitive setting where customers are typically homogeneous and their utility depends additionally on the service's congestion level. In contrast, we model customer heterogeneity both in their WTP and their sensitivity to density, and explore their dependence structure.

Operations literature has a long history of studying queuing-based service systems, where two opposite types of customer behavior are examined depending on the application. Dating back to Naor (1969), the majority of studies model delay-averse customers, who exhibit avoid-the-crowd behavior and hence prefer to join services with shorter queue length (see Hassin and Haviv 2003, Hassin 2016, for a comprehensive review). More recently, researchers (Debo et al. 2012, 2020, Guo et al. 2023) find that in the presence of information asymmetry, customers may exhibit the opposite, follow-the-crowd behavior and hence prefer to join longer queues. The underlying premise is that queue length can signal the unobservable service quality. Although the delay sensitivity in the former stream of research seems to resemble our notion of density sensitivity, two important distinctions exist. First, the delay sensitivity in queuing-based models refers to customers' waiting *time*, while our density sensitivity refers to the *simultaneous* crowdedness present in a service setting. Second, preferences for time and money are substitutable in the delay-sensitive models by monetize the waiting time as a cost, whereas our model explicitly separates the WTP and density sensitivity in a non-substitutable way and treats them as two heterogeneous preference dimension with a rich dependence structure.

Product Line Design. Multidimensional preferences, though understudied in service settings, have been featured prominently in the literature of product line design, whereby products exhibit multiple quality traits. Representative works include Chen (2001), Lacourbe et al. (2009), Lauga and Ofek (2011), Liu and Shuai (2019), Lin et al. (2020), and Rashkova and Dong (2022). Common in this stream of research is the *substitutability* between different quality dimensions in that consumers' utility along one quality dimension can be compensated by another dimension. Furthermore, the interdependence between different heterogeneous preference dimensions has not yet been explored. By contrast, we consider a customer's "rigid" preferences, whereby a purchase is made only if *both* dimensions of her preferences, namely the price and density, are satisfied. In addition, we investigate the effects of their dependence structure on the service provider's optimal strategies.

Copulas. To model such dependence structure between heterogeneous preferences, we leverage the notion of copulas (see Nelsen 2007 for a general introduction), whose applications in operations management seem to be limited, apart from a few notable studies. Clemen and Reilly (1999) studies the use of copulas to model dependence among variables in a decision analysis context and illustrates their use in conducting risk analysis for an airline. Related, Bagchi and Paul (2014) applies copulas to the optimal allocation of screening resources in airport security. Motivated by practical concerns in agribusiness (Bansal et al. 2017), Bansal and Wang (2019) uses copulas to approximate joint distributions for the uncertainty in subjective expert judgments of the means and standard deviations of random variables. From a decision-theoretic perspective, Abbas (2009) proposes the concept of *utility copulas* (as inspired by the *Sklar copulas* from multivariate statistics) to build a general framework to model preferences over dependent attributes. Our framework is based on the classical notion of Sklar copulas as we intend to incorporate preference heterogeneity and its dependence structure. In addition, our focus is on the service provider's decision problem rather than consumers'.

3. Model

We consider a market with a monopolistic service provider (e.g., a restaurant or movie theater, or a private social or country club, serving a specific area). The provider must determine a price $p \geq 0$ and a density level $c \geq 0$, where c specifies the number of customers the provider can serve simultaneously. Motivated by the service contexts discussed in the introduction, we consider a setting where the provider has already incurred the fixed cost associated with offering up to an established maximum density, and hence the provider can choose any density c up to the established maximum without incurring any additional cost. As discussed in the introduction, the cost of operating a given commercial flight is not very sensitive to whether the middle seats are occupied, and also the cost of operating a single screening at a movie theater is minimally affected by the percentage of seats that are filled. Similarly, the cost of upkeep of an existing golf course is relatively inelastic to the number of members playing the course. Without loss of generality, we normalize both the provider's maximum possible price and maximum possible density to one. For a specified density and price, the actual units sold (the *sales*) is the minimum of the density and the demand at that density and price.

The customer population consists of a continuum, which is normalized to a unit mass as the maximum density has been normalized to 1. First, each customer has a *valuation* for the service, represented by a random variable V. The valuation is essentially the maximum price that a customer is willing to pay for the service; we normalize the valuation V to be between 0 and 1. Second, each customer possesses a certain *density tolerance*, namely the density at the service facility below which she is willing to accept the service and above which she will decline the service. In an airplane or movie theater with density-sensitive customers who fear for their safety, a customer's density tolerance would reflect the maximum proportion of seats occupied for which the customer would be willing to board the plane or sit in the theater. On the other hand, for a private social or country club, the density tolerance would be the number of members above which a prospective member deems the service to be not exclusive or private enough and will decline to join. We denote this quantity by another random variable S and also normalize it to be between 0 and 1 as the maximum density has been normalized to 1. In short, valuation determines whether the customer is willing to *pay* the service price set by the provider, but density tolerance determines whether the customer is willing to *accept* the service at the provider's chosen density.

As a novel feature in our setting, customers demonstrate *rigid* preferences in that they will purchase the service if and only if the price does not exceed their valuation of the service (i.e., $V \geq p$) *and* the provider's maximum density does not exceed their tolerance (i.e., $S \geq c$). This specification captures the setting where customers' purchase decisions are primarily driven by the provider's announced density limit rather than the realized density, i.e., the actual number of customers who decide to purchase the service. The realized density is difficult for customers to observe or infer prior to consuming the service due to the lack of information. This is particularly apt for exclusivity-driven services where perceived exclusivity is indeed created by the provider's announced density cap (Yellowstone Club 2016); for a pandemic setting, it reflects customers' conservatism and their most pessimistic estimate of the realized density. Accordingly, the demand is given by

$$D(p,c) = \mathsf{P}(V \geq p, S \geq c). \tag{1}$$

Even though price and density appear to operate similarly from the customer's point of view, there is an important difference from the provider's perspective. To wit, the price enters the objective function of the firm by itself as well as through demand, but the density enters into the firm's objective only through the demand. The effects on the firm of using density versus price to regulate the demand are thus different, and we will show that which one is used in the optimal strategy depends on the density tolerance distribution.

Model Dependence Structure via Copula. The above-mentioned characteristics may not be independent of each other. For instance, customers with higher willingness to pay may tend to have lower density tolerance, or vice versa. We are interested in the role that the dependence structure between V and S plays in determining the provider's price and density. To model the wide range of possible dependence structures, we use copulas. By Sklar's theorem (Nelsen 2007, Theorem 2.10.9), *any* joint probability distribution can be represented by a copula, which is a tool to model a multivariate distribution that cleanly separates the marginal distributions from the dependence structure. The copula offers a general, flexible approach to model dependence among random variables. Specifically, in our setting with two random variables, a copula can be defined as follows. Let F and G denote the continuous, strictly increasing marginal cumulative distribution functions (CDFs) of customers' valuation V and density tolerance S, respectively. A copula in two variables is defined as a bivariate cumulative distribution function (CDF) $C(u_1, u_2)$ whose marginal distributions are both standard uniform. A copula C induces a joint distribution for V and S through

$$\mathsf{P}(V \leq v, S \leq s) = C(F(v), G(s)).$$

Clearly, the marginal distributions of the joint distribution defined above coincide with F and G. Moreover, as noted, Sklar's theorem implies that any bivariate joint distribution can be represented by the above equation with an appropriately chosen copula C.

An important property of copulas, and one that motivates our model, is the Fréchet-Hoeffding bounds. By this property, *any* copula C with continuous marginal distributions $F(p)$ and $G(c)$ is bounded between two extremes of dependence. One represents perfect positive dependence given by

$$M(F(v), G(s)) = \min\{F(v), G(s)\},$$

and the other represents perfect negative dependence given by

$$W(F(v),G(s)) = \max\{F(v)+G(s)-1,0\}.$$

In other words, any copula is bounded between perfect negative dependence and perfect positive dependence, i.e., $W(F(v),G(s)) \le C(F(v),G(s)) \le M(F(v),G(s))$ (Nelsen 2007, Theorem 2.2.3). If the copula of V and S is M (perfect positive dependence), then we have $\mathsf{P}(F(V)=G(S))=1$, that is, V almost surely increases in S. If the joint distribution is the copula W (perfect negative dependence), then $\mathsf{P}(F(V)+G(S)=1)=1$, that is, V almost surely decreases in S (Nelsen 2007, Section 2.5). We also define the product copula that represents independence, given by $\Pi(F(v),G(s)) = F(v)G(s)$.

Various families of copulas exist to parameterize dependence structures. In this paper, we adopt the two-parameter Fréchet family (Fréchet 1958), which is based on the three copulas $M(F(v),G(s))$, $W(F(v),G(s))$, and $\Pi(F(v),G(s))$ defined above. We adopt this family because it is (i) comprehensive in the sense of Nelsen (2007, Section 2.2) and (ii) easy to interpret by decomposing the dependence structure into three polar modes: perfect positive dependence, perfect negative dependence, and independence. Formally, and using different (but equivalent) notation from Nelsen (2007), the Fréchet copula in our setting is given by

$$C(F(v),G(s)) = \alpha M(F(v),G(s)) + (1-\lambda)\Pi(F(v),G(s)) + (\lambda-\alpha)W(F(v),G(s)), \tag{2}$$

for $0<\alpha<\lambda<1$ (equality can also be accommodated in this relation, but for ease of exposition we consider the strictly ordered case). Intuitively, the copula in (2) is a convex combination of three polar dependence relationships, namely perfect positive dependence, perfect negative dependence, and independence. It thus covers a wide range of dependence scenarios with only two parameters, including each of the three "vertices" corresponding to independence and the two extremes of dependence. In particular, given λ, an increase in the parameter $\alpha \in (0,\lambda)$ increases the weight on positive dependence and decreases the weight on negative dependence; thus, a higher (resp., lower) value of α implies that customer valuation and density tolerance are more positively (negatively) dependent.

For valuations and density tolerances distributed according to the copula C defined in (2), we can apply the inclusion-exclusion principle to express the demand function (1) as

$$\begin{aligned} D(p,c) = \mathsf{P}(V \ge p, S \ge c) &= \alpha \min\{1-F(p), 1-G(c)\} + (1-\lambda)(1-F(p))(1-G(c)) \\ &\quad + (\lambda-\alpha)\max\{1-F(p)-G(c),0\}. \end{aligned} \tag{3}$$

With this parameterization, the customer population can be visualized *as if* it consists of three clusters: α proportion of customers exhibit perfect positive dependence between valuation and density tolerance, $1-\lambda$ proportion of customers have independent valuation and density tolerance, and $\lambda-\alpha$ proportion of customers exhibit perfect negative dependence between these two preference dimensions. However, we emphasize that these "sub-populations" are used only for interpretation purpose: customers do not necessarily come from these three literal sub-populations.

Provider's Problem. Given the provider's decision (p, c) and the induced demand $D(p, c)$ given in (3), the actual service units sold is equal to $\min\{c, D(p, c)\}$. The provider's revenue is then given by

$$R(p,c) = p\min\{D(p,c), c\}, \tag{4}$$

and its optimization problem is

$$r^* = \max_{p,c\in[0,1]} \quad R(p,c). \tag{5}$$

We denote the service provider's optimal price and density by p^* and c^*. To see the impact of density sensitivity on the provider's optimal decisions, we also consider a benchmark case for comparison. In the benchmark case, customers are completely insensitive to the density level (i.e., the marginal distribution G becomes degenerate with a probability mass concentrated at 1), so the demand only depends on the price and reduces to $D^\circ(p) = 1 - F(p)$. In this case, for a given price p, any density that falls between $1 - F(p)$ and 1 is optimal for the provider. Without loss of optimality, we simply set the density equal to the actual demand, i.e., $1 - F(p)$. Therefore, the provider's benchmark problem is

$$\max_{p\in[0,1]} R^\circ(p) = p(1 - F(p)). \tag{6}$$

We denote the optimal price in the benchmark problem (6) by p°. The price p° implies an optimal density equal to $c^\circ = 1 - F(p^\circ)$ in the benchmark problem.

3.1. Preliminary Result

We now establish a crucial property of the provider's density decision and the induced demand, which allows us to further simplify the provider's problem. The following lemma demonstrates that it is optimal for the provider to induce demand equal to its chosen density.

LEMMA 1 **(Balance Density and Demand).** *(i) For each price $p \in [0,1]$, there exists a unique density $c(p)$ such that $c(p) = D(p, c(p))$. (ii) For each density $c \in [0,1]$, there exists a unique price $p(c)$ such that $c = D(p(c), c)$. (iii) The provider's optimal strategy (p^*, c^*) must satisfy $c^* = D(p^*, c^*)$.*

Proof. All proofs can be found in the appendix. □

To understand this result, we note that for a fixed price, demand is decreasing in the density c. Starting from the full density (i.e., $c = 1$), the demand will be zero for any price because density tolerances are continuously distributed over $[0, 1]$. If the provider decreases the density from 1, then its demand will increase, and so will its sales, which are the minimum of demand and density. As long as the density is strictly larger than the demand, further decreasing density will continue to increase sales, which will also increase revenue since the price is kept constant. Once density and demand are equal, to decrease density further would continue to increase demand, but it would now decrease sales because the sales are the minimum of demand and density. Thus, it is always optimal to exactly balance demand and density.[2]

Concretely, it might be that 80% of potential customers would feel safe flying on an airplane or visiting a movie theater that is operating at 20% density, but at 20% density the airplane or theater cannot serve all of these customers. Similarly, the lower the membership cap is at a private club, the more private and exclusive the club is and thus the more people would desire to join, but by definition the membership cap would keep the club from admitting all of these potential members. In either case, it is optimal for the provider to lower the density just enough that it can serve exactly as many customers as wish to purchase; otherwise it is either under- or over-serving the addressable market. The density level that achieves this outcome will be different for different prices, as demand decreases in price. Using Lemma 1, we essentially reduce the provider's problem that originally contained two decision variables into a problem with only one decision variable (we find it convenient to use the density c as our decision variable where the price $p(c)$ is set to achieve $c = D(p,c)$, but a similar approach could be followed with p and $c(p)$). In the following sections, we will apply this result to help us solve for the provider's optimal policy.

4. Solving the Provider's Problem

We now proceed to solve the provider's optimization problem. We consider a wide range of distributions for the density tolerance, focusing on power distributions $G(c) = c^k$ for $k > 0$. To obtain the sharpest insights, we first study the special case of $k = 1$, i.e., the standard uniform distribution, in Section 4.1. Then, in Section 4.2, we consider power distributions with $k \geq 1$, which subsumes the uniform case and covers most plausible degrees of customer sensitivity to density. This leaves only severely density-sensitive customer populations ($k < 1$), which we study in Section 5. Throughout the remainder of the paper, we consider customer valuations that follow a standard uniform distribution, i.e., $F(p) = p$. In Appendix A.2, we relax this assumption and the power distribution assumption on G, and we show that our main findings generalize to any F and G that satisfy appropriate conditions.

4.1. Uniformly Distributed Density Tolerance

In this section, we specialize the distribution G of the density tolerance to the standard uniform distribution, which allows us to demonstrate some of the key qualitative features of the provider's optimal decisions. With uniformly distributed density tolerance, it is always optimal for the provider to follow a certain strategy, as evidenced by the next lemma.

[2] Lemma 1(iii) is stated for the optimal solution (p^*, c^*), but in fact, for any price $p \in [0,1]$, the highest revenue at that price is achieved by setting c such that $c = D(p,c)$. See the proof in Appendix A.1 for details.

LEMMA 2 **(Full-Coverage, Price-Active Strategy: Uniform)**. *For uniformly distributed density tolerance (i.e., $G(c) = c$), the optimal price and density must fall within the region in which $G(c) \le F(p) \le 1 - G(c)$.*

A key to understanding Lemma 2 is the impact of the inequalities on the service demand. First, the right inequality in the lemma implies that it is optimal to set price and density to satisfy $F(p) + G(c) \le 1$. By equation (3), this implies a nonnegative (strictly positive, if the inequality holds strictly) demand is induced from customers whose valuation and density tolerance are negatively dependent. This cluster of customers is the most difficult for the service provider to attract because those willing to tolerate higher density are not willing to pay high prices. With uniformly distributed density tolerance, Lemma 2, however, informs us that it is optimal not to completely exclude such customers and the provider should set its price and density levels to obtain a *full coverage* of the market (we will later show that the same result holds for a much larger space of distributions).

Regarding the left inequality, we recall from (3) that the demand from customers with perfectly positive dependence between their valuation and density tolerance is determined by $\min\{1 - F(p), 1 - G(c)\}$ and hence can be regulated by either the density c or the price p. With uniformly distributed density tolerance, Lemma 2 finds it optimal to set $G(c) \le F(p)$, suggesting that the price acts to regulate the demand from this cluster of customers (i.e., $\min\{1 - F(p), 1 - G(c)\} = 1 - F(p)$). Thus, we refer to it as a *price-active* strategy (this result also extends to the larger space of distributions).

By Lemma 2, and substituting the uniform CDFs, we can restrict our attention to price-density pairs with $c \le p \le 1 - c$, which allows us to reduce the demand function (3) to

$$D(p, c) = 1 + \alpha c + (1 - \lambda)cp - c - p. \tag{7}$$

By Lemma 1, we can focus on price-density pairs with $c = D(p, c)$; substituting into equation (7) and rearranging then yields for each density c the unique price $p(c)$ that achieves this, namely

$$p(c) = \frac{1 + \alpha c - 2c}{1 - (1 - \lambda)c}. \tag{8}$$

Thus, the optimal solution to the provider's problem (5) must be of the form $(p(c), c)$ with $c \le p(c) \le 1 - c$, and it can therefore be found by solving the reduced problem

$$\max_{0 \le c \le 1} R(c) = p(c)c \quad \text{subject to} \quad c \le p(c) \le 1 - c, \tag{9}$$

where $p(c)$ is given in equation (8).

We are now in a position to derive the provider's optimal price and density in closed form.

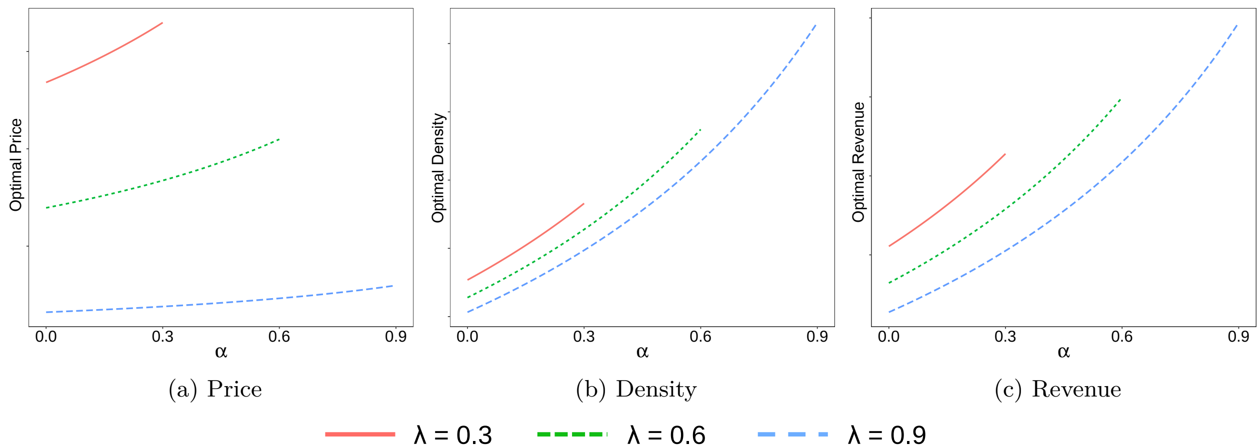


Figure 1 **Provider's optimal solution vs. α for $\lambda \in \{0.3, 0.6, 0.9\}$ with uniform F and G**

Proposition 1 **(Optimal Strategy: Uniform).** *With uniformly distributed density tolerance (i.e., $G(c) = c$), the provider's optimal price p^* and density c^* are given respectively by*

$$p^* = \frac{2-\alpha-\sqrt{(2-\alpha)\lambda+\alpha^2-3\alpha+2}}{1-\lambda} \quad \text{and} \quad c^* = \frac{2-\alpha-\sqrt{(2-\alpha)\lambda+\alpha^2-3\alpha+2}}{(1-\lambda)(2-\alpha)}.$$

The corresponding optimal revenue is

$$R(p^*, c^*) = \frac{3+\lambda-2\alpha-2\sqrt{(2-\alpha)\lambda+\alpha^2-3\alpha+2}}{(1-\lambda)^2}.$$

Figure 1 plots the optimal price, density, and revenue from Proposition 1 as functions of α for different values of λ. It can be easily seen that the optimal density, optimal price and, consequently, optimal revenue increase monotonically in α. The intuition behind the monotonicity of the optimal revenue is as follows. As the willingness to pay and density tolerance become more positively dependent, the fraction of customers with high realizations for both quantities increases, and therefore the demand will increase for all (p, c) pairs (although the sales will only weakly increase because the demand could exceed the density). For $\alpha = \alpha_1$, consider the optimal price-density pair (p_1^*, c_1^*) and the corresponding revenue R_1^*. At this price and density, for $\alpha = \alpha_2 > \alpha_1$, the sales will be weakly higher and thus the revenue will also be weakly higher than for $\alpha = \alpha_1$, implying that the optimal revenue must be increasing in α. Moreover, because the demand will be higher for each price-density pair, the service provider can afford to set a higher density and still achieve $c = D(p, c)$; hence, the optimal density (and optimal sales since the two must be equal at optimality by Lemma 1) is also increasing in α.

Corollary 1 **(Monotonicity: Uniform).** *For uniformly distributed density tolerance (i.e., $G(c) = c$), the optimal price, density, sales, and revenue are all increasing in the degree of positive dependence $\alpha < \lambda$ for fixed λ.*

The monotonicity of the optimal price as the positive dependence α increases is more nuanced. Indeed, by (8), the effect of an increase in α on the optimal price can be decomposed into two parts as

$$\frac{\mathrm{d}p^*}{\mathrm{d}\alpha} = \underbrace{\frac{c^*}{1-(1-\lambda)c^*}}_{\text{positive effect}} + \frac{\mathrm{d}c^*}{\mathrm{d}\alpha} \underbrace{\frac{\alpha-\lambda-1}{(1-(1-\lambda)c^*)^2}}_{\text{negative effect}}. \tag{10}$$

Specifically, an increase in α has a *direct positive* effect on the optimal price because demand increases with α for any given price-density pair and hence, all else being equal, a higher price is needed to bring the demand down so as to maintain $c = D(p,c)$ (see Lemma 1). However, there is also an *indirect negative* effect. Lemma 1 implies that it is always optimal to induce $c = D(p,c)$. Because the optimal density increases with α, keeping this balance for higher α will tend to pull the price down, all else being equal. So, the net change in the optimal price depends on which of these two effects dominates. Corollary 1 shows that for uniform G, the positive effect dominates the negative one, and therefore the optimal price is increasing in α. However, as will become evident in the next section, the optimal price is not always increasing in α if the density tolerance distribution G is non-uniform.

We now compare the provider's policy from Proposition 1 to the policy from the benchmark problem. We find that the provider sets a higher price and a lower density when customers are density-sensitive.

Corollary 2 **(Comparison with Benchmark: Uniform)**. *For uniformly distributed density tolerance (i.e., $G(c) = c$), the optimal price is higher than the benchmark price, and the optimal density is lower than the benchmark density, i.e., $p^* \geq p^\circ$ and $c^* \leq c^\circ$.*

It is perhaps intuitive that the provider should operate at a lower density when customers are density-sensitive than in the benchmark case, i.e., $c^* \leq c^\circ$. However, it is less obvious that $p^* \geq p^\circ$. We can understand why by looking closer at the demand function. For any $c > 0$ and $p < 1$, the demand is strictly less with density sensitivity than in the benchmark problem. In the example of a service provider like a movie theater or airline operating in a pandemic, this decrease in demand stems from customers' fear of infection leading them not to accept the service if the density is too high. For a privacy- or exclusivity-based service like a private club, this decrease in demand stems from the desire for an "elevated" or "bespoke" experience catering to a privileged few. In addition to changing the density, the service provider can also modify the price to adjust to density-sensitive customers. One option is to lower the price—relative to the benchmark p°—to recover some customers. The other is to increase the price to earn more revenue per customer. It is not clear a priori which option leads to higher total revenue.

Corollary 2 reveals that in the face of customer density sensitivity, a provider starting from the benchmark price should increase the price and serve fewer (but more lucrative) customers. This is partially because the preference dimensions are not substitutable from the customer's perspective. In an alternative model where congestion was translatable into monetary units, any and all customers could be recovered by sufficiently lowering the price. By contrast, in our setting, lowering the price (without lowering the density) cannot recover the customers for whom the density is unacceptable. Since a lower price can only attract those who were priced out but were willing to accept the service at the chosen density, it is better to raise the price above the benchmark and serve fewer, more lucrative customers. Although there are myriad contributing factors to the real-world phenomenon, our finding that prices should be higher when customers exhibit density sensitivity—which we will later show generalizes to a large class of density tolerance distributions—is indeed consistent with the historically high inflation observed in the United States in 2022 (Wallace 2022).

4.2. Generalized Distribution for Density Tolerance

Next, we consider a broad range of distributions for customer density tolerance. Specifically, we focus on a power distribution, i.e., $G(c) = c^k$ with $k \geq 1$ (we will later consider the case of severely density-sensitive customers, i.e., $k < 1$). For $k = 1$, the power distribution corresponds to the standard uniform distribution studied in the previous section. As k increases, the power distribution becomes larger in the usual stochastic order with the probability mass shifting to higher values of the density tolerance; that is, the customers become less sensitive to the density. In the extreme when $k \to \infty$, the power distribution becomes degenerate with all the probability mass concentrated at 1, in which case customers become completely insensitive to density.

We first show that the optimality of the full-coverage, price-active strategy identified in Section 4.1 generalizes to this broader class of distributions.

Lemma 3 **(Full-Coverage, Price-Active Strategy: Generalized)**. *For density tolerance with the CDF $G(c) = c^k$, $k \geq 1$, the optimal price p^* and density c^* satisfy $G(c^*) \leq F(p^*) \leq 1 - G(c^*)$.*

Lemma 3 and $G(c) = c^k$ allow us to simplify the demand function (3) to

$$D(p, c) = 1 + \alpha c^k + (1 - \lambda) p c^k - p - c^k, \tag{11}$$

and we can again use Lemma 1 to perform a reduction to a single-variable problem, analogous to that achieved in Section 4.1. Specifically, for $G(c) = c^k$ and $k \geq 1$, the solution to the platform's problem (5) can be found by solving the reduced problem

$$\max_{0 \leq c \leq 1} R(c) = p(c)c \quad \text{subject to} \quad c^k \leq p(c) \leq 1 - c^k, \tag{12}$$

where

$$p(c) = \frac{1 - (1 - \alpha)c^k - c}{1 - (1 - \lambda)c^k}. \tag{13}$$

As a key result, we now provide the complete solution to the provider's problem, which applies over the entire space of distributions considered in this section.

PROPOSITION 2 **(Optimal Strategy: Generalized)**. *For density tolerance with the CDF $G(c) = c^k$, $k \geq 1$, the provider's optimal density $c^* \in (0,1)$ is given by the unique solution in c to*

$$1 - 2c + (1-\alpha)(1-\lambda)c^{2k} - (k-2)(1-\lambda)c^{k+1} + c^k(\alpha + \alpha k - k\lambda + \lambda - 2) = 0. \tag{14}$$

The optimal price is given by

$$p^* = \frac{1 - (1-\alpha)(c^*)^k - c}{1 - (1-\lambda)(c^*)^k}, \tag{15}$$

and the optimal revenue is $R(p^*, c^*) = p^* c^*$.

This result allows us to compare the provider's optimal price and density with the optimal solutions of the benchmark problem (6). The following corollary reveals that the provider sets a higher price and a lower density when customers are density-sensitive, generalizing Corollary 2.

COROLLARY 3 **(Comparison with Benchmark: Generalized)**. *For density tolerance with the CDF $G(c) = c^k$, $k \geq 1$, the optimal price is higher than the benchmark price, and the optimal density is lower than the benchmark density, i.e., $p^* \geq p^\circ$ and $c^* \leq c^\circ$.*

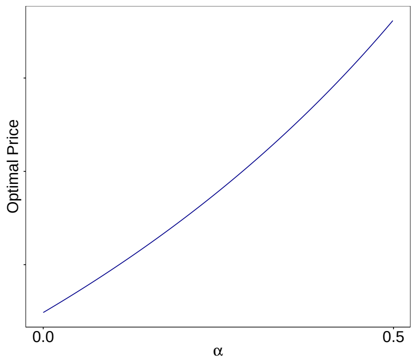
The intuition for this result is similar to that given in Section 4.1, which we will not repeat here.

Next, we show that Corollary 1 generalizes on all measures except price (which we consider below).

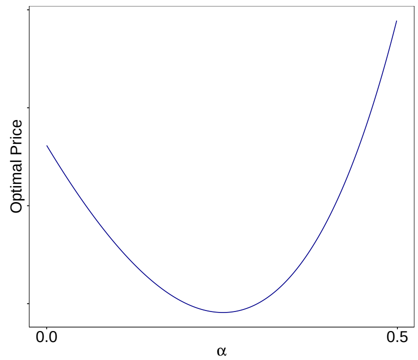
PROPOSITION 3 **(Monotonicity: Generalized)**. *For density tolerance with the CDF $G(c) = c^k$, $k \geq 1$, the optimal density, sales, and revenue are all increasing in the degree of positive dependence $\alpha < \lambda$ for fixed λ.*

Intuitively, an increase in α suggests more positive dependence between customers' valuation and density tolerance, thus increasing the demand for a given price-density pair. If the true positive dependence parameter is $\alpha' + \epsilon$, and if the provider sets the price and density at their optimal values based on a positive dependence parameter α', then his revenue will be the same as the optimal revenue for α', but the demand will exceed the set density. From this price-density pair, the provider can increase his revenue by slightly increasing the density and keeping the price fixed; this will achieve more sales at the same price. Thus, the directional effects of a change in α on the density, sales and revenue match those in the uniform case from Section 4.1 (cf. Corollary 1).

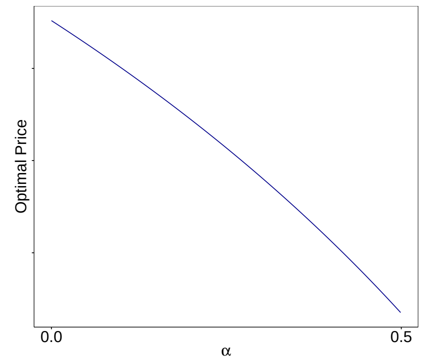
Proposition 3 does not offer a conclusion about the optimal price. For uniformly distributed customer preferences, Corollary 1 showed that the optimal price is monotonically increasing in α. However, when we consider a broader class of distributions, the optimal price may not be monotonic and may demonstrate novel behavior. We characterize the possibilities in the next proposition.



(a) $k = 1.03$



(b) $k = 1.26$



(c) $k = 1.43$

Figure 2 Optimal price vs. α for $\lambda = 0.5$ ($\underline{k} \approx 1.035$, $\overline{k} \approx 1.414$) with $G(c) = c^k$

Proposition 4 **(Price Comparative Statics: Generalized)**. *For density tolerance with the CDF $G(c) = c^k$, $k \geq 1$, the optimal price p^* is quasi-convex in α; that is, it can either increase, decrease, or first decrease then increase as α increases. In particular, there exist two thresholds $\underline{k}, \overline{k} \in [1,2]$ with $\underline{k} \leq \overline{k}$ such that p^* is increasing in α for any $k \leq \underline{k}$ and decreasing in α for any $k \geq \overline{k}$.*

Proposition 4 reveals that as customer valuation and density tolerance become more positively dependent, the behavior of the optimal price depends on the density tolerance distribution G, as illustrated in Figure 2. We recall from equation (10) that the effect of an increase in α on the provider's optimal price can be decomposed into a *direct positive* effect and an *indirect negative* effect. A similar decomposition based on (13) also applies for the power distribution $G(c)$. When k is smaller and close to 1 (i.e., $k \leq \underline{k}$), the optimal price behaves consistently with the case of the uniform distribution, namely increases with α. In this case, customers are sufficiently heterogenous in their density sensitivity, so an increase in α has a larger *direct* impact on demand and hence the positive effect dominates. On the other hand, for sufficiently large k (i.e., $k \geq \overline{k}$), the density sensitivity distribution becomes much more skewed and concentrated more on the higher tolerance level, so the demand is less sensitive to the density around the optimal solution. Thus, the direct effect is weakened and the indirect negative effect dominates, reversing the monotonicity of the optimal price with respect to α. During the transition phase (i.e., $\underline{k} < k < \overline{k}$), the optimal price exhibits non-monotonic behavior, namely first decreases and then increases in α.

Combining Propositions 3 and 4, we note that as the demand becomes more negatively dependent (i.e., α decreases), the provider would prefer to restrict the service capacity and raise the price (when consumers are not extremely density sensitive, i.e., $k \geq \bar{k}$). These findings resonate with the anecdotal case of the exclusive private clubs such as the previously-mentioned Augusta National and Yellowstone Club, and other elite private clubs like Cypress Point Club in California and Seminole Golf Club in Florida, with only 250 and 300 members, respectively (Dobson 2021).

5. Severe Density Sensitivity

In this section, we consider customer populations exhibiting severe density sensitivity, represented by distributions $G(c) = c^k$ with $k < 1$. As k decreases, the probability mass of the power distribution shifts toward smaller values of the density tolerance, and customers become more sensitive to the density. In particular, as $k \to 0$, the power distribution becomes degenerate with the probability mass concentrated at 0, i.e., customers become completely density-intolerant. At the other extreme, i.e., as $k \to 1$, the power distribution approaches the uniform distribution studied in Section 4.1.

For a severely density-sensitive customer population, the optimal strategy is not always full-coverage, price-active; that is, the optimal price and density do not necessarily satisfy $G(c) \le F(p)$ and $G(c) + F(p) \le 1$. Based on the result of $\min\{1 - F(p), 1 - G(c)\}$ and $\max\{0, 1 - F(p) - G(c)\}$ in the demand function (3), the provider has four possible qualitatively different strategies. We recall that, under price and density levels satisfying $G(c) + F(p) \le 1$, the provider achieves *full coverage* of the market by serving all three clusters of customers with independence, perfect positive and negative dependence between the valuation and density tolerance. Conversely, if $G(c) + F(p) \ge 1$, then the provider has only *partial coverage*, since the demand from the cluster with perfect negative dependence reduces to zero. Regarding the cluster with perfect positive dependence (represented by $\min\{1 - F(p), 1 - G(c)\}$), we recall that the *price* is *active* in regulating the demand from this cluster if $G(c) \le F(p)$; otherwise, the *density* would be *active* in doing so. Therefore, the provider's price and density decisions (p, c) can be categorized into four distinct strategies:

1. Full-Coverage, Price-Active (FP), if $G(c) + F(p) \le 1$ and $G(c) \le F(p)$;
2. Full-Coverage, Density-Active (FD), if $G(c) + F(p) \le 1$ and $G(c) \ge F(p)$;
3. Partial-Coverage, Price-Active (PP), if $G(c) + F(p) \ge 1$ and $G(c) \le F(p)$; and
4. Partial-Coverage, Density-Active (PD), if $G(c) + F(p) \ge 1$ and $G(c) \ge F(p)$.

When *both* price and density are active in regulating the demand from the cluster with perfect positive dependence, i.e., $G(c) = F(p)$, we refer to the strategy as FB or PB depending on whether the coverage is *full* or *partial*. On the other hand, when $G(c) + F(p) = 1$, the full coverage in effect degenerates to partial coverage, as the demand from the cluster with perfect negative dependence reduces to zero.[3] Our solution strategy is to first identify the optimal price and density decisions under full and partial coverage. Then, we compare their corresponding optimal revenues to determine the *overall* optimal strategy for the provider, which we characterize in the proposition below.

PROPOSITION 5 **(Optimal Strategy: Severe Density Sensitivity)**. *When the optimal strategy is full-coverage, there exist two thresholds* $0 < k_{\mathrm{FD}} < k_{\mathrm{FP}} \le 0$ *such that it is (i)* FD *if* $k \le k_{\mathrm{FD}}$*, (ii)* FB *if* $k_{\mathrm{FD}} < k < k_{\mathrm{FP}}$*, and (iii)* FP *if* $k \ge k_{\mathrm{FP}}$*. When the provider's optimal strategy is partial-coverage,*

[3] We can show that the optimal strategy falls on this degenerate case only for parameters of zero measure.

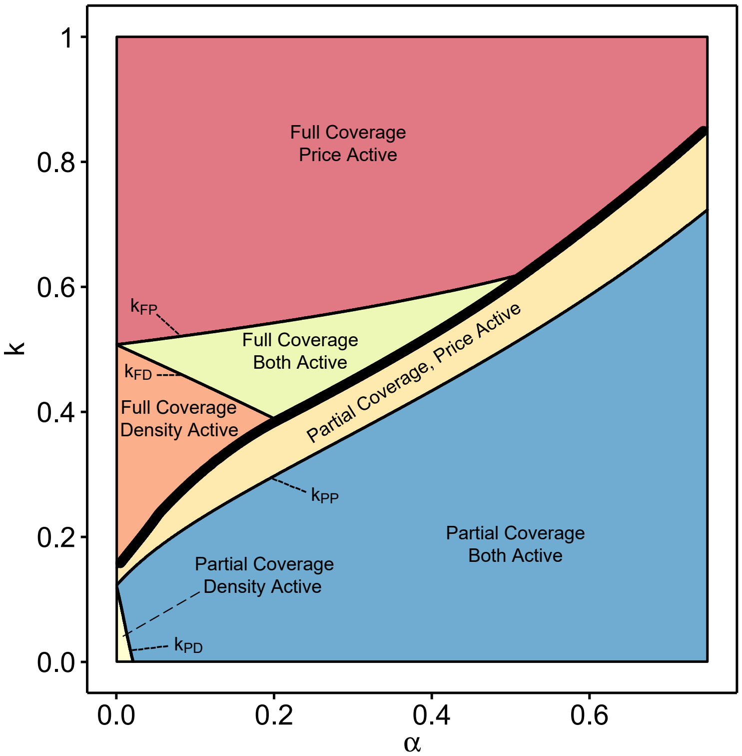


Figure 3 **Optimal strategy in (α, k) space with $\lambda = 0.75$.**

there exist two thresholds $0 < k_{\text{PD}} < k_{\text{PP}} \le 1$ *such that it is (i)* PD *if* $k \le k_{\text{PD}}$*, (ii)* PB *if* $k_{\text{PD}} < k < k_{\text{PP}}$*, and (iii)* PP *if* $k \ge k_{\text{PP}}$*.*

(The explicit characterization of the thresholds and the optimal price and density are given in Lemmas EC.3 and EC.4 and Proposition EC.2 in Appendix D.)

Proposition 5 shows which lever (price versus density) the full-coverage and partial-coverage strategy would activate in the optimum to regulate the demand from the customers with positively dependent preferences. In both cases, we find that density (resp., price) should be the active lever when customers become highly (resp., relatively less) sensitive to density, i.e., k is below a lower threshold k_{FD} or k_{PD} (resp., above a higher threshold k_{FP} or k_{PP}). The intuition is as follows. For highly density-sensitive customers (i.e., for k sufficiently small), very few of them are willing to accept the service even if the provider sets a small density, i.e., $G(c)$ will be relatively close to 1. Setting $F(p) > G(c)$ would further dampen the limited demand, so for small k, the provider chooses to regulate the demand using the density by setting $F(p) \le G(c)$. On the other hand, when k is larger, i.e., when the density tolerance distribution is more evenly spread, there are more customers with relatively high density tolerance. With more customers willing to accept the service at higher densities, the potential demand is higher. Thus, the provider can set a higher price to earn more revenue per customer because setting $F(p) > G(c)$ can still achieve a healthy demand. In other words, a price-active strategy is optimal for larger k. Naturally, in between these two polar cases, the density and price become equally effective and should be both activated to regulate the demand from the customers with positively dependent preferences.

While having characterized the optimal structure for both full- and partial-coverage strategies in the optimum, Proposition 5 remains agnostic about when each of these two strategies is optimal. The answer to this question depends on the comparison of their respective optimal revenues, which turns out to be analytically intractable. Nonetheless, our numerical experiment suggests that the full-coverage strategy identified for $k \geq 1$ continues to be optimal for $k < 1$ above a threshold; the partial-coverage strategy becomes optimal only when k falls below that threshold, i.e., when customers become extremely density-sensitive. To visualize the provider's optimal strategy, we illustrate it in the (α, k) space, as shown by Figure 3, where the thick line partitions the (α, k) space into a full-coverage region (upper left) and a partial-coverage region (lower right). Intuitively, when the customers become extremely sensitive to the density, the provider must set a low density in order to attract any customer. Alternatively, the provider prefers not to serve the negative-dependent cluster but only focuses on the more lucrative customers from the independent and positively dependent clusters. Indeed, as shown in Figure 3, the partial-coverage strategy is optimal for sufficiently large α.

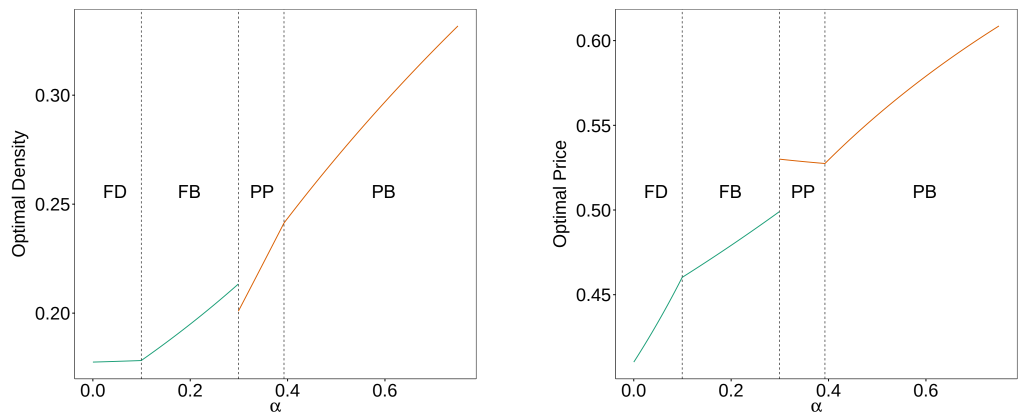
As our last analytical result, the following proposition characterizes how the optimal price and density are affected by the positive dependence parameter α.

PROPOSITION 6 **(Effects of α on Optimal Density and Price)**. *The optimal density is increasing in α under both full- and partial-coverage optimal strategies; the optimal price is increasing in α under all optimal strategies (*$\mathrm{FP}, \mathrm{FD}, \mathrm{PD}, \mathrm{FB}$*, and* PB*) but is decreasing when the optimal strategy is* PP*.*

Figure 4 illustrates the findings in Proposition 6. Consistent with our findings for $k = 1$, the optimal density is increasing in α under both the full-coverage and partial-coverage strategies (albeit with a discontinuity as the optimal strategy transits from full- to partial-coverage). As previously discussed, an increase in α implies higher demand for each price-density pair, which allows the provider to operate at a higher density. Also in line with our earlier findings, the optimal price is increasing in α except under strategy PP. Intuitively, the partial-coverage strategy eschews the negatively dependent cluster and limits the demand to independent and positively dependent clusters. Since price is the active lever for regulating demand under strategy PP, the provider thus *lowers* the price to attract more customers from the growing positively dependent cluster.

6. Calibrated Numerical Study

In this section, we leverage a real dataset to numerically validate our distributional assumptions, calibrate the model parameters, and subsequently evaluate the provider's optimal strategy. The data

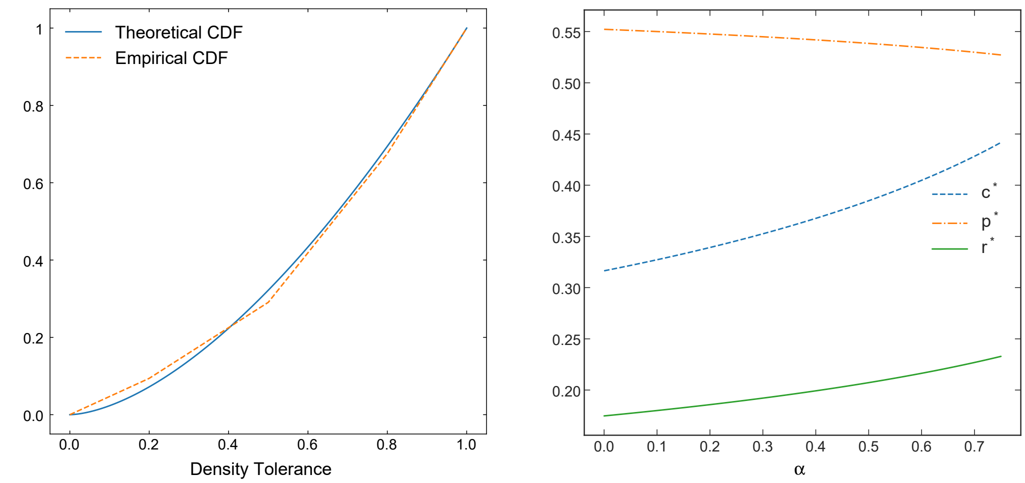


(a) Density (b) Price

Figure 4 Optimal density and price vs. α for $\lambda = 0.75$ and $k = 0.45$

set is drawn from Shelat et al. (2022b), which conducted a choice experiment measuring the impact of COVID-19 risk perceptions on train travel decisions in the Netherlands. In the experiment, researchers first asked respondents to provide a subjective assessment of the risk of different train trips based on multiple factors including the service density (measured by percent of seats occupied), face-mask mandate, extra cleaning of contact surfaces, and the infection level in the population. Then, they were asked to choose between pairs of train trips specified by risk level, travel time, and ticket price; after choosing one of the two trips, respondents were subsequently asked to indicate whether they would prefer to opt out of a train trip entirely rather than take their chosen trip.

Since the level of crowdedness or density is one of the primary factors determining the risk level—and because respondents were primed to consider the density since it was part of their subjective assessment of risk—we use risk level as a proxy measure for density and thus risk tolerance as a proxy for density tolerance. To isolate the impact of risk levels on respondents' decisions to travel by train or not, we choose a subset of the original survey data, namely the responses for trip pairs that both had the same risk level: one with low price, long duration and the other with high price, short duration. For these pairs, we calculate what percentage of respondents prefers not travelling by train at all to both trip options. This allows us to determine whether the respondent is willing to travel with the given risk level or not (i.e., whether they are willing to accept the service). The subset includes the three risk levels 1, 3 and 5 corresponding to very low risk, medium risk, and very high risk. Using a respondent's decisions for each risk level, we find the maximum risk level the respondent is willing to tolerate on a train trip. We then map the risk levels to numbers in the interval $[0, 1]$. Letting 0 and 1 represent no risk and maximum risk, we map risk levels 1, 3, and 5 to 0.2, 0.5, and 0.8, respectively. We divide respondents into 4 groups, corresponding to tolerances in each of the intervals $[0, 0.2)$, $(0.2, 0.5]$, $[0, 5, 0.8)$, and $[0.8, 1]$. Assuming density tolerance is uniformly distributed



(a) Empirical and theoretical CDFs (b) Optimal decisions and revenue

Figure 5 Optimal decisions and revenue as a function of α for $\lambda = 0.75$ and $k = \tilde{k} = 1.635$

within each group, we construct an empirical distribution for density tolerance across all groups—i.e., over the entire range $[0,1]$—and plot it in Figure 5.

We can then use this empirical distribution to estimate the power distribution parameter by the method of moments. Because the mean of a power distribution is one-to-one with the parameter k, there is a unique $\tilde{k}$ such that the mean of the corresponding power distribution matches the mean of the empirically constructed distribution, namely $\tilde{k} = 1.635$. Figure 5a depicts both the empirical CDF constructed from the data and the theoretical CDF of the estimated power distribution, revealing a close visual match. We also assess the goodness of fit of the estimated distribution with the Kolmogorov–Smirnov (K-S) test. Since in practice, we would likely not have access to the true distribution, we generate 500 independent draws from the empirically constructed distribution to use as our sample data for density tolerances. We then use the method of moments to match the mean of a power distribution to the sample mean of the data. Finally, we conduct a K-S test of the sample data against the fitted power distribution. We repeat this process 20 times, and the average p-value is 0.286, i.e., strongly failing to reject the hypothesis that the data comes from the estimated power distribution. Overall, we thus conclude that a power distribution is indeed a good fit for customer preferences, and that the best-fitting parameter ($\tilde{k} = 1.635$) falls comfortably within the parameter range ($k \geq 1$) to which our main results from Section 4 apply.

Next, using the estimated power parameter, we compute the optimal decisions and revenue by varying the degree of positive dependence α between valuation and density tolerance. As shown in Figure 5b, the optimal density and optimal revenue are both increasing in α (consistent with Proposition 3), while the optimal price is decreasing in α (consistent with Proposition 4). Indeed, the estimated power $\tilde{k} = 1.635$ is sufficiently large (above the threshold $\bar{k}$ identified in Proposition 4) so that the indirect negative effect is dominant (see the discussion following Proposition 4).

For additional robustness tests, we consider, in Appendix E, another dataset—from related work by Shelat et al. 2022a—that also measures traveller preferences during the COVID-19 pandemic. On this alternate dataset, we again find that the power family of distributions is a reasonable approximation for the customer density tolerance distribution, and we conclude that this family is an appropriate choice for our analytical study.

7. Conclusion

Motivated by the practices in the service industry in the aftermath of a pandemic, as well as exclusivity- and privacy-based club operations, we have introduced a novel, copula-based framework for studying a service provider's problem whose customers are sensitive to both the service price and the crowd *density.* A novel and crucial feature of our framework is the statistical dependence between customers' valuation and density tolerance. Under this framework, we characterize the provider's joint optimal price and density cap decisions in relation to the marginal density tolerance distribution and the degree of (positive or negative) dependence between the two customer attributes.

We find that a full-coverage, price-active strategy—in which the provider serves all segments of the market and activates the price (rather than the density) to regulate demand—is optimal for the service provider over a broad range of dependence structures and marginal density tolerance distributions. In particular, we demonstrate the nuanced interaction between the dependence structure and the marginal density tolerance distribution. As a result, while the optimal density cap and revenue are increasing as the valuation and density tolerance become more positively dependent, the optimal price may be non-monotonic. For severely density-sensitive customer populations, we find that the full-coverage, price-active strategy may no longer be optimal. It may be optimal for the service provider to drop the demand cluster of negative dependence or/and to activate the density to regulate the demand from the positively dependent cluster.

Our findings not only offer prescriptive guidelines for service providers operating in the presence of density-sensitive demand (especially in the aftermath of a pandemic), but they also speak to some highly debated economic and social issues. For instance, our results shed some light on the stringent access restrictions imposed by exclusive private clubs, as their customer base (e.g., celebrities) is likely to exhibit strong negative dependence between their willingness to pay and density tolerance. Our results on the optimal price suggest a fresh perspective to combat the recent rampant inflation: reshaping the dependence structure between customers' valuation and density tolerance can be an effective lever, but private enterprises and governments may have divergent incentives.

Our framework and results open several avenues for future work. We have assumed uniformly distributed WTP and a broader class of distributions for density tolerance, allowing more richness for the latter as a novel feature of our work. For more general WTP distributions, we believe that our main insights would likely continue to hold, but the analysis would be technically challenging if both $F(p)$ and $G(c)$ took general forms. To capture a wide range of dependence structures, we adopted the Fréchet family of copulas. It would be an interesting future direction to explore alternative families of copulas, with the caveat that clean analytical results could prove difficult to obtain. As one of the first works on service operations to employ copulas, we hope that our work can inspire further use of this oft-overlooked tool in related settings.

References

Abbas AE (2009) Multiattribute utility copulas. *Oper. Res.* 57(6):1367–1383.

Anstey C (2023) Strains in services. *Bloomberg* URL https://www.bloomberg.com/news/newsletters/2023-02-23/what-s-happening-in-the-world-economy-services-inflation-is-proving-tough, accessed on 04/19/2023.

Associated Press (2023) Biden ends COVID national emergency after Congress acts. *NPR* URL https://www.npr.org/2023/04/11/1169191865/biden-ends-covid-national-emergency, accessed on 04/19/2023.

Bagchi A, Paul JA (2014) Optimal allocation of resources in airport security: Profiling vs. screening. *Oper. Res.* 62(2):219–233.

Bai M, Cui Y, Kong G, Zhang AZ (2022) No panic in pandemic: The impact of individual choice on public health policy and vaccine priority. *Available at SSRN 3763514* .

Bansal S, Gutierrez GJ, Keiser JR (2017) Using experts' noisy quantile judgments to quantify risks: Theory and application to agribusiness. *Operations Research* 65(5):1115–1130.

Bansal S, Wang T (2019) Using subjective probability distributions to support supply chain decisions for innovative agribusiness products. *Working Paper, Pennsylvania State University* .

BTS (2020) Preliminary air traffic data, April 2020: 96% reduction in U.S. airline passengers from 2019. URL https://www.bts.gov/newsroom/preliminary-air-traffic-data-april-2020-96-reduction-us-airline-passengers-2019, accessed on August 14, 2021.

BTS (2021) Average air fares dropped to all-time low in 2020. URL https://www.bts.gov/newsroom/average-air-fares-dropped-all-time-low-2020, accessed on 04/20/2023.

Buchanan J (1965) An economic theory of clubs. *Economica* 32(1):1–14.

Burton M (2017) Soho House, explained. *Eater* URL https://www.eater.com/2017/8/15/16136272/soho-house-nick-jones-expansion-history-membership-rules, accessed on 11/10/2022.

Buteau M, Paskin J (2015) 118 rich and powerful people who are members of Augusta National. *Bloomberg* URL https://www.bloomberg.com/graphics/2015-augusta-national-golf-club-members/, accessed on 11/10/2022.

Chen C (2001) Design for the environment: A quality-based model for green product development. *Manage. Sci.* 47(2):250–263.

Clemen RT, Reilly T (1999) Correlations and copulas for decision and risk analysis. *Manage. Sci.* 45(2):208–224.

Daniels LD (2021) Even though restaurants now allowed to increase capacity to 75%, most can't. *Dallas Observer* URL https://www.dallasobserver.com/restaurants/dallas-restaurants-can-up-capacity-to-75-but-many-lack-space-11989246, accessed on August 14, 2021.

de Véricourt F, Gurkan H, Wang S (2021) Informing the public about a pandemic. *Manage. Sci.* 67(10):6350–6357.

Debo L, Rajan U, Veeraraghavan SK (2020) Signaling quality via long lines and uninformative prices. *M&SOM* 22(3):513–527.

Debo LG, Parlour C, Rajan U (2012) Signaling quality via queues. *Manage. Sci.* 58(5):876–891.

Delta (2021) Delta extends middle seat blocking through April 2021; Only U.S. airline to continue providing more space for customers. URL `https://news.delta.com/delta-extends-middle-seat-blocking-through-april-2021-only-us-airline-continue-providing-more-space`, accessed on August 14, 2021.

Derby MS (2023) Fed's Williams: Inflation still 'too high,' Fed will act to lower it. *Reuters* URL `https://www.reuters.com/business/retail-consumer/feds-williams-inflation-still-too-high-fed-will-act-lower-it-2023-04-19/`, accessed on 04/19/2023.

Dobson J (2021) The top 20 most exclusive golf and country clubs in America honored with Platinum Status. *Forbes* URL `https://www.forbes.com/sites/jimdobson/2021/05/21/the-top-20-most-exclusive-golf-and-country-clubs-in-america-honored-with-platinum-status/`, accessed on 11/22/2022.

FAA (2022) Benefit-cost analysis. *Federal Aviation Administration* URL `https://www.faa.gov/regulations_policies/policy_guidance/benefit_cost`, accessed on 11/21/2022.

Fandango (2022) Important theater safety information regarding COVID-19. *Fandango* URL `https://www.fandango.com/covid-theater-safety-update`, accessed on 11/18/2022.

Fréchet (1958) Remarques au sujet de la note précédente. *C R Acad Sci Paris Sér I Math* 246:2719–2720.

Ginsberg M (2010) Hollywood hideouts: Secrets of the Soho House. *The Hollywood Reporter* URL `https://www.hollywoodreporter.com/movies/movie-news/hollywood-hideouts-secrets-soho-house-34822/`, accessed on 01/18/2023.

Griff Z (2021) It's official: The era of blocked middle seats just ended. URL `https://thepointsguy.com/news/blocked-middle-seats-are-over/`, accessed on August 14, 2021.

Guo P, Haviv M, Luo Z, Wang Y (2023) Signaling service quality through queue disclosure. *M&SOM* 25(2):543–562.

Gupta S, Starr MK, Farahani RZ, Asgari N (2022) OM Forum—Pandemics/epidemics: Challenges and opportunities for Operations Management research. *M&SOM* 24(1):1–23.

Hassin R (2016) *Rational Queueing* (CRC Press).

Hassin R, Haviv M (2003) *To Queue or Not to Queue: Equilibrium Behavior in Queueing Systems* (Springer Science & Business Media).

Hassin R, Meilijson I, Perlman Y (2023) Queueing with negative network effects. *Manufacturing & Service Operations Management* .

Hillstone (2023a) Hillstone — Houston URL `https://hillstonerestaurant.com/locations/houston/`, accessed on 03/29/2023.

Hillstone (2023b) Hillstone — San Francisco URL `https://hillstonerestaurant.com/locations/sanfrancisco/`, accessed on 03/29/2023.

Hillstone Restaurant Group (2023) Dining Guidelines and Attire Standards URL `https://hillstone.com/dininginfo/`, accessed on 03/29/2023.

Honig Z (2021) Delta is now the only US airline blocking all middle seats. *Points Guy* URL `https://thepointsguy.com/guide/airline-social-distancing-policies/`, accessed on August 14, 2021.

Jain A, Rayal S (2022) Managing medical equipment capacity with early spread of infection in a region. *Prod. Oper. Manag.* (Forthcoming).

Kaufman SM, Moss ML, McGuinness KB, Cowan NR, Rudner CE, Schwartz SI, Joshi M, Schack DL, Rogers J, Limone O, et al. (2020) Transportation during Coronavirus in New York City. *NYU Wagner Rudin Center for Transportation Report* .

Kay G (2023) Elon Musk belongs to an exclusive hollywood club that forbids members from taking pictures. Here's everything we know about the San Vicente Bungalows. *Insider* URL `https://www.businessinsider.com/photos-elon-musk-member-of-exclusive-hollywood-club-see-inside-2022-11`, accessed on 04/20/2023.

Koenig D (2022) US airlines blame China for some flight cancellations as restrictions tighten ahead of Olympics. USA Today, URL `https://www.usatoday.com/story/travel/airline-news/2022/01/12/us-airlines-china-travel-restrictions-forcing-some-flights-cancel/9183333002/`, accessed on December 20, 2022.

Lacourbe P, Loch CH, Kavadias S (2009) Product positioning in a two-dimensional market space. *Prod. Oper. Manag.* 18(3):315–332.

Lauga DO, Ofek E (2011) Product positioning in a two-dimensional vertical differentiation model: The role of quality costs. *Marketing Sci.* 30(5):903–923.

Lin YT, Sun H, Wang S (2020) Designing sustainable products under coproduction technology. *M&SOM* 22(6):1181–1198.

Liu Q, Shuai J (2019) Multi-dimensional product differentiation. *Available at SSRN 1975191* .

McClintock P (2022) Box office ends year 60 percent behind 2019 with $4.5B domestically. *The Hollywood Reporter* URL `https://www.hollywoodreporter.com/business/business-news/2021-box-office-revenue-stormy-year-1235067966/`, accessed on 11/20/2022.

Mutikani L, Nomiyama C (2023) U.S. service sector slows in March; inflation cooling - ISM survey. *Reuters* URL `https://www.reuters.com/markets/us/us-service-sector-slows-march-inflation-cooling-ism-survey-2023-04-05/`, accessed on 04/19/2023.

Nageswaran L (2022) Implications of vaccine shopping during pandemic. *Prod. Oper. Manag.* (Forthcoming).

Naor P (1969) The regulation of queue size by levying tolls. *Econometrica* 15–24.

Nelsen RB (2007) *An Introduction to Copulas* (Springer Science & Business Media).

Pallini T (2021) Delta CEO Ed Bastian declares 'It's absolutely safe to sit in the middle seat' in defiance of CDC suggesting airlines should block them. *Business Insider* URL `https://www.businessinsider.com/delta-ceo-bastian-says-safe-to-sit-in-middle-seat-2021-4`, accessed on August 14, 2021.

Pino I (2022) The average net worth of Americans at every age, and 5 ways to boost yours. *Fortune* URL `https://fortune.com/recommends/investing/average-net-worth-by-age/`, accessed on 04/29/2023.

Rashkova I, Dong L (2022) Role of innovation valuation uncertainty and operational cost structure on product line design. *Available at SSRN* .

Rubin R, Maddaus G (2020) Movie theaters in these states are closed due to Coronavirus. *Variety* URL `https://variety.com/2020/film/news/movie-theaters-close-u-s-coronavirus-1203536126/`, accessed on 11/19/2022.

Sandler T, Tschirhart J (1997) Club theory: Thirty years later. *Public Choice* 93(3-4):335–355.

Scotchmer S (1985) Profit-maximizing clubs. *J. Public Econ.* 27(1):25–45.

Shaked M, Shanthikumar JG (2007) *Stochastic Orders* (Springer Science & Business Media).

Shelat S, Cats O, van Cranenburgh S (2022a) Traveller behaviour in public transport in the early stages of the COVID-19 pandemic in the Netherlands. *Transportation Research Part A: Policy and Practice* 159:357–371.

Shelat S, Van De Wiel T, Molin E, van Lint J, Cats O (2022b) Analysing the impact of COVID-19 risk perceptions on route choice behaviour in train networks. *PLOS One* 17(3):e0264805.

Smith R (2015) Finance returns the favor and snubs Soho House. *Reuters* URL `https://www.reuters.com/article/soho-house-bonds/finance-returns-the-favour-and-snubs-soho-house-idUSL5N11U3ZS20150925`, accessed on 11/10/2022.

Sontag E (2020) Where restaurants have reopened across the U.S. *Eater* URL `https://www.eater.com/21264229/where-restaurants-reopened-across-the-u-s`, accessed on August 14, 2021.

Tang CS, Yoo OS, Zhan D (2023) When should grocery stores adopt time-based pricing? Impact of competition and negative congestion externality. *Prod. Oper. Manag.* (Forthcoming).

Tong A (2022) The 'world's only private ski' resort, where Mark Zuckerberg and Bill Gates hit the slopes. *The San Francisco Standard* URL `https://sfstandard.com/arts-culture/parks-outdoors/the-worlds-only-private-ski-resort-where-mark-zuckerberg-and-bill-gates-hit-the-slopes/`.

Topkis DM (1998) *Supermodularity and Complementarity* (Princeton University Press).

Trebay G (2019) The San Vicente Bungalows: Hollywood's new clubhouse. *New York Times* URL `https://www.nytimes.com/2019/02/23/style/san-vicente-bungalows-jeff-klein.html`, accessed on 04/20/2023.

Wallace A (2022) Inflation data shows US prices were still uncomfortably high last month. *CNN* URL `https://www.cnn.com/2022/10/28/economy/us-pce-inflation-september`, accessed on 11/16/2022.

Yang L, Cui S, Wang Z (2022) Design of Covid-19 testing queues. *Prod. Oper. Manag.* 31:2204–2221.

Yellowstone Club (2016) Fact Sheet and Dues URL `https://yellowstoneclub.com/wp-content/uploads/2016/03/Fact-Sheet-and-Dues.pdf`, accessed on 01/06/2023.

Zhong H, Wang G, Dai T (2021) Wheels on the bus: Impact of vaccine rollouts on demand for public transportation. *Available at SSRN 3874150* .

Appendix. Preliminary Results, Proofs, and Calibrated Numerical Study

This appendix is divided into sections. We state and prove preliminary results in Appendix A. Then, we give the proofs for results in Sections 4.1 and 4.2 in Appendices B and C, respectively. In Appendix D, we provide a more detailed characterization of the solution to the provider's problem with severely density-sensitive customers considered in Section 5, as well as the proofs for the results in that section. We provide details of the second calibrated numerical study in Appendix E. Finally, we provide the proofs of supporting claims in Appendix F.

A. Preliminary Results and Proofs

We first introduce some preliminary results that are used repeatedly in the rest of the appendix.

A.1. Proof for Section 3.1.

Proof of Lemma 1. To prove part (i), first consider the extreme case of $p=1$. From (3), we have $D(1,c)=0$ for any $c\in[0,1]$. Therefore, $c(1)=0$ is the unique density such that $c(1)=D(1,c)$, and part (i) holds for $p=1$.

Next, fix $p\in[0,1)$. The expression (3) for the demand $D(p,c)$ has three terms. The first term $\alpha\min\{1-F(p),1-G(c)\}$ is weakly decreasing in c because G is an increasing function. Similarly, the third term $(\lambda-\alpha)\max\{1-F(p)-G(c),0\}$ is weakly decreasing in c. Since $0<\lambda<1$ and $p\in[0,1)$, the remaining term $(1-\lambda)(1-F(p))(1-G(c))$ is strictly decreasing in c. Therefore, the demand $D(p,c)$ is strictly decreasing in c. Because $G(0)=0$ and $G(1)=1$, from equation (3) we get $D(p,0)=1-F(p)>0$ and $D(p,1)=0<1$. Since $D(p,c)$ is continuous in c for the given p, there thus exists c' such that $c'=D(p,c')$ by the Intermediate Value Theorem, and moreover this c' is unique because $D(p,c)$ is strictly decreasing in c. Thus, part (i) holds for $p\in[0,1)$, and since we showed above that it holds for $p=1$, the proof of part (i) is complete.

Part (ii) follows from part (i) by symmetry because $D(p,c)$ in (3) is symmetric in $F(p)$ and $G(c)$.

For part (iii), first note that it is possible to achieve strictly positive revenue because since $\lambda<1$, the expression $(1-\lambda)(1-F(p))(1-G(c))$ in the demand function (3) implies that there will be strictly positive demand for any $0<p,c<1$. Therefore, $p=0$ or $p=1$ cannot be optimal, since either price achieves a revenue of zero: by (4) we have $R(0,c)\le 0\cdot D(0,c)=0$ and $R(1,c)\le 1\cdot D(1,c)=0$. Similarly, $c=0$ or $c=1$ cannot be optimal, since either will achieve a sales (and thus revenue) of zero. For the remainder of the proof, we restrict our attention to p,c with $0<p,c<1$.

Now, consider a $p\in(0,1)$, and denote by $c(p)$ the unique density that satisfies $c(p)=D(p,c(p))$ (such $c(p)$ can be found by part (ii) that was proved above). Since $p\in(0,1)$, the demand $D(p,c)$ is strictly decreasing in c by the arguments above for part (i). Thus, for any density $c_1<c(p)$, we have $D(p,c_1)>D(p,c(p))=c(p)>c_1$. The actual units sold at a density of c_1 is $\min\{c_1,D(p,c_1)\}=c_1$, which is therefore strictly less than the units sold at a density of $c(p)$, namely $c(p)$. Similar reasoning implies that the units sold is strictly less at a density $c_2>c(p)$ than at $c(p)$. Therefore, for any $\tilde c\ne c(p)$, we have $R(p,\tilde c)=p\min\{\tilde c,D(p,\tilde c)\}<pc(p)=R(p,c(p))$. This implies that any pair (p,c) with $c\ne D(p,c)$ is strictly sub-optimal, and thus at optimality we must have $c^*=D(p^*,c^*)$. □

A.2. Results for General Distributions

We now derive results for distributions F and G satisfying certain technical conditions (note that here we do not necessarily require the valuation distribution F to be uniform). These results facilitate the proofs of the results in Sections 4 and 5. Where convenient, we use the notation $\bar F(p)=1-F(p)$ and $\bar G(c)=1-G(c)$.

One of the technical conditions relates to the star order $\le_*$ (Shaked and Shanthikumar 2007, Section 4.B). For distributions F and G, we have $G\le_* F$ if and only if $F^{-1}(u)/G^{-1}(u)$ is (weakly or strictly) increasing in u. The star order is also related to the dispersive order $\le_{disp}$: for two nonnegative random variables X and Y, $X\le_* Y$ is equivalent to $\log X\le_{disp}\log Y$ (Shaked and Shanthikumar 2007, Theorem 4.B.1); that is, $\log Y$ has a higher variance than $\log X$.

Lemma EC.1. *If $G(c)\le c$ for all $c\in[0,1]$ and $G\le_* F$, then the optimal price p^* and density c^* must satisfy $G(c^*)\le F(p^*)\le \bar G(c^*)$.*

After restricting our attention to prices and densities satisfying the inequalities in Lemma EC.1, we can simplify the demand function (3) to

$$D(p,c) = \bar{F}(p)\left(\lambda + (1-\lambda)\,\bar{G}(c)\right) - (\lambda-\alpha)G(c). \tag{EC.1}$$

Therefore, for distributions F and G satisfying the hypotheses of Lemma EC.1, the provider's problem (5) can be reduced to

$$\begin{aligned} \max_{p,c\in[0,1]} \; & R(p,c) = p\min\{c, D(p,c)\} \\ \text{s.t. } & G(c) \le F(p) \le \bar{G}(c), \end{aligned} \tag{EC.2}$$

where the second constraint applies by Lemma EC.1 and $D(p,c)$ is defined by (EC.1).

Using Lemma 1, we can further reduce the provider's problem (EC.2) to a single-variable optimization problem where we express price as a function of density. Recall that by Lemma 1, we must have $c = D(p,c)$ at the optimal solution and a unique price $p(c)$ exists for each $c \in [0,1]$ such that $c = D(p(c),c)$. We can thus restrict our attention to price-density pairs of the form $(c, p(c))$. Using (EC.1), setting $c = D(p,c)$ and rearranging gives

$$F(p) = 1 - \frac{(\lambda-\alpha)G(c) + c}{1-(1-\lambda)G(c)}, \tag{EC.3}$$

which implies

$$p(c) = F^{-1}\left(1 - \frac{(\lambda-\alpha)G(c) + c}{1-(1-\lambda)G(c)}\right). \tag{EC.4}$$

Using equations (EC.2)–(EC.4) and Lemma 1, the provider's problem (EC.2) thus reduces to

$$\max_{0\le c\le 1} R(c) = p(c)c \quad \text{subject to} \quad G(c) \le F\left(p(c)\right) \le \bar{G}(c), \tag{EC.5}$$

where $p(c)$ is given in equation (EC.4). This reduction allows us to prove the following proposition, in which we use the probability density functions $f(p)$ and $g(c)$ corresponding to the CDFs $F(p)$ and $G(c)$, respectively.

Proposition EC.1. *If $G(c) \le c$ for all $c \in [0,1]$, $f(p)$ is weakly increasing in p, and $G \le_* F$, then the provider's optimal density, sales volume, and revenue are all increasing in the degree of positive dependence α.*

A.3. Proofs of Results for General Distributions

Proof of Lemma EC.1. By arguments in the proof of Lemma 1, $p = 0$ or $p = 1$ cannot be optimal, nor can $c = 0$ or $c = 1$. For the remainder of the proof, we consider prices and densities with $0 < p, c < 1$, which is also equivalent to $0 < F(p), G(c) < 1$ since the CDFs are assumed strictly increasing on [0,1].

First, let us show that it is optimal to set $G(c) \le F(p)$. Consider an arbitrary pair of values $a_1, a_2 \in (0,1)$ with $a_1 < a_2$. For $i \in \{1,2\}$, define $p_i := F^{-1}(a_i)$ and $c_i := G^{-1}(a_i)$, and observe that $p_1 < p_2$ and $c_1 < c_2$ because F^{-1} and G^{-1} are increasing functions.

From (3), we have

$$D_{12} := D(p_1,c_2) = \alpha\min\{1-a_1, 1-a_2\} + (1-\lambda)(1-a_1)(1-a_2) + (\lambda-\alpha)\max\{0, 1-a_1-a_2\} = D(p_2,c_1).$$

Case 1: $D_{12} \le c_1 < c_2$. We have $R(p_1,c_2) = p_1 D_{12} < p_2 D_{12} = R(p_2,c_1)$.

Case 2: $c_1 < D_{12}$. From (4) and the definitions of p_1, p_2, c_1, c_2, we have

$$R(p_1,c_2) = p_1\min\{D_{12}, c_2\} = F^{-1}(a_1)\min\{D_{12}, G^{-1}(a_2)\} \quad \text{and} \quad R(p_2,c_1) = p_2 c_1 = F^{-1}(a_2)G^{-1}(a_1).$$

Combined with $G \le_* F$, this implies

$$R(p_1,c_2) = F^{-1}(a_1)\min\{D_{12}, G^{-1}(a_2)\} \le F^{-1}(a_1)G^{-1}(a_2) \le F^{-1}(a_2)G^{-1}(a_1) = R(p_2,c_1),$$

where the last inequality follows from the definition of the star order $\le_*$ given in Appendix A.2.

Now, suppose (p_1,c_2) is optimal (which is only possible if the inequalities above hold at equality). Then, by Lemma 1, we must have $D(p_1,c_2) = c_2$. Given $G \le_* F$, we have already shown that $R(p_1,c_2) \le R(p_2,c_1)$. By Lemma 1, for a price of p_2, there exists a unique density $c(p_2)$ with $D(p_2, c(p_2)) = c(p_2)$. Since $D(p_2, c_1) = D(p_1, c_2) = c_2 > c_1$, we must have $c(p_2) > c_1$ because $D(p, c)$ is strictly decreasing in c for $p \in (0,1)$. We then have $R(p_1, c_2) \le R(p_2, c_1) = p_2 c_1 < p_2 c(p_2) = R(p_2, c(p_2))$, contradicting the optimality of (p_1, c_2). Thus, in both Case 1 and Case 2 we have $R(p_1, c_2) < R(p_2, c_1)$, i.e., strictly higher revenue is achieved by setting price p_2 and density c_1 (where $G(c_1) = a_1 < a_2 = F(p_2)$) than by setting price p_1 and density c_2 (where $G(c_2) = a_2 > a_1 = F(p_1)$). Since a_1 and a_2 were chosen arbitrarily, it therefore cannot be optimal to set the price and density such that $G(c) > F(p)$, implying that the optimal solution must satisfy $G(c^*) \le F(p^*)$.

We showed that the optimal solution must satisfy $G(c^*) \le F(p^*)$. So, to prove that the optimal solution must satisfy $G(c^*) \le F(p^*) \le \bar{G}(c^*)$, it suffices to show that a solution with both $G(c) \le F(p)$ and $\bar{G}(c) < F(p)$ cannot be optimal.

Suppose $G(c) \le F(p)$ and $\bar{G}(c) < F(p)$. In this case, equation (3) implies

$$\begin{aligned} D(p,c) &= \alpha \min\{\bar{F}(p), \bar{G}(c)\} + (1-\lambda)\bar{F}(p)\bar{G}(c) + (\lambda - \alpha)\max\{1 - F(p) - G(c), 0\} \\ &= \bar{F}(p)((1-\lambda)\bar{G}(c) + \alpha). \end{aligned}$$

Observing that $\bar{G}(c) < F(p)$ implies $\bar{F}(p) < G(c)$, we then have

$$D(p,c) = \bar{F}(p)((1-\lambda)\bar{G}(c) + \alpha) \le \bar{F}(p)((1-\lambda)\bar{G}(c) + \lambda) = \bar{F}(p)(\bar{G}(c) + \lambda G(c)) \le \bar{F}(p) < G(c) \le c, \tag{EC.6}$$

where the last inequality follows from the hypothesis of the lemma. Equation (EC.6) implies that if $G(c) \le F(p)$ and $\bar{G}(c) < F(p)$, then $D(p,c) < c$, which by Lemma 1 cannot be optimal. We conclude that the optimal solution must satisfy $G(c^*) \le F(p^*) \le \bar{G}(c^*)$. □

Proof of Proposition EC.1. We show the result for revenue first, then density, and finally sales.

- We first show that the optimal revenue is increasing in α.

By equations (EC.1) and (EC.2), the objective function is increasing in α for any given (p, c), while the constraint is independent of α. Hence, we obtain the result.

- We next show that the optimal density is increasing in α.

We use the reduced problem (EC.5) to accomplish this. First, we prove that as α increases, the set defining the feasible region—i.e., densities c such that $G(c) \le F(p(c)) \le 1 - G(c)$—increases, that is, the lower bound of the feasible region stays the same while the upper bound increases in α. Second, we show that the revenue function is supermodular and has increasing differences in (c, α). These two results imply that the optimal density is increasing in α by Topkis (1998, Theorem 2.8.1).

Using the formula for $p(c)$ given in (EC.4), the constraint in (EC.5) can be rearranged to give

$$G(c) - (1-\lambda)G(c)^2 \le 1 + \alpha G(c) - G(c) - c \le 1 + \lambda G(c) + (1-\lambda)G(c)^2 - 2G(c). \tag{EC.7}$$

Since $G(c) \le c$ and $\alpha < \lambda$, the second inequality in (EC.7) holds for any value of c and α; equivalently, for $p(c)$ given in (EC.4), we have $F(p(c)) \le \bar{G}(c)$ for all $c \in [0,1]$. Rearranging the first inequality in (EC.7) yields the equivalent inequality

$$S_\alpha(c) := c + 2G(c) - \alpha G(c) - (1-\lambda)G(c)^2 \le 1. \tag{EC.8}$$

We have $S_\alpha(0) = 0 < 1$, $S_\alpha(1) = 3 - \alpha - (1-\lambda) > 1$, and

$$S'_\alpha(c) = 1 + g(c)(2 - \alpha - 2(1-\lambda)G(c)) > 0. \tag{EC.9}$$

Therefore, the function $S_\alpha(c)$ is increasing in c for $c \in [0,1]$ and crosses 1 within this interval. There must then exist a unique $\bar{c}_\alpha$ such that $S_\alpha(c) \le 1$ for any $c \in [0, \bar{c}_\alpha]$ and $S_\alpha(c) > 1$ for $c > \bar{c}_\alpha$. Since $S_\alpha(c) \le 1$ is equivalent to $G(c) \le F(p(c))$ and $F(p(c)) \le 1 - G(c)$ for all $c \in [0,1]$, the single inequality $c \le \bar{c}_\alpha$ gives the same feasible set as the constraint in (EC.5).

Now, for α_1, α_2 with $\alpha_1 < \alpha_2$, let $\bar{c}_{\alpha_1}$ ($\bar{c}_{\alpha_2}$) be the density satisfying $S_{\alpha_1}(\bar{c}_{\alpha_1}) = 1$ (resp. $S_{\alpha_2}(\bar{c}_{\alpha_2}) = 1$). Because $0 < \bar{c}_{\alpha_1} < 1$ and $\partial S_\alpha(c)/\partial\alpha < 0$ for $c \in (0,1]$ by (EC.8), we have $S_{\alpha_2}(\bar{c}_{\alpha_1}) < 1$. Therefore, since $S_\alpha(c)$ increases in c by

(EC.9), we must have $\bar{c}_{\alpha_1} < \bar{c}_{\alpha_2}$. Hence, for arbitrary $c_1 \in [0, \bar{c}_{\alpha_1}]$ and $c_2 \in [0, \bar{c}_{\alpha_2}]$, we have $\min\{c_1, c_2\} \in [0, \bar{c}_{\alpha_1}]$ and $\max\{c_1, c_2\} \in [0, \bar{c}_{\alpha_2}]$, so $[0, \bar{c}_{\alpha_1}] \le [0, \bar{c}_{\alpha_2}]$, i.e., the feasible set increases in α in the sense of Topkis (1998, Section 2.4).

Next, let us show that the revenue function is supermodular in (c, α). From (EC.5), we have $R'(c) = p(c) + cp'(c)$, and we can show that both $p(c)$ and $p'(c)$ increase in α for given c. The former follows from direct inspection of (EC.4). Second, we have $\mathrm{d}p/\mathrm{d}c \le 0$ from a direct inspection of (EC.3). Using equations (EC.3) and (EC.4), direct computation reveals

$$\frac{\mathrm{d}p(c)}{\mathrm{d}\alpha} = \underbrace{\frac{1}{f(p(c))}}_{\uparrow c} \underbrace{\frac{G(c)}{1-(1-\lambda)G(c)}}_{\uparrow c} \implies \frac{\partial p(c)}{\partial \alpha \partial c} \ge 0, \tag{EC.10}$$

where $1/f(p(c))$ is increasing in c because $f(p(c))$ is increasing in $p(c)$ and $p(c)$ is decreasing in c. Therefore, because both $p(c)$ and $p'(c)$ are increasing in α, we have that $R'(c) = p(c) + cp'(c)$ is also increasing in α, which together with the increasing feasible region implies that R is supermodular in (c, α). Topkis (1998, Theorem 2.8.1) then gives us that the optimal density increases in α.

• Finally, that the optimal density is increasing in turn implies that the sales also increase in α since we must have $c^* = D(p, c^*)$ by Lemma 1. □

B. Proofs for Section 4.1

Proof of Lemma 2. We have $G(c) = c$, which implies $G(c) \le c$. We also have $G \le_* F$ because $F^{-1}(u)/G^{-1}(u) = 1$ is weakly increasing in u (see the definition of the star order in Appendix A). Therefore, the result follows immediately from Lemma EC.1. □

Proof of Proposition 1. As shown in the main body, we can find the solution to (5) by solving the reduced problem (9). Differentiating the objective function $R(c)$ in (9) gives the first-order condition (FOC)

$$c = \frac{2-\alpha \pm \sqrt{(2-\alpha)\lambda + \alpha^2 - 3\alpha + 2}}{(1-\lambda)(2-\alpha)}. \tag{EC.11}$$

The expression under the radical is bounded below by $2 - 2\alpha > 0$, implying that the FOC solutions are real. The larger solution can be seen to be strictly larger than 1. We can thus ignore it and focus on the smaller solution, which for brevity we refer to as merely as "the solution" or "the solution to (EC.11)" for the rest of the proof. The second derivative of the objective function is

$$R''(c) = -\frac{2(1+\lambda-\alpha)}{(1-(1-\lambda)c)^3} < 0,$$

where the inequality holds for all $0 \le c \le 1$. Thus, the objective function is concave and the FOC (EC.11) is sufficient for a global maximum, provided that the solution satisfies the constraint in (9), namely $c \le p(c) \le 1 - c$, with $p(c)$ given in (8). By the proof of Proposition EC.1, we have $p(c) \le 1 - c$ for all $c \in [0, 1]$. The remaining inequality $c \le p(c)$ can be shown to be equivalent to

$$c \le \frac{3-\alpha-\sqrt{4\lambda+\alpha^2-6\alpha+5}}{2(1-\lambda)}. \tag{EC.12}$$

Because $0 < \alpha < \lambda < 1$, we have $(1-\alpha)^2(1-\lambda)(1+\lambda-\alpha) \ge 0$. After some algebraic manipulation, this gives

$$(3-\alpha)(2-\alpha) - 2(2-\alpha) \ge (2-\alpha)\sqrt{4\lambda+\alpha^2-6\alpha+5} - 2\sqrt{(2-\alpha)\lambda+\alpha^2-3\alpha+2},$$

which is equivalent to

$$\frac{2-\alpha-\sqrt{(2-\alpha)\lambda+\alpha^2-3\alpha+2}}{(1-\lambda)(2-\alpha)} \le \frac{3-\alpha-\sqrt{4\lambda+\alpha^2-6\alpha+5}}{2(1-\lambda)},$$

i.e., the solution to (EC.11) satisfies (EC.12).

Because $\lambda < 1$ and $\alpha < 1$, we have $\sqrt{(2-\alpha)\lambda+\alpha^2-3\alpha+2} \le \sqrt{2-\alpha+\alpha^2-3\alpha+2} = \sqrt{(2-\alpha)^2} = 2-\alpha$, and this implies $(2-\alpha-\sqrt{(2-\alpha)\lambda+\alpha^2-3\alpha+2})/((1-\lambda)(2-\alpha)) \ge 0$. So, the solution to (EC.11) satisfies the constraint $c \le p(c) \le 1-c$ and also falls within the domain of the problem (9), i.e., the interval $[0, 1]$. We conclude that the optimal density c^* for (9) and thus also for (5) is given by the solution to (EC.11).

Finally, substituting c^* from (EC.11) into (8) gives the optimal price, namely

$$p^* = p(c^*) = \frac{1+\alpha c^* - 2c^*}{1-(1-\lambda)c^*} = \frac{2-\alpha-\sqrt{(2-\alpha)\lambda+\alpha^2-3\alpha+2}}{1-\lambda},$$

and substituting c^* and $p(c^*)$ into (5) or into (9) yields the optimal revenue in the proposition statement. □

Proof of Corollary 1. We prove the first part of this result directly by taking the derivative of p^* with respect to α. Differentiating the expression for p^* given in Proposition 1, we have

$$\frac{\mathrm{d}p^*}{\mathrm{d}\alpha} = \frac{(3-2\alpha+\lambda)-2\sqrt{(2-\alpha)\lambda+\alpha^2-3\alpha+2}}{(1-\lambda)\sqrt{(2-\alpha)\lambda+\alpha^2-3\alpha+2}},$$

where the expression under the radicals is positive as argued after equation (EC.11). The denominator of the above derivative is positive since $\lambda<1$. We also have

$$(3-2\alpha+\lambda)^2-4\left((2-\alpha)\lambda+\alpha^2-3\alpha+2\right)=(1-\lambda)^2>0$$
$$\Longrightarrow 3-2\alpha+\lambda>2\sqrt{(2-\alpha)\lambda+\alpha^2-3\alpha+2},$$

implying that the numerator is positive as well. As both the denominator and the numerator are positive, we have $\mathrm{d}p^*/\mathrm{d}\alpha>0$; that is, the optimal price is increasing in α.

For the monotonicity of density, sales, and revenue, recall from the proof of Lemma 2 that we have $G\leq_* F$ and $G(c)=c\leq c$. Additionally, the uniform density $f(p)$ is constant and hence weakly increasing in p. Thus, the conditions of Proposition EC.1 are satisfied, and the result follows. □

Proof of Corollary 2. Consider a fixed but arbitrary $0<\lambda<1$. First, with uniformly distributed valuations, the provider's benchmark problem (6) becomes $\max_p R^\circ(p)=p(1-p)$, whose optimal solution is $p^\circ=1/2$. When $\alpha=0$, the optimal price p^* becomes $p^*=(2-\sqrt{2+2\lambda})/(1-\lambda)$. We have $(1-\lambda)^2\geq 0$, which is equivalent to $(3+\lambda)^2\geq 4(2+2\lambda)$. Taking the square root of both sides of this inequality and rearranging yields $4-2\sqrt{2+2\lambda}\geq 1-\lambda$, which implies $p^*=(2-\sqrt{2+2\lambda})/(1-\lambda)\geq 1/2=p^\circ$. Since we know from Corollary 1 that p^* is increasing in α, we will also have $p^*\geq p^\circ$ for any $0<\alpha<\lambda$.

In the benchmark problem, the optimal density is equal to the demand at the optimal price, i.e., $c^\circ=1-p^\circ=1/2$. Lemma 2 and the above derivations then imply $c^*\leq 1-p^*\leq 1/2=c^\circ$. □

C. Proofs for Section 4.2

Proof of Lemma 3. The result follows immediately from Lemma EC.1. To see this, note that $G\leq_* F$ because $F^{-1}(u)/G^{-1}(u)=u^{1-1/k}$ increases in u (see the definition of the star order in Appendix A), and since $k\geq 1$, we have $G(c)=c^k\leq c$. Thus, the conditions of Lemma EC.1 are satisfied, and the result follows. □

Proof of Proposition 2. As shown in the main body, we can find the solution to (5) by solving the reduced problem (12). Differentiating the objective function $R(c)$ in (12) gives the first-order-condition (FOC)

$$R'(c)=\frac{k(1-\lambda)c^k\left(\alpha c^k-c^k-c+1\right)}{(1-(1-\lambda)c^k)^2}+\frac{\alpha c^k-c^k-c+1}{1-(1-\lambda)c^k}+\frac{c\left(\alpha k c^{k-1}-kc^{k-1}-1\right)}{1-(1-\lambda)c^k}=0. \quad \text{(EC.13)}$$

We have $1-(1-\lambda)c^k>0$ since $\lambda>0$ and $c^k\leq 1$. Therefore, we can multiply (EC.13) through by $(1-(1-\lambda)c^k)^2$ to get the equivalent FOC

$$w(c):=1-2c+(1-\alpha)(1-\lambda)c^{2k}-(k-2)(1-\lambda)c^{k+1}+c^k(\alpha+\alpha k-k\lambda+\lambda-2)=0. \quad \text{(EC.14)}$$

Note that $w(c)$ has the same sign as $R'(c)$. Differentiating a second time gives

$$\begin{aligned} R''(c)=&-\frac{c^{k+1}(1-\lambda)(\left(k^2+3k-4\right)+\left(k^2-3k+2\right)(1-\lambda)c^{k+1})}{c\,(1-(1-\lambda)c^k)^3}\\ &-\frac{(k-1)k(1-\lambda)(\lambda-\alpha)c^{2k}+k(k+1)(\lambda-\alpha)c^k+2c}{c\,(1-(1-\lambda)c^k)^3}<0. \end{aligned} \quad \text{(EC.15)}$$

Therefore, the FOC (EC.14) is sufficient for the global maximum, provided that it satisfies $c^k \le p(c) \le 1 - c^k$ for $p(c)$ defined in (13). By the proof of Proposition EC.1, we have $p(c) \le 1 - c^k$ for all $c \in [0,1]$. The left-hand inequality $c^k \le p(c)$ can be rearranged to give

$$c + 2c^k - \alpha c^k - (1-\lambda)c^{2k} \le 1, \tag{EC.16}$$

the LHS of which is easily shown to be increasing in c for $c \in [0,1]$. Because the LHS of (EC.16) is monotonic, equal to 0 for $c = 0$, and equal to $2 + \lambda - \alpha > 1$ for $c = 1$, there exists a unique $\bar{c}_\alpha \in (0,1)$ such that (EC.16) holds at equality, i.e., $\bar{c}_\alpha$ is the unique solution to

$$1 - \bar{c}_\alpha - (2-\alpha)\bar{c}_\alpha^k + (1-\lambda)\bar{c}_\alpha^2 k = 0. \tag{EC.17}$$

The set of densities satisfying (EC.16) can thus be expressed as the interval $[0, \bar{c}_\alpha]$.

From (EC.13), we have $R'(0) = 1$, so concavity of the objective function implies $c^* > 0$. We proceed to show that $R'(\bar{c}_\alpha) < 0$. Evaluating at $\bar{c}_\alpha$ and simplifying using equation (EC.17), we get

$$w(\bar{c}_\alpha) = -\bar{c}_\alpha - \bar{c}_\alpha^{1+k}(1-\lambda)(k-2) - \alpha\bar{c}_\alpha^{2k}(1-\lambda) - \bar{c}_\alpha^k(k(\lambda-\alpha) - \lambda).$$

If $k \ge 2$, then we have

$$w(\bar{c}_\alpha) = -\bar{c}_\alpha - \bar{c}_\alpha^{1+k}(1-\lambda)(k-2) - \alpha\bar{c}_\alpha^{2k}(1-\lambda) - \bar{c}_\alpha^k(k(\lambda-\alpha)-\lambda) \le -\bar{c}_\alpha - \bar{c}_\alpha^k(k(\lambda-\alpha)-\lambda).$$

Combined with $k(\lambda-\alpha) - \lambda \ge -1$, the above implies $w(\bar{c}_\alpha) \le -\bar{c}_\alpha - \bar{c}_\alpha^k(k(\lambda-\alpha)-\lambda) \le -\bar{c}_\alpha + \bar{c}_\alpha^k < 0$, and thus we have $R'(\bar{c}_\alpha) < 0$ since w and R' have the same sign.

If instead $1 \le k \le 2$, then we have

$$\begin{aligned} w(\bar{c}_\alpha) &= -\bar{c}_\alpha - \bar{c}_\alpha^{1+k}(1-\lambda)(k-2) - \alpha\bar{c}_\alpha^{2k}(1-\lambda) - \bar{c}_\alpha^k(k(\lambda-\alpha)-\lambda) \\ &\le -\bar{c}_\alpha - \bar{c}_\alpha^{1+k}(1-\lambda)(k-2) - \bar{c}_\alpha^k(k(\lambda-\alpha)-\lambda) \\ &= -\bar{c}_\alpha + \bar{c}_\alpha^k(\bar{c}_\alpha(1-\lambda)(2-k) + \lambda - k(\lambda-\alpha)). \end{aligned}$$

As $\bar{c}_\alpha(1-\lambda)(2-k) + \lambda - k(\lambda-\alpha)$ is decreasing in $k \in [1,2]$, we then get

$$\begin{aligned} w(\bar{c}_\alpha) &\le -\bar{c}_\alpha + \bar{c}_\alpha^k(\bar{c}_\alpha(1-\lambda)(2-k) + \lambda - k(\lambda-\alpha)) \\ &= -\bar{c}_\alpha + \bar{c}_\alpha^k(\bar{c}_\alpha(1-\lambda)+\alpha) < -\bar{c}_\alpha + \bar{c}_\alpha^k(1-\lambda+\lambda) = -(\bar{c}_\alpha - \bar{c}_\alpha^k) \le 0. \end{aligned}$$

That $w(\bar{c}_\alpha) < 0$ implies $R'(\bar{c}_\alpha) < 0$ in this case also. Moreover, in both cases we have $w(\bar{c}_\alpha) < 0$, and since $w(0) = 1$, a solution must exist to (EC.14), and moreover this solution is unique because $R(c)$ is strictly concave by (EC.15).

We conclude that the solution to the FOC (EC.14) (equiv. (EC.13)) satisfies $c^k \le p(c) \le 1 - c^k$ and is thus feasible for the reduced problem (12). Since the objective function is concave, this solution achieves the global maximum, and we have that the optimal density c^* is given by the unique solution to (EC.14), which is the same as (14) in the proposition statement. ☐

Proof of Corollary 3. Since customer valuations are uniformly distributed, we know from the proof of Corollary 2 that the benchmark price $p^\circ = 1/2$ and benchmark density $c^\circ = 1 - p^\circ = 1/2$.

We first show $c^* \le 1/2$. To begin, observe that

$$2 - \alpha - \lambda + k(\lambda-\alpha) - 2(1-\alpha)(1-\lambda)c^k \ge 2 - \alpha - \lambda - 2(1-\alpha)(1-\lambda) = \alpha + \lambda - 2\alpha\lambda \ge (\lambda-\alpha)^2. \tag{EC.18}$$

We also have

$$\begin{aligned} 2 - (2-k)(1+k)(1-\lambda)c^k &\ge 2 - (2-k)(1+k) = k^2 - k \ge 0 \quad \text{for } k \in [1,2], \\ \text{and} \quad 2 - (2-k)(1+k)(1-\lambda)c^k &> 2 \quad \text{for } k > 2 \end{aligned} \tag{EC.19}$$

By (EC.14), differentiating $w(c)$ then gives

$$w'(c) = -(2 - (2-k)(1+k)(1-\lambda)c^k) - kc^{k-1}(2 - \alpha - \lambda + k(\lambda-\alpha) - 2(1-\alpha)(1-\lambda)c^k) \le 0,$$

where the inequality follows from (EC.18) and (EC.19). That $w(c)$ is decreasing in c implies that at the solution $w(c)=0$, the function is crossing from positive to negative. Note that this argument is also valid for $\alpha=\lambda$ (which we do not formally include in our other analysis but is useful in this case because of monotonicity), and that for $\alpha=\lambda$, equation (EC.14) reduces to

$$w(c)=1-2c+(1-\lambda)^2c^{2k}-(k-2)(1-\lambda)c^{k+1}+c^k(2\lambda-2)=0. \tag{EC.20}$$

By (EC.20), for $\alpha=\lambda$ we have $w(0)=1>0$ and $w(1)\le -1-\lambda(1-\lambda)<0$, implying together with the monotonicity of $w(c)$ that a unique solution exists to $w(c)=0$ for $\alpha=\lambda$, as it does for $\alpha<\lambda$ as shown in the proof of Proposition 2.

Inspection of (EC.14) reveals that $w(c)$ is increasing in α for fixed c. To reflect the dependence on α, in this paragraph we augment our notation by writing $w_\alpha(c)$ to signify the particular form of $w(c)$ for a given α, and c^*_α for the solution to $w_\alpha(c)=0$. For $0<\alpha_1\le\alpha_2\le\lambda$, the previous paragraph implies that $w_{\alpha_2}(c^*_{\alpha_1})\ge w_{\alpha_1}(c^*_{\alpha_1})=0$, and therefore $c^*_{\alpha_2}\ge c^*_{\alpha_1}$ because $w_{\alpha_2}(c)$ is decreasing in c. Rearrangement of (EC.20) gives $2c=1+(1-\lambda)c^k((1-\lambda)c^k-(k-2)c-2)\le 1+(1-\lambda)c^k((1-\lambda)c+c-2)\le 1$, which in turn implies that the solution to (EC.20) is bounded above by $1/2=c^\circ$. As we have just shown that c^*_α is increasing in α, this implies that for $\alpha<\lambda$ we will also have $c^*_\alpha\le 1/2=c^\circ$. We now resume writing c^* instead of c^*_α.

We now show the result for the optimal price p^*. That $k\ge 1$ implies

$$z(c^*):=2+k(\lambda-\alpha)-\lambda-\alpha+(k-2)(1-\lambda)c^*-(1-\alpha)(1-\lambda)(c^*)^k\ge 2-2\alpha-(1-\lambda)(c^*+(1-\alpha)c^*). \tag{EC.21}$$

Then, by Proposition 2, the optimal density c^* must satisfy

$$1-2c^*-z(c^*)(c^*)^k=0. \tag{EC.22}$$

As argued above, we have $c^*\le 1/2$, which together with equation (EC.21) implies

$$z(c^*)\ge 2-2\alpha-(1-\lambda)(c^*+(1-\alpha)c^*)\ge 2-2\alpha-(1-\lambda)=1+\lambda-2\alpha\ge 0. \tag{EC.23}$$

Equations (EC.22) and (EC.23) together imply $1-2c^*-(1+\lambda-2\alpha)(c^*)^k\ge 1-2c^*-z(c^*)(c^*)^k=0$, and $1-2c^*-(1+\lambda-2\alpha)(c^*)^k\ge 0$ can be rearranged to give $2(1-(1-\alpha)(c^*)^k-c)\ge 1-(1-\lambda)(c^*)^k$. Using the expression for p^* in Proposition 2, the above inequality implies

$$p^*=\frac{1-(1-\alpha)(c^*)^k-c}{1-(1-\lambda)(c^*)^k}\ge\frac{1}{2}=p^\circ. \quad\square$$

Proof of Proposition 3. The result follows immediately from Proposition EC.1. As shown in the proof of Lemma 3, the distributions G and F satisfy $G\le_* F$ and $G(c)\le c$, and the uniform density $f(p)$ is constant and hence weakly increasing in p. Thus, the conditions of Proposition EC.1 are satisfied, and the result follows. $\square$

Proof of Proposition 4. The proof has three parts. First, we show that the optimal price p^* is quasi-convex in α by showing that its derivative cannot change from positive to negative. Second and third, respectively, we characterize $\overline{k}$ and $\underline{k}$ such that p^* is decreasing in α for any $k\ge\overline{k}$ and increasing in α for any $k\le\underline{k}$.

- We first show that the optimal price $p^*=p(c^*)$ is quasi-convex in α.

By Proposition 2, the optimal price is

$$p^*=p(c^*)=\frac{1-(1-\alpha)(c^*)^k-c^*}{1-(1-\lambda)(c^*)^k},$$

where c^* is defined in the same proposition as the solution to (14). By differentiating $p(c^*)$ in α, we get

$$\frac{\mathrm{d}p(c^*)}{\mathrm{d}\alpha}=\frac{(c^*)^k}{1-(1-\lambda)(c^*)^k}+\frac{\mathrm{d}c^*}{\mathrm{d}\alpha}\left(\frac{k(1-\lambda)(c^*)^{k-1}\left(1-(1-\alpha)(c^*)^k-c^*\right)}{\left(1-(1-\lambda)(c^*)^k\right)^2}+\frac{\alpha k(c^*)^{k-1}-k(c^*)^{k-1}-1}{1-(1-\lambda)(c^*)^k}\right). \tag{EC.24}$$

By the Implicit Function Theorem, equation (14) implies

$$\frac{\mathrm{d}c^*}{\mathrm{d}\alpha}=\frac{\left(1-(1-\lambda)(c^*)^k\right)\left(1+k-(1-\lambda)(c^*)^k\right)(c^*)^{k+1}}{L_1(c^*)+L_2(c^*)},$$

where

$$L_1(c^*) = (1-\lambda)(k^2+3k-4)(c^*)^{k+1} + (\lambda-\alpha)k(k+1)(c^*)^k + 2c^*,$$
$$\text{and}\quad L_2(c^*) = (1-\lambda)^2(k^2-3k+2)(c^*)^{2k+1} + (1-\lambda)(\lambda-\alpha)(k-1)k(c^*)^{2k}.$$

Inserting our expression for $\mathrm{d}c^*/\mathrm{d}\alpha$ into (EC.24), we get

$$\frac{\mathrm{d}p(c^*)}{\mathrm{d}\alpha} = \frac{\big((1-\lambda)^2(k-1)^2(c^*)^{2k} + (1-\lambda)(\lambda-\alpha)k^2(c^*)^{2k-1} + (1-\lambda)(3k-2)(c^*)^k + 1-k\big)(c^*)^{k+1}}{(1-(1-\lambda)(c^*)^k)(L_1(c^*)+L_2(c^*))}. \tag{EC.25}$$

It is straightforward to show that

$$\begin{aligned} L_1(c^*) + L_2(c^*) &= 2(1-(1-\lambda)(c^*)^k)^2 + (1-\lambda)\big(3k+k^2-(1-\lambda)(3k-k^2)(c^*)^k\big)(c^*)^k \\ &\quad + k(\lambda-\alpha)\big(1+k-(1-k)(1-\lambda)(c^*)^k\big)(c^*)^k \\ &> 0. \end{aligned} \tag{EC.26}$$

Since $(c^*)^{k+1} > 0$ and $1-(1-\lambda)(c^*)^k > 0$, equations (EC.25) and (EC.26) imply that $\mathrm{d}p(c^*)/\mathrm{d}\alpha$ has the same sign as

$$\Theta(c^*) = (1-\lambda)^2(k-1)^2(c^*)^{2k} + (1-\lambda)(\lambda-\alpha)k^2(c^*)^{2k-1} + (1-\lambda)(3k-2)(c^*)^k + 1-k. \tag{EC.27}$$

Now, we want to show that the sign of $\Theta(c^*)$ cannot change from positive to negative as α increases. We know from Proposition 3 that c^* increases in α. Therefore, it is sufficient to show that the sign of $\Theta(c^*)$ cannot change from positive to negative as c^* increases. To do so, we use equation (14) to express α in terms of c^*.

Isolating α in equation (14) yields

$$\alpha = \frac{-1+2c^* + (2-(1-k)\lambda)(c^*)^k - (2-k)(1-\lambda)(c^*)^{k+1} - (1-\lambda)(c^*)^{2k}}{(c^*)^k(1+k-(1-\lambda)(c^*)^k)}.$$

Inserting this expression for α into (EC.27), we get

$$\Theta(c^*) = \frac{\big(1-(1-\lambda)(c^*)^k\big)^2\big(k^2(1-\lambda)(c^*)^{k-1} - (k-1)^2(1-\lambda)(c^*)^k - k^2+1\big)}{1+k-(1-\lambda)(c^*)^k}. \tag{EC.28}$$

Because $(1-(1-\lambda)(c^*)^k)^2 > 0$ and $1+k-(1-\lambda)(c^*)^k > 0$, the sign of $\Theta(c^*)$ in (EC.28) is the same as the sign of

$$v(c^*) = k^2(1-\lambda)(c^*)^{k-1} - (k-1)^2(1-\lambda)(c^*)^k - k^2 + 1. \tag{EC.29}$$

We have

$$v'(c^*) = (1-\lambda)k(k-1)(c^*)^{k-2}(k-(k-1)c^*) \geq 0.$$

Since $v(c^*)$ is increasing in c^*, its sign and therefore the sign of $\Theta(c^*)$ cannot change from positive to negative as c^* increases. Since c^* increases in α, this implies that the sign of $\Theta(c^*)$ cannot change from positive to negative as α increases. Finally, since $\mathrm{d}p(c^*)/\mathrm{d}\alpha$ has the same sign as $\Theta(c^*)$, we conclude that the sign of $\mathrm{d}p(c^*)/\mathrm{d}\alpha$ also cannot change from positive to negative as α increases. Therefore, the optimal price $p(c^*)$ is quasi-convex in α.

• We next show that there exists $\overline{k} \in [1,2]$ such that p^* is decreasing in α for any $k \geq \overline{k}$. As argued above surrounding the definition of $v(c^*)$ in (EC.29), the sign of $\Theta(c^*)$ is the same as that of $v(c^*)$. By dropping the negative term $-(k-1)^2(1-\lambda)(c^*)^k$, we get

$$v(c^*) \leq k^2(1-\lambda)(c^*)^{k-1} - k^2 + 1 = -k^2(1-(1-\lambda)(c^*)^{k-1}) + 1. \tag{EC.30}$$

Recalling that $c^* \leq 1/2$ by Corollary 3, by (EC.30) we have for $k \geq 2$ that

$$v(c^*) \leq 1 - k^2(1-(1-\lambda)(c^*)^{k-1}) \leq 1 - 4(1-(1-\lambda)c^*) \leq 1 - 4(1-\frac{1}{2}) = -1.$$

Thus, for $k \geq 2$ we have $v(c^*) \leq 0$, which implies that also $\Theta(c^*) \leq 0$ and $\mathrm{d}p(c^*)/\mathrm{d}\alpha \leq 0$ since all three have the same sign. We conclude that the optimal price p^* is decreasing in α for all $k \geq 2$.

Alternatively, suppose that $k \geq \lambda^{-1/2}$. This implies $k^2\lambda \geq 1$, which in turn gives

$$v(c^*) \leq -k^2(1-(1-\lambda)(c^*)^{k-1}) + 1 \leq -k^2(1-(1-\lambda)) + 1 = -k^2\lambda + 1 \leq 0.$$

Thus, similarly, the optimal price p^* is decreasing in α for all $k \geq \lambda^{-1/2}$, which can be a tighter or looser bound than $k = 2$, depending on λ; importantly, we must have $\lambda^{-1/2} \geq 1$ since $\lambda < 1$. We then let $\overline{k} := \min\{2, \lambda^{-1/2}\} \in [1, 2]$, and we have p^* decreasing in α for any $k \geq \overline{k}$, as desired.

• Finally, we show that there exists $\underline{k} \in [1, 2]$ such that p^* is increasing in α for any $k \leq \underline{k}$.

We first find the lower bound of the optimal density c^*. As argued in the proof of Corollary 3, the LHS of (14) (equivalent to (EC.14)), which we denote by $w(c)$, is decreasing in c. Importantly, this monotonicity can be seen by inspection of the argument to be valid also for $\alpha = 0$, and thus a unique solution to $w(c) = 0$ indeed exists for $\alpha = 0$ because in this case we have $w(0) = 1$ and $w(1) = -3 + (1-\alpha)(1-\lambda) - (1-\lambda)(k-2) \leq -2 + 1 - \lambda < 0$. We again augment our notation as c^*_α and w_α. By the monotonicity of $w_\alpha(c)$ in c and the existence of a solution c^*_0 with $w_0(c^*_0) = 0$, we have that the monotonicity of c^*_α extends to c^*_0, i.e., $c^*_{\alpha_2} \geq c^*_{\alpha_1}$ for all $0 \leq \alpha_1 \leq \alpha_2 \leq \lambda$.

For $\alpha = 0$, $w_0(c^*_0) = 0$ can be rearranged to give

$$2c^*_0 = 1 - (c^*_0)^k(2 + (k-2)c^*_0 + \lambda(k-1-(k-2)c^*_0) - (1-\lambda)(c^*_0)^k).$$

Since the RHS of the above equality is decreasing in λ, we can substitute $\lambda = 1$ to get

$$2c^*_0 \geq 1 - (1+k)(c^*_0)^k. \tag{EC.31}$$

The RHS of (EC.31) is increasing in k for fixed c^*_0. To see this, note that its partial derivative in k is

$$\frac{\partial\left(1-(c^*_0)^k(1+k)\right)}{\partial k} = -(c^*_0)^k\left((1+k)\log(c^*_0)+1\right) \geq -(c^*_0)^k((1+k)\log(\frac{1}{2})+1) \geq 0,$$

where the first inequality holds because for any $\alpha > 0$ we have $c^*_0 \leq c^*_\alpha \leq 1/2$, with $c^*_\alpha \leq 1/2$ following from the proof of Corollary 3. Therefore, substituting $k = 1$ into equation (EC.31) gives

$$2c^*_0 \geq 1 - (1+k)(c^*_0)^k \geq 1 - 2c^*_0 \implies c^*_0 \geq \frac{1}{4},$$

which because c^*_α is increasing in α also implies $c^*_\alpha \geq 1/4$ for any $0 < \alpha < \lambda$. We now resume writing c^* instead of c^*_α.

Now, using equation (EC.27) and $c^* \geq 1/4$, respectively, we get

$$\begin{aligned}\Theta(c^*) &\geq (1-\lambda)^2(k-1)^2(c^*)^{2k} + (1-\lambda)(3k-2)(c^*)^k + 1 - k\\ &\geq (1-\lambda)(3k-2)(\frac{1}{4})^k + (1-\lambda)^2(k-1)^2(\frac{1}{4})^{2k} + 1 - k.\end{aligned}$$

Since we are considering $k < \overline{k} \leq 2$, we must have

$$\Theta(c^*) \geq (1-\lambda)(3k-2)(\frac{1}{4})^2 + (1-\lambda)^2(k-1)^2(\frac{1}{4})^4 + 1 - k =: t(k). \tag{EC.32}$$

Because $\Theta(c^*) \geq t(k)$, the inequality $t(k) \geq 0$ is sufficient to get $\mathrm{d}p(c^*)/\mathrm{d}\alpha \geq 0$ because $\Theta(c^*)$ and $\mathrm{d}p(c^*)/\mathrm{d}\alpha$ have the same sign. The inequality $t(k) \geq 0$ can be reduced to

$$k \leq \underline{k} := \frac{\lambda^2 + 22\lambda + 105 - 4\sqrt{\lambda^3 + 33\lambda^2 + 315\lambda + 675}}{(1-\lambda)^2}, \tag{EC.33}$$

and we now have that p^* is increasing in α for any $1 \leq k \leq \underline{k}$, provided that $\underline{k} \in [1, 2]$.

The final step is to confirm that $\underline{k} \in [1, 2]$, where $\underline{k}$ is given in (EC.33). Differentiating $\underline{k}$ in λ gives

$$\frac{\mathrm{d}\underline{k}}{\mathrm{d}\lambda} = -\frac{2}{(1-\lambda)^3}\left(\frac{\lambda^2 + 54\lambda + 201}{\sqrt{\lambda+3}} - 12\lambda - 116\right). \tag{EC.34}$$

We also have

$$\left(\lambda^2 + 54\lambda + 201\right)^2 - (12\lambda + 116)^2(\lambda + 3) = (1-\lambda)(33-\lambda) \geq 0, \tag{EC.35}$$

which is equivalent to $(\lambda^2 + 54\lambda + 201)/\sqrt{\lambda+3} \geq 12\lambda + 116$. Since $-2/(1-\lambda)^3 < 0$, equations (EC.34) and (EC.35) together imply that $\mathrm{d}\underline{k}/\mathrm{d}\lambda \leq 0$, i.e., that $\underline{k}$ is decreasing in λ. Therefore, substituting $\lambda = 0$ into equation (EC.33) gives the highest possible $\underline{k}$, and we get

$$\underline{k} \leq 105 - 4\sqrt{675} \approx 1.08 < 2. \tag{EC.36}$$

To confirm that $\underline{k} \geq 1$, we again use the fact that it is decreasing in λ and take the limit as λ approaches 1. For $\lambda = 1$, (EC.33) is indeterminate because both numerator and denominator become zero, but two successive applications of L'Hôpital's Rule give $\lim_{\lambda \uparrow 1} \underline{k} = 1$. Since $\underline{k}$ is decreasing in λ, this and (EC.36) together imply that we have $1 \leq \underline{k} \leq 105 - 4\sqrt{675} \approx 1.08 < 2$, as desired. □

D. Additional Technical Details and Proofs for Section 5

In this section, we first provide technical details and results leading to a more explicit characterization of the solution to the provider's problem with severely density-sensitive customers considered in Section 5. These results also lead us toward the structural results in the main body of Section 5, which we then prove. Recall that $k<1$ throughout this section.

D.1. Technical Details for Section 5

Similar to Section 4, we can reduce the provider's problem into four sub-problems—each corresponding to one of the strategies FP, FD, PP, or PD—with the density as the single decision variable (by leveraging Lemma 1):

$$\max_{0\le c\le 1} R_{\mathrm{FP}}(c)=p_{\mathrm{FP}}(c)c \quad \text{subject to} \quad c^k\le p_{\mathrm{FP}}(c) \quad \text{and} \quad c^k+p_{\mathrm{FP}}(c)\le 1, \qquad \text{where} \quad p_{\mathrm{FP}}(c):=\frac{1-(1-\alpha)c^k-c}{1-(1-\lambda)c^k}, \tag{FP}$$

$$\max_{0\le c\le 1} R_{\mathrm{FD}}(c)=p_{\mathrm{FD}}(c)c \quad \text{subject to} \quad p_{\mathrm{FD}}(c)\le c^k \quad \text{and} \quad c^k+p_{\mathrm{FP}}(c)\le 1, \qquad \text{where} \quad p_{\mathrm{FD}}(c):=\frac{1-c^k-c}{1-\alpha-(1-\lambda)c^k}, \tag{FD}$$

$$\max_{0\le c\le 1} R_{\mathrm{PP}}(c)=p_{\mathrm{PP}}(c)c \quad \text{subject to} \quad c^k\le p_{\mathrm{PP}}(c) \quad \text{and} \quad c^k+p_{\mathrm{FP}}(c)\ge 1, \qquad \text{where} \quad p_{\mathrm{PP}}(c):=\frac{1-\lambda+\alpha-(1-\lambda)c^k-c}{1-\lambda+\alpha-(1-\lambda)c^k}, \tag{PP}$$

$$\max_{0\le c\le 1} R_{\mathrm{PD}}(c)=p_{\mathrm{PD}}(c)c \quad \text{subject to} \quad p_{\mathrm{PD}}(c)\le c^k \quad \text{and} \quad c^k+p_{\mathrm{PD}}(c)\ge 1, \qquad \text{where} \quad p_{\mathrm{PD}}(c):=\frac{(1-\lambda+\alpha)(1-c^k)-c}{(1-\lambda)(1-c^k)}. \tag{PD}$$

With the next lemma, we characterize the solutions to each of these sub-problems.

Lemma EC.2 **(Properties of Sub-Problems)**. *For $i\in\{\mathrm{FP},\mathrm{FD},\mathrm{PP},\mathrm{PD}\}$, the functions $R_i(c)$ defined in equations* (FP), (FD), (PP), *and* (PD) *are concave, and there exists $c_i^*\in(0,1)$ such that $R_i'(c_i^*)=0$. Furthermore, two unique constants $c_{\mathrm{FB}},c_{\mathrm{PB}}\in(0,1)$ exist such that*

$$c^k \lesseqqgtr p_{\mathrm{FP}}(c) \iff c\lesseqqgtr c_{\mathrm{FB}} \iff c^k\lesseqqgtr p_{\mathrm{FD}}(c), \quad \textit{and} \quad c^k\lesseqqgtr p_{\mathrm{PP}}(c)\iff c\lesseqqgtr c_{\mathrm{PB}}\iff c^k\lesseqqgtr p_{\mathrm{PD}}(c). \tag{EC.37}$$

There also exist unique constants $c_{\mathrm{FPP}}\in(0,1)$ and $c_{\mathrm{FPD}}\in(0,1)$ such that

$$c^k+p_{\mathrm{FP}}(c)\lesseqqgtr 1 \iff c\gtreqqless c_{\mathrm{FPP}} \iff c^k+p_{\mathrm{PP}}(c)\lesseqqgtr 1 \quad \textit{and} \quad c^k+p_{\mathrm{FD}}(c)\lesseqqgtr 1\iff c\gtreqqless c_{\mathrm{FPD}}\iff c^k+p_{\mathrm{PD}}(c)\lesseqqgtr 1, \tag{EC.38}$$

and

$$c_{\mathrm{FPP}}\lesseqqgtr c_{\mathrm{FB}}\iff c_{\mathrm{FPD}}\lesseqqgtr c_{\mathrm{FB}} \quad \textit{and} \quad c_{\mathrm{FPP}}\lesseqqgtr c_{\mathrm{PB}}\iff c_{\mathrm{FPD}}\lesseqqgtr c_{\mathrm{PB}}. \tag{EC.39}$$

Lemma EC.2 implies that $c_{\mathrm{FB}}^k=p_{\mathrm{FP}}(c_{\mathrm{FB}})=p_{\mathrm{FD}}(c_{\mathrm{FB}})$, meaning that at this point *both* price and density are active in regulating the demand (hence the subscript B) and that $R_{\mathrm{FP}}(c_{\mathrm{FB}})=R_{\mathrm{FD}}(c_{\mathrm{FB}})$. This property enables us to group sub-problems (FP) and (FD) to form the combined problem (labeled F because it encompasses both full-coverage strategies) given by

$$\max_{0\le c\le 1} R_{\mathrm{F}}(c) \quad \text{subject to} \quad G(c)+F(p_{\mathrm{F}}(c))\le 1, \tag{F}$$

where

$$R_{\mathrm{F}}(c)=\begin{cases}R_{\mathrm{FP}}(c) \text{ if } c\le c_{\mathrm{FB}}\\ R_{\mathrm{FD}}(c) \text{ if } c\ge c_{\mathrm{FB}},\end{cases} \quad \text{and} \quad p_{\mathrm{F}}(c)=\begin{cases}p_{\mathrm{FP}}(c) \text{ if } c\le c_{\mathrm{FB}}\\ p_{\mathrm{FD}}(c) \text{ if } c\ge c_{\mathrm{FB}}.\end{cases} \tag{EC.40}$$

We also define the relaxed problem $(\widetilde{\mathrm{F}})$ without the constraint as

$$R_{\widetilde{\mathrm{F}}}(c):=\max_{0\le c\le 1} R_{\mathrm{F}}(c). \tag{$\widetilde{\mathrm{F}}$}$$

Lemma EC.2 also implies that $c_{\mathrm{PB}}^k = p_{\mathrm{PP}}(c_{\mathrm{PB}}) = p_{\mathrm{PD}}(c_{\mathrm{PB}})$ and $R_{\mathrm{PP}}(c_{\mathrm{PB}}) = R_{\mathrm{PD}}(c_{\mathrm{PB}})$. We can therefore similarly group sub-problems (PP) and (PD) to form the combined problem (labeled P because it encompasses both partial-coverage strategies) given by

$$\max_{0\le c\le 1} R_{\mathrm{P}}(c) \quad \text{subject to} \quad G(c) + F(p_{\mathrm{P}}(c)) \ge 1, \tag{P}$$

where

$$R_{\mathrm{P}}(c) = \begin{cases} R_{\mathrm{PP}}(c) \text{ if } c \le c_{\mathrm{PB}} \\ R_{\mathrm{PD}}(c) \text{ if } c \ge c_{\mathrm{PB}}, \end{cases} \quad \text{and} \quad p_{\mathrm{P}}(c) = \begin{cases} p_{\mathrm{PP}}(c) \text{ if } c \le c_{\mathrm{PB}} \\ p_{\mathrm{PD}}(c) \text{ if } c \ge c_{\mathrm{PB}}. \end{cases} \tag{EC.41}$$

We also define the relaxed problem $(\widetilde{\mathrm{P}})$ without the constraint as

$$R_{\widetilde{\mathrm{P}}}(c) := \max_{0\le c\le 1} R_{\mathrm{P}}(c). \tag{$\widetilde{\mathrm{P}}$}$$

We denote the optimal solutions of problems (F) and (P) by $(c_{\mathrm{F}}^*, p_{\mathrm{F}}^*)$ and $(c_{\mathrm{P}}^*, p_{\mathrm{P}}^*)$, respectively.

The relaxed problems $(\widetilde{\mathrm{F}})$ and $(\widetilde{\mathrm{P}})$ play an important role in the optimal policy, and with the next two lemmas, we provide a full characterization of the solution to each. We let c_{FP}^*, c_{FD}^*, c_{PP}^*, and c_{PD}^* denote the FOC solutions to the revenue functions of subproblems (FP), (FD), (PP), and (PD), respectively.

Lemma EC.3 **(Solution to Problem $(\widetilde{\mathrm{F}})$).** *Let $k_{\mathrm{FP}} \in (0,1)$ be the unique solution to*

$$1 - \lambda + \alpha - (3 - \alpha - k + \alpha k)c_{\mathrm{FB}}^k + (2-k)(1-\lambda)c_{\mathrm{FB}}^{2k} = 0, \tag{EC.42}$$

and $k_{\mathrm{FD}} \in (0,1]$ the unique solution to

$$1 - \lambda + \alpha - (3 - 2\alpha - k)c_{\mathrm{FB}}^k + (2-k)(1-\lambda)c_{\mathrm{FB}}^{2k} = 0. \tag{EC.43}$$

The optimal solution $c_{\widetilde{\mathrm{F}}}$ to the relaxed problem $(\widetilde{\mathrm{F}})$ is given by

$$c_{\widetilde{\mathrm{F}}} = \begin{cases} c_{\mathrm{FD}}^* & \text{if } k \le k_{\mathrm{FD}} \\ c_{\mathrm{FB}} & \text{if } k_{\mathrm{FD}} < k < k_{\mathrm{FP}} \\ c_{\mathrm{FP}}^* & \text{if } k \ge k_{\mathrm{FP}}, \end{cases} \quad \textit{and} \quad p_{\widetilde{\mathrm{F}}} = \begin{cases} p_{\mathrm{FD}}^* \triangleq p_{\mathrm{FD}}(c_{\mathrm{FD}}^*) & \text{if } k \le k_{\mathrm{FD}} \\ p_{\mathrm{FB}}^* \triangleq p_{\mathrm{FP}}(c_{\mathrm{FB}}) & \text{if } k_{\mathrm{FD}} < k < k_{\mathrm{FP}} \\ p_{\mathrm{FP}}^* \triangleq p_{\mathrm{FP}}(c_{\mathrm{FP}}^*) & \text{if } k \ge k_{\mathrm{FP}}, \end{cases} \tag{EC.44}$$

where $c_{\mathrm{FP}}^*, c_{\mathrm{FD}}^*$ and c_{FB} are defined in Lemma EC.2 and $p_{\mathrm{FP}}(c)$, $p_{\mathrm{FD}}(c)$ are defined in (FP) *and* (FD).

Lemma EC.4 **(Solution to Problem $(\widetilde{\mathrm{P}})$).** *Let $k_{\mathrm{PP}} \in (0,1)$ be the unique solution to*

$$1 - \lambda + \alpha - (2\alpha + (1-\lambda)(3-k))c_{\mathrm{PB}}^k + (2-k)(1-\lambda)c_{\mathrm{PB}}^{2k} = 0, \tag{EC.45}$$

Let $k_{\mathrm{PD}} \in (0,1]$ be the unique solution to

$$1 - \lambda + \alpha - ((1-k)\alpha + (1-\lambda)(3-k))c_{\mathrm{PB}}^k + (2-k)(1-\lambda)c_{34}^{2k} = 0, \tag{EC.46}$$

if $\alpha < 1 - \lambda$, otherwise $k_{\mathrm{PD}} = 0$. The optimal solution $c_{\widetilde{\mathrm{P}}}$ to the relaxed problem $(\widetilde{\mathrm{P}})$ is given by

$$c_{\widetilde{\mathrm{P}}} = \begin{cases} c_{\mathrm{PD}}^* & \text{if } k \le k_{\mathrm{PD}} \\ c_{\mathrm{PB}} & \text{if } k_{\mathrm{PD}} < k < k_{\mathrm{PP}} \\ c_{\mathrm{PP}}^* & \text{if } k \ge k_{\mathrm{PP}}, \end{cases} \quad \textit{and} \quad p_{\widetilde{\mathrm{P}}} = \begin{cases} p_{\mathrm{PD}}^* \triangleq p_{\mathrm{PD}}(c_{\mathrm{PD}}^*) & \text{if } k \le k_{\mathrm{PD}} \\ p_{\mathrm{PB}}^* \triangleq p_{\mathrm{PP}}(c_{\mathrm{PB}}) & \text{if } k_{\mathrm{PD}} < k < k_{\mathrm{PP}} \\ p_{\mathrm{PP}}^* \triangleq p_{\mathrm{PP}}(c_{\mathrm{PP}}^*) & \text{if } k \ge k_{\mathrm{PP}}, \end{cases} \tag{EC.47}$$

where $c_{\mathrm{PP}}^*, c_{\mathrm{PD}}^*$ and c_{PB} are defined in Lemma EC.2 and $p_{\mathrm{PP}}(c)$, $p_{\mathrm{PD}}(c)$ are defined in (PP) *and* (PD).

As shown by the following proposition, a thorough analysis of the shape of each piece of the revenue functions and the location of the boundaries allows us to prove that the global optimal solution can be characterized in terms of the solutions to the relaxed problems $(\widetilde{\mathrm{F}})$ and $(\widetilde{\mathrm{P}})$ given in Lemmas EC.3 and EC.4.

Proposition EC.2 **(Global Optimal Solution: Severe Density Sensitivity).** *Let $c_{\widetilde{\mathrm{F}}}$ and $c_{\widetilde{\mathrm{P}}}$ be the optimal densities for problems $(\widetilde{\mathrm{F}})$ and $(\widetilde{\mathrm{P}})$, respectively. If both $c_{\widetilde{\mathrm{F}}}$ and $c_{\widetilde{\mathrm{P}}}$ are feasible in* (F) *and* (P)*, respectively, then the globally optimal density c^* is given by*

$$c^* = \begin{cases} c_{\widetilde{\mathrm{F}}} \textit{ if } R_{\widetilde{\mathrm{F}}} \ge R_{\widetilde{\mathrm{P}}} \\ c_{\widetilde{\mathrm{P}}} \textit{ otherwise.} \end{cases}$$

If $c_{\widetilde{\mathrm{F}}}$ (resp., $c_{\widetilde{\mathrm{P}}}$) is infeasible in (F) *(resp.,* P*), then $c_{\widetilde{\mathrm{P}}}$ (resp., $c_{\widetilde{\mathrm{F}}}$) must be feasible in* (P) *(resp.,* F*) and hence globally optimal.*

D.2. Proofs for Appendix D.1

Here we give the proofs of the results in the previous subsection. Several of these proofs require supporting claims, whose proofs themselves are somewhat extensive. We state claims where they are needed within the proof of the result that they support, relegating the proofs of claims to Appendix F.

Proof of Lemma EC.2. The argument for the concavity of $R_i(c)$ and the existence of $c_i^* \in (0,1)$ is as follows. We first show $R_i'(0) > 0 > R_i'(1)$, so there exists $c_i^* \in (0,1)$ that solves the FOC of $R_i(c)$, i.e., $R_i'(c_i^*) = 0$. Then, we show $R_i''(c) < 0$ for $c \in [0,1]$, implying its concavity. Thus, c_i^* is the maximizer of the function $R_i(c)$.

• Differentiating $R_{\mathrm{FP}}(c)$ gives

$$R'_{\mathrm{FP}}(c) = \frac{1-2c+(1-\alpha)(1-\lambda)c^{2k} - c^k(2-\lambda-\alpha+k\lambda-k\alpha)+(2-k)(1-\lambda)c^{k+1}}{\left(1-c^k(1-\lambda)\right)^2}, \tag{EC.48}$$

and substitution of $c=0$ and $c=1$ then yields

$$R'_{\mathrm{FP}}(0) = 1 > 0, \quad \text{and} \quad R'_{\mathrm{FP}}(1) = \frac{-(1-\alpha)k-(2-\alpha)\lambda}{\lambda^2} < 0.$$

Differentiating $R'_{\mathrm{FP}}(c)$ again gives

$$R''_{\mathrm{FP}}(c) = \frac{-k(\lambda-\alpha)(k+1-c^k(1-k)(1-\lambda))c^{k-1} - 2(1-(1-\lambda)c^k)^2}{(1-(1-\lambda)c^k)^3} - \frac{(3k\lambda+k^2(2-\lambda))(1-\lambda)}{(1-(1-\lambda)c^k)^3} < 0.$$

• Differentiating $R_{\mathrm{FD}}(c)$ gives

$$R'_{\mathrm{FD}}(c) = \frac{1-2c-\alpha(1-(1+k)c^k-2c)+(1-\lambda)c^{2k}+(2-k)(1-\lambda)c^{k+1}-(2-\lambda+\lambda k)c^k}{\left(1-\alpha-c^k(1-\lambda)\right)^2}, \tag{EC.49}$$

and substitution of $c=0$ and $c=1$ then yields

$$R'_{\mathrm{FD}}(0) = \frac{1}{1-\alpha} > 0, \quad \text{and} \quad R'_{\mathrm{FD}}(1) = \frac{-(1-\alpha)k-2(\lambda-\alpha)}{(\lambda-\alpha)^2} < 0.$$

Differentiating $R'_{\mathrm{FD}}(c)$ again gives

$$\begin{aligned} R''_{\mathrm{FD}}(c) = & \frac{-k(\lambda-\alpha)((1-\alpha)(k+1)-(1-\lambda)(1-k)c^k)c^{k-1} - 2(1-\alpha-(1-\lambda)c^k)^2}{(1-\alpha-(1-\lambda)c^k)^3} \\ & - \frac{(3k(\lambda-\alpha)+k^2(2-\alpha-\lambda))(1-\lambda)}{(1-\alpha-(1-\lambda)c^k)^3} < 0. \end{aligned}$$

• Differentiating $R_{\mathrm{PP}}(c)$ gives

$$R'_{\mathrm{PP}}(c) = \frac{\alpha^2+2\alpha(1-\lambda-(1-\lambda)c^k-c)+(1-\lambda)(1-\lambda-2c+(2-k)c^{1+k}-2(1-\lambda)c^k+(1-\lambda)c^{2k})}{\left(1-\lambda+\alpha-c^k(1-\lambda)\right)^2}, \tag{EC.50}$$

and substitution of $c=0$ and $c=1$ then yields

$$R'_{\mathrm{PP}}(0) = 1 > 0, \quad \text{and} \quad R'_{\mathrm{PP}}(1) = \frac{-(2-\alpha)\alpha-(1-\lambda)k}{\alpha^2} < 0.$$

Differentiating $R'_{\mathrm{PP}}(c)$ again gives

$$R''_{\mathrm{PP}}(c) = \frac{-2\alpha^2-\alpha(1-\lambda)(4-(4-3k-k^2)c^k)}{(1-\lambda+\alpha-(1-\lambda)c^k)^3} - \frac{2(1-\lambda)^2(1-c^k)^2+(1-\lambda)^2(3k(1-c^k)+k^2(1+c^k))c^k}{(1-\lambda+\alpha-(1-\lambda)c^k)^3} < 0.$$

• Finally, differentiating $R_{\mathrm{PD}}(c)$ gives

$$R'_{\mathrm{PD}}(c) = \frac{1-\lambda-2c+\alpha(1-c^k)^2+(2-k)c^{1+k}-2(1-\lambda)c^k+(1-\lambda)c^{2k}}{(1-c^k)^2(1-\lambda)}, \tag{EC.51}$$

and substitution of $c=0$ and $c=1$ then yields

$$R'_{\mathrm{PD}}(0) = \frac{1-\lambda+\alpha}{1-\lambda} > 0, \quad \text{and} \quad \lim_{c\to 1} R'_{\mathrm{PD}}(c) = -\infty.$$

Differentiating $R'_{\mathrm{PD}}(c)$ again time gives

$$R''_{\mathrm{PD}}(c) = \frac{-(1-\lambda)^2(2(1-c^k)^2+(3k(1-c^k)+k^2(1+c^k))c^k)}{(1-\lambda-(1-\lambda)c^k)^3} \le 0,$$

which implies that $R_{\text{PD}}(c)$ is concave. Thus, there exists a unique $c^*_{\text{PD}} \in (0,1)$ with $R'_{\text{PD}}(c^*_{\text{PD}}) = 0$, and it is the maximizer of $R_{\text{PD}}(c)$.

The above arguments establish the first sentence in the lemma statement. In the remainder of the proof, we derive equations (EC.37)–(EC.39).

- We now show the first part of equation (EC.37), that $c^k \lesseqqgtr p_{\text{FP}}(c) \iff c \lesseqqgtr c_{\text{FB}} \iff c^k \lesseqqgtr p_{\text{FD}}(c)$.

Using (FP), we can write $c^k \lesseqqgtr p_{\text{FP}}(c)$ as

$$c^k \lesseqqgtr \frac{1-(1-\alpha)c^k - c}{1-(1-\lambda)c^k}.$$

Since $1-(1-\lambda)c^k > 0$, we can rearrange the above inequality to yield

$$c^k \lesseqqgtr \frac{1-(1-\alpha)c^k - c}{1-(1-\lambda)c^k} \iff c^k(1-(1-\lambda)c^k) \lesseqqgtr 1-(1-\alpha)c^k - c$$
$$\iff g_{\text{FB}}(c) := 1 - c - (2-\alpha)c^k + c^{2k}(1-\lambda) \gtreqqless 0. \tag{EC.52}$$

By similar steps, we can rearrange $c^k \lesseqqgtr p_{\text{FD}}(c)$, where $p_{\text{FD}}(c)$ is defined in (FD), to give the same inequality (EC.52).

Thus, we have $c^k \lesseqqgtr p_{\text{FP}}(c) \iff g_{\text{FB}}(c) \gtreqqless 0 \iff c^k \lesseqqgtr p_{\text{FD}}(c)$, which, together with the following claim, implies the first part of equation (EC.37).

Claim EC.1. *There exists a unique constant $c_{\text{FB}} \in (0,1)$ such that $g_{\text{FB}}(c) \gtreqqless 0 \iff c \lesseqqgtr c_{\text{FB}}$.*

- We next show the second part of equation (EC.37), that $c^k \lesseqqgtr p_{\text{PP}}(c) \iff c \lesseqqgtr c_{\text{PB}} \iff c^k \lesseqqgtr p_{\text{PD}}(c)$.

Using (PP), we can write $c^k \lesseqqgtr p_{\text{PP}}(c)$ as

$$c^k \lesseqqgtr \frac{1-\lambda+\alpha-(1-\lambda)c^k - c}{1-\lambda+\alpha-(1-\lambda)c^k}.$$

Since $1-\lambda+\alpha-(1-\lambda)c^k > 0$, with some rearrangements, we can simplify this inequality to

$$1-\lambda+\alpha-c-\alpha c^k - 2c^k(1-\lambda) + c^{2k}(1-\lambda) \gtreqqless 0. \tag{EC.53}$$

By similar steps, we can rearrange $c^k \lesseqqgtr p_{\text{PD}}(c)$, where $p_{\text{PD}}(c)$ is defined in (PD), to give the same inequality (EC.53).

We denote the left hand side of (EC.53) as $g_{\text{PB}}(c)$, i.e.,

$$g_{\text{PB}}(c) = 1-\lambda+\alpha-c-\alpha c^k - 2c^k(1-\lambda) + c^{2k}(1-\lambda). \tag{EC.54}$$

Combining the above results, we have $c^k \lesseqqgtr p_{\text{PP}}(c) \iff g_{\text{PB}}(c) \gtreqqless 0 \iff c^k \lesseqqgtr p_{\text{PD}}(c)$, which, together with the following claim, implies the second part of equation (EC.37).

Claim EC.2. *There exists a unique constant $c_{\text{PB}} \in (0,1)$ such that $g_{\text{PB}}(c) \gtreqqless 0 \iff c \lesseqqgtr c_{\text{PB}}$.*

- We now show the first part of equation (EC.38), that $c^k + p_{\text{FP}}(c) \lesseqqgtr 1 \iff c \gtreqqless c_{\text{FPP}} \iff c^k + p_{\text{PP}}(c) \lesseqqgtr 1$.

Using (FP), we can write $c^k + p_{\text{FP}}(c) \lesseqqgtr 1$ as

$$c^k + \frac{1-(1-\alpha)c^k - c}{1-(1-\lambda)c^k} \lesseqqgtr 1.$$

Since $1-(1-\lambda)c^k > 0$, with some rearrangements, we can simplify this inequality to

$$c + c^{2k}(1-\lambda) - c^k(1-\lambda+\alpha) \gtreqqless 0. \tag{EC.55}$$

By similar steps, we can rearrange $c^k + p_{\text{PP}}(c) \lesseqqgtr 1$ to give the same inequality (EC.55).

We denote the left hand side of (EC.55) as $g_{\text{FPP}}(c)$, i.e.,

$$g_{\text{FPP}}(c) = c + c^{2k}(1-\lambda) - c^k(1-\lambda+\alpha). \tag{EC.56}$$

Thus, we have $c^k + p_{\text{FP}}(c) \lesseqqgtr 1 \iff g_{\text{FPP}}(c) \gtreqqless 0 \iff c^k + p_{\text{PP}}(c) \lesseqqgtr 1$, which, together with the following claim, implies the first part of equation (EC.38).

Claim EC.3. *There exists a unique constant $c_{\text{FPP}} \in (0,1)$ such that $g_{\text{FPP}}(c) \gtreqless 0 \iff c \gtreqless c_{\text{FPP}}$.*

- We now show the second part of equation (EC.38), that $c^k + p_{\text{FD}}(c) \lesseqgtr 1 \iff c \gtreqless c_{\text{FPD}} \iff c^k + p_{\text{PD}}(c) \lesseqgtr 1$. Using (FD), we can write $c^k + p_{\text{FD}}(c) \lesseqgtr 1$ as

$$c^k + \frac{1 - c^k - c}{1 - \alpha - (1-\lambda)c^k} \lesseqgtr 1.$$

Since $1 - \alpha - (1-\lambda)c^k > 0$, with some rearrangements, we can simplify this inequality to

$$c - \alpha(1 - c^k) - c^k(1-\lambda) + c^{2k}(1-\lambda) \gtreqless 0. \tag{EC.57}$$

By similar steps, we can rearrange $c^k + p_{\text{PD}}(c) \lesseqgtr 1$ to give the same inequality (EC.57).

$$c - \alpha(1 - c^k) - c^k(1-\lambda) + c^{2k}(1-\lambda) \gtreqless 0.$$

We denote the left hand side of the last inequality as $g_{\text{FPD}}(c)$, i.e.,

$$g_{\text{FPD}}(c) = c - \alpha(1 - c^k) - c^k(1-\lambda) + c^{2k}(1-\lambda). \tag{EC.58}$$

Thus, we have $c^k + p_{\text{FD}}(c) \lesseqgtr 1 \iff g_{\text{FPD}}(c) \gtreqless 0 \iff c^k + p_{\text{PD}}(c) \lesseqgtr 1$, which, together with the following claim, implies the second part of equation (EC.38).

Claim EC.4. *There exists a unique constant $c_{\text{FPD}} \in (0,1)$ such that $g_{\text{FPD}}(c) \gtreqless 0 \iff c \gtreqless c_{\text{FPD}}$.*

Finally, the following claim uses the previous results to establish equation (EC.39), completing the proof of the lemma.

Claim EC.5. $c_{\text{FPP}} \lesseqgtr c_{\text{FB}} \iff c_{\text{FPD}} \lesseqgtr c_{\text{FB}}$ *and* $c_{\text{FPP}} \lesseqgtr c_{\text{PB}} \iff c_{\text{FPD}} \lesseqgtr c_{\text{PB}}$. □

Proof of Lemma EC.3. First, Lemma EC.2 implies that problem ($\widetilde{\text{F}}$) is equivalent to maximizing $R_{\text{F}}(c)$ over $c \in [0,1]$, where $R_{\text{F}}(c)$ is defined in equation (EC.40).

Claim EC.6. *The function $R_{\text{F}}(c)$ is continuous and unimodal, whose unique maximizer $c_{\widetilde{\text{F}}} \in (0,1)$ solves the relaxed problem ($\widetilde{\text{F}}$).*

Claim EC.6 and the concavity of $R_{\text{FP}}(c)$ and $R_{\text{FD}}(c)$ from Lemma EC.2 together imply that $c_{\widetilde{\text{F}}}$ must take one of three possible values, determined as follows. If $R'_{\text{FP}}(c_{\text{FB}}) \le 0$ (and thus $c^*_{\text{FP}} \le c_{\text{FB}}$), then Claim EC.6 implies $R'_{\text{FD}}(c_{\text{FB}}) \le 0$ and hence $c_{\widetilde{\text{F}}} = c^*_{\text{FP}}$; if $R'_{\text{FP}}(c_{\text{FB}}) > 0$ and $R'_{\text{FD}}(c_{\text{FB}}) \ge 0$, then $c_{\widetilde{\text{F}}} = c^*_{\text{FD}} \ge c_{\text{FB}}$; and if $R'_{\text{FP}}(c_{\text{FB}}) > 0$ and $R'_{\text{FD}}(c_{\text{FB}}) < 0$, then $c_{\widetilde{\text{F}}} = c_{\text{FB}}$.

Let $x(k) := c^k_{\text{FB}}$, where c_{FB} as a function of k, is implicitly defined via Claim EC.1 in the proof of Lemma EC.2. Then, by (EC.37) and (EC.52), we have that $x(k) \in [0,1]$ is the unique solution to

$$1 - x^{\frac{1}{k}} - (2-\alpha)x + (1-\lambda)x^2 = 0. \tag{EC.59}$$

Using equation (EC.59), we can show that the signs of $R'_{\text{FP}}(c_{\text{FB}})$ and $R'_{\text{FD}}(c_{\text{FB}})$ defined in (EC.48) and (EC.49) are given by those of $M_1(k)$ and $M_2(k)$, respectively, where

$$\begin{aligned} M_1(k) &:= -1 + (3 - \alpha - k + \alpha k)x(k) - (2-k)(1-\lambda)x(k)^2 \\ M_2(k) &:= -1 + (3 - 2\alpha - k)x(k) - (2-k)(1-\lambda)x(k)^2. \end{aligned} \tag{EC.60}$$

We demonstrate (EC.60) by showing the case of $M_1(k)$; the case of $M_2(k)$ follows from a similar argument by (EC.49). By equation (EC.48), the sign of $R'_{\text{FP}}(c)$ is the same as the sign of its numerator. By the definition of c_{FB} in Claim EC.1, the numerator of $R'_{\text{FP}}(c)$ above evaluated at $c = c_{\text{FB}}$ can be simplified to

$$\begin{aligned} &1 - 2c_{\text{FB}} + (1-\alpha)(1-\lambda)c^{2k}_{\text{FB}} - c^k_{\text{FB}}(2 - \lambda - \alpha + k\lambda - k\alpha) + (2-k)(1-\lambda)c^{k+1}_{\text{FB}} \\ &= 1 - 2c_{\text{FB}} + (1-\alpha)(1-\lambda)c^{2k}_{\text{FB}} - c^k_{\text{FB}}(2 - \lambda - \alpha + k\lambda - k\alpha) + (2-k)(1-\lambda)c^{k+1}_{\text{FB}} - g_{\text{FB}}(c_{\text{FB}}) \\ &= -\alpha(1-\lambda)c^{2k}_{\text{FB}} + c^k_{\text{FB}}(k\alpha - k\lambda + \lambda) + ((2-k)(1-\lambda)c^k_{\text{FB}} - 1)c_{\text{FB}} \\ &= -\alpha(1-\lambda)c^{2k}_{\text{FB}} + c^k_{\text{FB}}(k\alpha - k\lambda + \lambda) + ((2-k)(1-\lambda)c^k_{\text{FB}} - 1)(1 - (2-\alpha)c^k_{\text{FB}} + (1-\lambda)c^{2k}_{\text{FB}}) \\ &= -\left(1 - (1-\lambda)c^k_{\text{FB}}\right)\left(1 - (3 - \alpha - k + k\alpha)c^k_{\text{FB}} + (2-k)(1-\lambda)c^{2k}_{\text{FB}}\right). \end{aligned}$$

Since $1-(1-\lambda)c_{\mathrm{FB}}^k > 0$, the sign of $R'_{\mathrm{FP}}(c_{\mathrm{FB}})$ is determined by the sign of

$$-1+(3-\alpha-k+k\alpha)x(k)-(2-k)(1-\lambda)x(k)^2 = M_1(k),$$

by using $x(k)=c_{\mathrm{FB}}^k$.

We therefore have

$$c_{\widetilde{\mathrm{F}}} = \begin{cases} c^*_{\mathrm{FP}} & \text{if } M_1(k)\le 0 \\ c^*_{\mathrm{FD}} & \text{if } M_1(k)>0 \text{ and } M_2(k)\ge 0 \\ c_{\mathrm{FB}} & \text{otherwise,} \end{cases} \tag{EC.61}$$

from which (EC.44) immediately follows and completes the proof of this lemma, once the following properties regarding $x(k)$, $M_1(k)$ and $M_2(k)$ are established.

Claim EC.7. *$x(k)$ is strictly decreasing in k.*

Claim EC.8. *For given λ and α, there exists a unique threshold k_{FP} such that $M_1(k) \lesseqqgtr 0$ for $k \gtreqqless k_{\mathrm{FP}}$. Namely, k_{FP} is the unique solution to* (EC.42).

Claim EC.9. *For given λ and α, there exists a unique threshold k_{FD} such that $M_2(k) \lesseqqgtr 0$ for $k \gtreqqless k_{\mathrm{FD}}$. Namely, k_{FD} is the unique solution to* (EC.43).

Claim EC.10. *For given λ and α, the two thresholds given in Claims EC.8 and EC.9 must satisfy $k_{\mathrm{FD}} < k_{\mathrm{FP}}$.* □

Proof of Lemma EC.4. First, Lemma EC.2 implies that problem $(\widetilde{\mathrm{P}})$ is equivalent to maximizing $R_{\mathrm{P}}(c)$ over $c\in[0,1]$, where $R_{\mathrm{P}}(c)$ is defined in equation (EC.41).

Claim EC.11. *The function $R_{\mathrm{P}}(c)$ is continuous and unimodal, and its unique maximizer $c_{\widetilde{\mathrm{P}}}\in(0,1)$ solves the relaxed problem* $(\widetilde{\mathrm{P}})$.

Claim EC.11 and the concavity of $R_{\mathrm{PP}}(c)$ and $R_{\mathrm{PD}}(c)$ from Lemma EC.2 together imply that $c_{\widetilde{\mathrm{P}}}$ must take one of three possible values, determined as follows. If $R'_{\mathrm{PP}}(c_{\mathrm{PB}})\le 0$ (and thus $c^*_{\mathrm{PP}}\le c_{\mathrm{PB}}$), then Claim EC.11 implies $R'_{\mathrm{PD}}(c_{\mathrm{PB}})\le 0$ and hence $c_{\widetilde{\mathrm{P}}}=c^*_{\mathrm{PP}}$; if $R'_{\mathrm{PP}}(c_{\mathrm{PB}})>0$ and $R'_{\mathrm{PD}}(c_{\mathrm{PB}})\ge 0$, then $c_{\widetilde{\mathrm{P}}}=c^*_{\mathrm{PD}}\ge c_{\mathrm{PB}}$; and if $R'_{\mathrm{PP}}(c_{\mathrm{PB}})>0$ and $R'_{\mathrm{PD}}(c_{\mathrm{PB}})<0$, then $c_{\widetilde{\mathrm{P}}}=c_{\mathrm{PB}}$.

Let $y(k):=c_{\mathrm{PB}}^k$, where c_{PB} as a function of k, is implicitly defined via Claim EC.2 in the proof of Lemma EC.2. Then, by (EC.37) and (EC.54), we have that $y(k)\in(0,1]$ is the unique solution to

$$1-\lambda+\alpha-y(k)^{\frac{1}{k}}-\alpha y(k)-2(1-\lambda)y(k)+(1-\lambda)y(k)^2=0. \tag{EC.62}$$

Using equation (EC.62), we can show that the signs of $R'_{\mathrm{PP}}(c_{\mathrm{PB}})$ and $R'_{\mathrm{PD}}(c_{\mathrm{PB}})$ are given by those of $M_3(k)$ and $M_4(k)$, respectively, where

$$\begin{aligned} M_3(k) &:= -1+\lambda-\alpha+y(k)(2\alpha+(1-\lambda)(3-k))-y(k)^2(1-\lambda)(2-k), \quad \text{and} \\ M_4(k) &:= -1+\lambda-\alpha+y(k)((1-k)\alpha+(1-\lambda)(3-k))-y(k)^2(1-\lambda)(2-k). \end{aligned} \tag{EC.63}$$

We demonstrate (EC.63) by showing the case of $M_3(k)$; the case of $M_4(k)$ follows from a similar argument. By equation (EC.50), the sign of $R'_{\mathrm{PP}}(c)$ is the same as the sign of its numerator. By the definition of c_{PB} in Claim EC.2, i.e., $g_{\mathrm{PB}}(c_{\mathrm{PB}})=0$, the numerator of $R'_{\mathrm{PP}}(c)$ above evaluated at $c=c_{\mathrm{PB}}$ can be simplified to

$$\begin{aligned} &\alpha^2+2\alpha(1-\lambda-c_{\mathrm{PB}}-(1-\lambda)c_{\mathrm{PB}}^k)+(1-\lambda)(1-\lambda-2c_{\mathrm{PB}}+(2-k)c_{\mathrm{PB}}^{k+1}-2(1-\lambda)c_{\mathrm{PB}}^k+(1-\lambda)c_{\mathrm{PB}}^{2k}) \\ &=\alpha^2+2\alpha(1-\lambda-c_{\mathrm{PB}}-(1-\lambda)c_{\mathrm{PB}}^k)+(1-\lambda)(1-\lambda-2c_{\mathrm{PB}}+(2-k)c_{\mathrm{PB}}^{k+1} \\ &\quad -2(1-\lambda)c_{\mathrm{PB}}^k+(1-\lambda)c_{\mathrm{PB}}^{2k})-(1-\lambda+\alpha)g_{\mathrm{PB}}(c_{\mathrm{PB}}) \\ &=\alpha(1-\lambda+\alpha)c_{\mathrm{PB}}^k-\alpha(1-\lambda)c_{\mathrm{PB}}^{2k}-(1-\lambda+\alpha)c_{\mathrm{PB}}+(2-k)(1-\lambda)c_{\mathrm{PB}}^{k+1} \\ &=\alpha(1-\lambda+\alpha)c_{\mathrm{PB}}^k-\alpha(1-\lambda)c_{\mathrm{PB}}^{2k}-(1-\lambda+\alpha)(1-\lambda+\alpha-\alpha c_{\mathrm{PB}}^k \\ &\quad -2(1-\lambda)c_{\mathrm{PB}}^k+(1-\lambda)c_{\mathrm{PB}}^k)+(2-k)(1-\lambda)c_{\mathrm{PB}}^k(1-\lambda+\alpha-\alpha c_{\mathrm{PB}}^k-2(1-\lambda)c_{\mathrm{PB}}^k+(1-\lambda)c_{\mathrm{PB}}^k) \\ &=(\alpha+(1-\lambda)(1-c_{\mathrm{PB}}^k))(-1+\lambda-\alpha+c_{\mathrm{PB}}^k(2\alpha+(1-\lambda)(3-k))-c_{\mathrm{PB}}^{2k}(1-\lambda)(2-k)). \end{aligned}$$

Since $\alpha+(1-\lambda)(1-c_{\mathrm{PB}}^k)>0$, the sign of $R'_{\mathrm{PP}}(c_{\mathrm{PB}})$ is determined by the sign of

$$-1+\lambda-\alpha+c_{\mathrm{PB}}^k(2\alpha+(1-\lambda)(3-k))-c_{\mathrm{PB}}^{2k}(1-\lambda)(2-k)=M_3(k),$$

by using $y(k)=c_{\mathrm{PB}}^k$.

We therefore have

$$c_{\widetilde{\mathrm{P}}}=\begin{cases} c^*_{\mathrm{PP}} & \text{if } M_3(k)\le 0\\ c^*_{\mathrm{PD}} & \text{if } M_3(k)>0 \text{ and } M_4(k)\ge 0\\ c_{\mathrm{PB}} & \text{otherwise,}\end{cases} \tag{EC.64}$$

from which (EC.47) immediately follows and completes the proof of this lemma, once the following properties regarding $y(k)$, $M_3(k)$ and $M_4(k)$ are established.

Claim EC.12. *$y(k)$ is strictly decreasing and convex in k.*

Claim EC.13. *For given λ and α, there exists a unique threshold k_{PP} such that $M_3(k)\lesseqqgtr 0$ for $k\gtreqqless k_{\mathrm{PP}}$. Namely, k_{PP} is the unique solution to* (EC.45).

Claim EC.14. *For given λ and α, there exists a unique threshold k_{PD} such that $M_4(k)\lesseqqgtr 0$ for $k\gtreqqless k_{\mathrm{PD}}$. Namely, k_{PD} is the unique solution to* (EC.46).

Claim EC.15. *For given λ and α, the two thresholds given in Claims EC.13 and EC.14 must satisfy $k_{\mathrm{PD}}<k_{\mathrm{PP}}$.*

□

Proof of Proposition EC.2. We first state the supporting claims.

Claim EC.16. *$G(c_{\widetilde{\mathrm{F}}})+F(p_{\mathrm{F}}(c_{\widetilde{\mathrm{F}}}))>1 \implies G(c^*_{\mathrm{F}})+F(p_{\mathrm{F}}(c^*_{\mathrm{F}}))=1$.*

Claim EC.17. *$G(c_{\widetilde{\mathrm{P}}})+F(p_{\mathrm{P}}(c_{\widetilde{\mathrm{P}}}))<1 \implies G(c^*_{\mathrm{P}})+F(p_{\mathrm{P}}(c^*_{\mathrm{P}}))=1$.*

Claim EC.18. *$F(p^*)+G(c^*)=1 \implies F(p^*)=G(c^*)$.*

Claim EC.19. *If the global optimal policy satisfies $F(p^*)+G(c^*)=1$ and $F(p^*)=G(c^*)$, then the solutions to $(\widetilde{\mathrm{F}})$ and $(\widetilde{\mathrm{P}})$ coincide with (c^*,p^*).*

We now prove the proposition using Claims EC.16-EC.19.

Case 1: $G(c_{\widetilde{\mathrm{F}}})+F(p_{\mathrm{F}}(c_{\widetilde{\mathrm{F}}}))\le 1$ and $G(c_{\widetilde{\mathrm{P}}})+F(p_{\mathrm{P}}(c_{\widetilde{\mathrm{P}}}))\ge 1$. In this case, $c_{\widetilde{\mathrm{F}}}$ ($c_{\widetilde{\mathrm{P}}}$) is feasible and therefore optimal for problem (F) (problem (P)), and therefore the global optimal solution is determined by a revenue comparison between $R_{\mathrm{F}}(c_{\widetilde{\mathrm{F}}},p_{\mathrm{F}}(c_{\widetilde{\mathrm{F}}}))$ and $R_{\mathrm{P}}(c_{\widetilde{\mathrm{P}}},p_{\mathrm{P}}(c_{\widetilde{\mathrm{P}}}))$. This includes the special case of $F(p^*)+G(c^*)=1$ and $F(p^*)=G(c^*)$, in which case (c^*,p^*) is the solution to both $(\widetilde{\mathrm{F}})$ and $(\widetilde{\mathrm{P}})$ by Claim EC.19, i.e., $(c_{\widetilde{\mathrm{F}}},p_{\mathrm{F}}(c_{\widetilde{\mathrm{F}}}))=(c_{\widetilde{\mathrm{P}}},p_{\mathrm{P}}(c_{\widetilde{\mathrm{P}}}))=(c^*,p^*)$.

Case 2: Otherwise. In this case, either $c_{\widetilde{\mathrm{F}}}$ is infeasible for (F), or $c_{\widetilde{\mathrm{P}}}$ is infeasible for (P), or both. As shown in Case 1, this is impossible if $F(p^*)+G(c^*)=1$ and $F(p^*)=G(c^*)$; also, by Claim EC.18, we cannot have $F(p^*)+G(c^*)=1$ and $F(p^*)\ne G(c^*)$. Therefore, we must have $F(p^*)+G(c^*)\ne 1$.

Suppose that $c_{\widetilde{\mathrm{F}}}$ and $c_{\widetilde{\mathrm{P}}}$ are infeasible in (F) and (P), respectively, which is equivalent to $G(c_{\widetilde{\mathrm{F}}})+F(p_{\mathrm{F}}(c_{\widetilde{\mathrm{F}}}))>1$ and $G(c_{\widetilde{\mathrm{P}}})+F(p_{\mathrm{P}}(c_{\widetilde{\mathrm{P}}}))<1$. By Claims EC.16 and EC.17, this would imply that $G(c^*_{\mathrm{F}})+F(p_{\mathrm{F}}(c^*_{\mathrm{F}}))=G(c^*_{\mathrm{P}})+F(p_{\mathrm{P}}(c^*_{\mathrm{P}}))=1$, contradicting that $F(p^*)+G(c^*)\ne 1$, because (c^*,p^*) is always either $(c^*_{\mathrm{F}},p_{\mathrm{F}}(c^*_{\mathrm{F}}))$ or $(c^*_{\mathrm{P}},p_{\mathrm{P}}(c^*_{\mathrm{P}}))$. Thus, at least one of $c_{\widetilde{\mathrm{F}}}$ or $c_{\widetilde{\mathrm{P}}}$ must be feasible in its constrained counterpart (F) or (P), respectively.

Now, suppose that $c_{\widetilde{\mathrm{F}}}$ is infeasible for (F). By the above argument, this implies that $c_{\widetilde{\mathrm{P}}}$ is feasible and therefore optimal for (P), i.e., we have $c^*_{\mathrm{P}}=c_{\widetilde{\mathrm{P}}}$. We also must have $G(c^*_{\mathrm{F}})+F(p_{\mathrm{F}}(c^*_{\mathrm{F}}))=1$ by Claim EC.16. Therefore, $(c^*_{\mathrm{F}},p_{\mathrm{F}}(c^*_{\mathrm{F}}))$ cannot be the global optimal solution because we have $F(p^*)+G(c^*)\ne 1$. Thus, the global optimal solution must be the optimal solution to (P) and the corresponding price, which we have just shown to be $(c_{\widetilde{\mathrm{P}}},p_{\mathrm{P}}(c_{\widetilde{\mathrm{P}}}))$. An exactly analogous argument implies that $c_{\widetilde{\mathrm{F}}}$ is the global optimum if $c_{\widetilde{\mathrm{P}}}$ is infeasible for (P). □

D.3. Proofs for Section 5

Proof of Proposition 5. The result follows immediately from Lemmas EC.3 and EC.4. □

Proof of Proposition 6. In this proof, we make the dependence on α explicit whenever necessary. The proof consists of six parts covering the monotonicity results for the six possible global optimal solution pairs found in Lemmas EC.3 and EC.4.

• **Sub-problem FP: $(c^*_{\mathrm{FP}}, p^*_{\mathrm{FP}})$.** Differentiating (EC.48) in the proof of Lemma EC.2 with respect to α yields

$$\frac{\partial^2 R_{\mathrm{FP}}(c)}{\partial c \partial \alpha} = \frac{c^k(1+k-(1-\lambda)c^k)}{(1-(1-\lambda)c^k)} > 0,$$

and hence c^*_{FP} is increasing in α by supermodularity.

Recall from Lemma EC.3 that $p^*_{\mathrm{FP}} \triangleq p_{\mathrm{FP}}(c^*_{\mathrm{FP}})$. Furthermore, $p_{\mathrm{FP}}(c^*_{\mathrm{FP}})$ is the same as (15) which represents the optimal price for $k \geq 1$. In the the proof of Proposition 4, we have already shown that $\partial p^*_{\mathrm{FP}}/\partial \alpha$ is given by (EC.25), and that $\partial p^*_{\mathrm{FP}}/\partial \alpha \lesseqgtr 0 \iff \Theta(c^*_{\mathrm{FP}}) \lesseqgtr 0$, where $\Theta(c^*_{\mathrm{FP}})$ is given by (EC.28). Rearranging (EC.28) and simplifying gives

$$\Theta(c^*_{\mathrm{FP}}) = \frac{\left(1-(1-\lambda)\left(c^*_{\mathrm{FP}}\right)^k\right)^2 \left(k^2(1-\lambda)\left(c^*_{\mathrm{FP}}\right)^{k-1} + (1-k)\left(1+k-(1-k)(1-\lambda)\left(c^*_{\mathrm{FP}}\right)^k\right)\right)}{1+k-(1-\lambda)(c^*_{\mathrm{FP}})^k} > 0,$$

where the inequality follows from $\lambda < 1$ and $k \in (0,1)$. Hence, we have $\partial p^*_{\mathrm{FP}}/\partial \alpha > 0$.

• **Sub-problem FD: $(c^*_{\mathrm{FD}}, p^*_{\mathrm{FD}})$.** By the FOC of $R_{\mathrm{FD}}(c)$, i.e., $R'_{\mathrm{FD}}(c^*_{\mathrm{FD}}) = 0$, and (EC.49), we have

$$\alpha\left((k+1)\left(c^*_{\mathrm{FD}}\right)^k + 2c^*_{\mathrm{FD}} - 1\right) + (1-\lambda)\left(c^*_{\mathrm{FD}}\right)^{2k} + (2-k)(1-\lambda)\left(c^*_{\mathrm{FD}}\right)^{k+1} - \left(c^*_{\mathrm{FD}}\right)^k(2+k\lambda-\lambda) - 2c^*_{\mathrm{FD}} + 1 = 0. \tag{EC.65}$$

Differentiating (EC.49) with respect to α, we get

$$\frac{\partial^2 R_{\mathrm{FD}}(c)}{\partial c \partial \alpha} = \frac{\alpha\left((k+1)c^k + 2c - 1\right) + (1-k)(1-\lambda)c^{2k} + 2(1-k)(1-\lambda)c^{k+1} - c^k(2+2k\lambda-k-\lambda) - 2c + 1}{(1-\alpha-(1-\lambda)c^k)^3}.$$

First, note that $1 - c^*_{\mathrm{FD}} - (c^*_{\mathrm{FD}})^k > 0$, otherwise, we would have $R_{\mathrm{FD}}(c^*_{\mathrm{FD}}) \leq 0$ by (FD), and this would contradict c^*_{FD} being the optimal solution because $R_{\mathrm{FD}}(0) = 0$ and $R'_{\mathrm{FD}}(0) > 0$, as shown in the proof of Lemma EC.2. Using (EC.65), evaluating this cross-derivative at $c = c^*_{\mathrm{FD}}$ yields

$$\left.\frac{\partial^2 R_{\mathrm{FD}}(c)}{\partial c \partial \alpha}\right|_{c=c^*_2} = \frac{(c^*_{\mathrm{FD}})^k(1-\lambda)\left(1-c^*_{\mathrm{FD}}-(c^*_{\mathrm{FD}})^k\right)}{\left(1-\alpha-(1-\lambda)\left(c^*_{\mathrm{FD}}\right)^k\right)^3} > 0.$$

which follows from $\alpha < \lambda < 1$. Hence, $R_{\mathrm{FD}}(c)$ has increasing differences in (c, α) at $c = c^*_{\mathrm{FD}}$, implying that $\mathrm{d}c^*_{\mathrm{FD}}/\mathrm{d}\alpha > 0$.

Using the definition of c^*_{FD} given in (EC.65) and the Implicit Function Theorem, we get

$$\frac{\mathrm{d}c^*_{\mathrm{FD}}}{\mathrm{d}\alpha} = -\frac{c^*_{\mathrm{FD}}\left(-1+2c^*_{\mathrm{FD}}+\left(c^*_{\mathrm{FD}}\right)^k(1+k)\right)}{k(c^*_{\mathrm{FD}})^k(\alpha k+\alpha-k\lambda+\lambda-2) - 2(1-\alpha)c^*_{\mathrm{FD}} + (2+k-k^2)(1-\lambda)(c^*_{\mathrm{FD}})^{k+1} + 2k(1-\lambda)(c^*_{\mathrm{FD}})^{2k}}. \tag{EC.66}$$

With some algebraic manipulations on (EC.65), we also get

$$c^*_{\mathrm{FD}} = \frac{\alpha\left(1-(1+k)\left(c^*_{\mathrm{FD}}\right)^k\right) - \lambda\left(c^*_{\mathrm{FD}}\right)^k\left(1-k-\left(c^*_{\mathrm{FD}}\right)^k\right) - \left(1-\left(c^*_{\mathrm{FD}}\right)^k\right)^2}{(2-k)(1-\lambda)(c^*_{\mathrm{FD}})^k + \alpha - 2} \tag{EC.67}$$

which allows us to show

$$I_1(c^*_{\mathrm{FD}}) = \underbrace{-1+2c^*_{\mathrm{FD}}}_{\text{replace } c^*_{\mathrm{FD}} \text{ with RHS of (EC.67)}} + (c^*_{\mathrm{FD}})^k(1+k) = \frac{k(1-\lambda)\left(1-(1-k)\left(c^*_{\mathrm{FD}}\right)^k\right)\left(c^*_{\mathrm{FD}}\right)^k}{2(1-\alpha)-(2-k)(1-\lambda)\left(c^*_{\mathrm{FD}}\right)^k} > 0. \tag{EC.68}$$

Together with $\mathrm{d}c^*_{\mathrm{FD}}/\mathrm{d}\alpha > 0$ and (EC.66), this implies

$$I_2(c^*_{\mathrm{FD}}) = k(c^*_{\mathrm{FD}})^k(\alpha k+\alpha-k\lambda+\lambda-2) - 2(1-\alpha)c^*_{\mathrm{FD}} + \left(2+k-k^2\right)(1-\lambda)(c^*_{\mathrm{FD}})^{k+1} + 2k(1-\lambda)(c^*_{\mathrm{FD}})^{2k} < 0. \tag{EC.69}$$

Using (EC.65), we also get

$$\alpha = \frac{c^*_{\mathrm{FD}}\left(2-(2-k)(1-\lambda)\left(c^*_{\mathrm{FD}}\right)^k\right) + \left(c^*_{\mathrm{FD}}\right)^k(2-(1-k)\lambda) - (1-\lambda)(c^*_{\mathrm{FD}})^{2k} - 1}{-1+2c^*_{\mathrm{FD}}+(c^*_{\mathrm{FD}})^k(1+k)}. \tag{EC.70}$$

Recall from Lemma EC.3 that $p^*_{\text{FD}} \triangleq p_{\text{FD}}(c^*_{\text{FD}})$. Next, using (EC.44) and (EC.70), it is straightforward to show that

$$\frac{\mathrm{d}p^*_{\text{FD}}}{\mathrm{d}\alpha} = \frac{\partial p_{\text{FD}}(c^*_{\text{FD}})}{\partial \alpha} + \frac{\partial p_{\text{FD}}(c^*_{\text{FD}})}{\partial c}\frac{\mathrm{d}c^*_{\text{FD}}}{\mathrm{d}\alpha} = -\frac{k\,(1-\lambda)\,(2c^*_{\text{FD}}\,(1-k)+k)\left(1-(c^*_{\text{FD}})^k - c^*_{\text{FD}}\right)^2 (c^*_{\text{FD}})^k}{I_1(c^*_{\text{FD}})I_2(c^*_{\text{FD}})\left(1-\alpha-(1-\lambda)\,(c^*_{\text{FD}})^k\right)^2} > 0$$

which follows from (EC.68), (EC.69), and $k \in (0,1)$ and $0 < \alpha < \lambda < 1$.

- $\boldsymbol{(c_{\text{FB}}, p^*_{\text{FB}})}$**.** Using $g_{\text{FB}}(c_{\text{FB}}) = 0$ by Claim EC.1 and the Implicit Function Theorem, we get

$$\frac{\mathrm{d}c_{\text{FB}}}{\mathrm{d}\alpha} = \frac{c_{\text{FB}}^{k+1}}{k(2-\alpha)c^k_{\text{FB}} - 2k(1-\lambda)c^{2k}_{\text{FB}} + c_{\text{FB}}} > 0,$$

which holds because

$$k(2-\alpha)c^k_{\text{FB}} - 2k(1-\lambda)c^{2k}_{\text{FB}} + c_{\text{FB}} = 2kc^k_{\text{FB}}(1-\frac{\alpha}{2}-(1-\lambda)c^k_{\text{FB}}) + c_{\text{FB}} > 0.$$

Since $p^*_{\text{FB}} = p_{\text{FP}}(c_{\text{FB}}) = c^k_{\text{FB}}$ by Lemma EC.2, p^*_{FB} is also increasing in α.

- **Sub-problem PP:** $\boldsymbol{(c^*_{\text{PP}}, p^*_{\text{PP}})}$**.** By the FOC of sub-problem 3, i.e., $R'_{\text{PP}}(c^*_{\text{PP}}) = 0$, and (EC.50), we have

$$(1+\alpha-\lambda)^2 + (1-\lambda)^2(c^*_{\text{PP}})^{2k} + (2-k)(1-\lambda)(c^*_{\text{PP}})^{k+1} - 2(1-\lambda)(1+\alpha-\lambda)(c^*_{\text{PP}})^k - 2c^*_{\text{PP}}(1+\alpha-\lambda) = 0. \quad \text{(EC.71)}$$

Note that $1-\lambda+\alpha-(1-\lambda)(c^*_{\text{PP}})^k - c^*_{\text{PP}} > 0$, otherwise, we would have $R_{\text{PP}}(c^*_{\text{PP}}) \le 0$ by (PP), which would be a contradiction to c^*_{PP} being the optimal solution because $R_{\text{PP}}(0) = 0$ and $R'_{\text{PP}}(0) > 0$, as shown in the proof of Lemma EC.2. Now using (EC.50), direct calculation reveals that

$$\frac{\partial^2 R_{\text{PP}}(c)}{\partial c \partial \alpha} = \frac{2c\left(\alpha+(1-\lambda)\left(1-(1-k)\,c^k\right)\right)}{\left(\alpha+(1-\lambda)\,(1-c^k)\right)^3} > 0 \implies \frac{\mathrm{d}c^*_{\text{PP}}}{\mathrm{d}\alpha} > 0.$$

Furthermore, using this result and Implicit Function Theorem on (EC.71), we get

$$0 < \frac{\mathrm{d}c^*_{\text{PP}}}{\mathrm{d}\alpha} = -\frac{2c\left(1-\lambda+\alpha-(1-\lambda)\,(c^*_{\text{PP}})^k - c^*_{\text{PP}}\right)}{2k(\lambda-1)(\alpha-\lambda+1)(c^*_{\text{PP}})^k - 2c^*_{\text{PP}}(\alpha-\lambda+1) + (k^2-k-2)\,(\lambda-1)(c^*_{\text{PP}})^{k+1} + 2k(\lambda-1)^2(c^*_{\text{PP}})^{2k}},$$

which implies

$$I_3(c^*_{\text{PP}}) = 2k(\lambda-1)(\alpha-\lambda+1)(c^*_{\text{PP}})^k - 2c^*_{\text{PP}}(\alpha-\lambda+1) + \left(k^2-k-2\right)(\lambda-1)(c^*_{\text{PP}})^{k+1} + 2k(\lambda-1)^2(c^*_{\text{PP}})^{2k} < 0. \quad \text{(EC.72)}$$

Next, differentiating $p^*_{\text{PP}} = p_{\text{PP}}(c^*_{\text{PP}})$ defined in Lemma EC.4 w.r.t. α, a similar argument to that used for p^*_{FD} yields

$$\frac{\mathrm{d}p^*_{\text{PP}}}{\mathrm{d}\alpha} = \frac{\partial p_{\text{PP}}(c^*_{\text{PP}})}{\partial \alpha} + \frac{\partial p_{\text{PP}}(c^*_{\text{PP}})}{\partial c}\frac{\mathrm{d}c^*_{\text{PP}}}{\mathrm{d}\alpha} = \frac{k(1-k)(1-\lambda)(c^*_{\text{PP}})^{2+k}}{I_3(c^*_{\text{PP}})\,(\alpha+(1-\lambda)\,(1-(c^*_{\text{PP}})^k))^2} < 0$$

by (EC.72) and $k < 1$, $\lambda < 1$.

- **Sub-problem PD:** $\boldsymbol{(c^*_{\text{PD}}, p^*_{\text{PD}})}$**.** Using (EC.51), direct calculation reveals that

$$\frac{\partial^2 R_{\text{PD}}(c)}{\partial c \partial \alpha} = \frac{1}{1-\lambda} > 0,$$

which implies that c^*_{PD} is increasing in α by supermodularity.

Note that by the FOC of $R_{\text{PD}}(c)$, i.e., $R'_{\text{PD}}(c^*_{\text{PD}}) = 0$, and (EC.51), we have

$$(1-\lambda+\alpha)\left(1-(c^*_{\text{PD}})^k\right)^2 - 2c^*_{\text{PD}} + (c^*_{\text{PD}})^{1+k}(2-k) = 0. \quad \text{(EC.73)}$$

Using this and the Implicit Function Theorem, we get

$$\frac{\mathrm{d}c^*_{\text{PD}}}{\mathrm{d}\alpha} = \frac{c^*_{\text{PD}}\left(1-(c^*_{\text{PD}})^k\right)^2}{2c^*_{\text{PD}} - (c^*_{\text{PD}})^{1+k}(2+k-k^2) + 2k(c^*_{\text{PD}})^k(1-\lambda+\alpha) - 2k(c^*_{\text{PD}})^{2k}(1-\lambda+\alpha)}. \quad \text{(EC.74)}$$

Since c^*_{PD} is increasing in α, we have $\mathrm{d}c^*_{\text{PD}}/\mathrm{d}\alpha > 0$, which implies

$$I_4(c^*_{\text{PD}}) = 2c^*_{\text{PD}} - (c^*_{\text{PD}})^{1+k}(2+k-k^2) - 2k(c^*_{\text{PD}})^k(1-\lambda+\alpha) + 2k(c^*_{\text{PD}})^{2k}(1-\lambda+\alpha) > 0. \quad \text{(EC.75)}$$

By (EC.73), we also get

$$\alpha = \frac{c^*_{\mathrm{PD}}\left(2-(2-k)\left(c^*_{\mathrm{PD}}\right)^k\right)}{\left(1-(c^*_{\mathrm{PD}})^k\right)^2} + \lambda - 1,$$

which allows us to compute

$$\begin{aligned} I_5(c^*_{\mathrm{PD}}) &= c^*_{\mathrm{PD}} - (c^*_{\mathrm{PD}})^{1+k}(1+2k-k^2) + 2k(c^*_{\mathrm{PD}})^k(1-\lambda+\alpha) - 2k(c^*_{\mathrm{PD}})^{2k}(1-\lambda+\alpha) \\ &= \frac{c^*_{\mathrm{PD}}\left(1-(1-k)\left(c^*_{\mathrm{PD}}\right)^k\right)^2 + k^2(c^*_{\mathrm{PD}})^k}{1-(c^*_{\mathrm{PD}})^k} > 0. \end{aligned} \tag{EC.76}$$

Using (EC.74) and the definition of p^*_{PD} given in (EC.47), direct calculation reveals that

$$\frac{\mathrm{d}p^*_{\mathrm{PD}}}{\mathrm{d}\alpha} = \frac{\partial p_{\mathrm{PD}}(c^*_{\mathrm{PD}})}{\partial \alpha} + \frac{\partial p_{\mathrm{PD}}(c^*_{\mathrm{PD}})}{\partial c}\frac{\mathrm{d}c^*_{\mathrm{PD}}}{\mathrm{d}\alpha} = \frac{I_5(c^*_{\mathrm{PD}})}{(1-\lambda)I_4(c^*_{\mathrm{PD}})} > 0, \tag{EC.77}$$

where the inequality follows from (EC.76) and (EC.75) since $0 < \lambda < 1$.

- $(\boldsymbol{c_{\mathrm{PB}}}, \boldsymbol{p^*_{\mathrm{PB}}})$**.** Using $g_{\mathrm{PB}}(c_{\mathrm{PB}}) = 0$ by Claim EC.2 and the Implicit Function Theorem, we find

$$\frac{\mathrm{d}c_{\mathrm{PB}}}{\mathrm{d}\alpha} = \frac{c_{\mathrm{PB}}\left(1-c^k_{\mathrm{PB}}\right)}{2k(1-\lambda+\alpha/2)c^k_{\mathrm{PB}} - 2k(1-\lambda)c^{2k}_{\mathrm{PB}} + c_{\mathrm{PB}}} > 0.$$

Since $p^*_{\mathrm{PB}} = p_{\mathrm{PP}}(c_{\mathrm{PB}}) = c^k_{\mathrm{PB}}$ by Lemma EC.2, p^*_{PB} is also increasing in α.

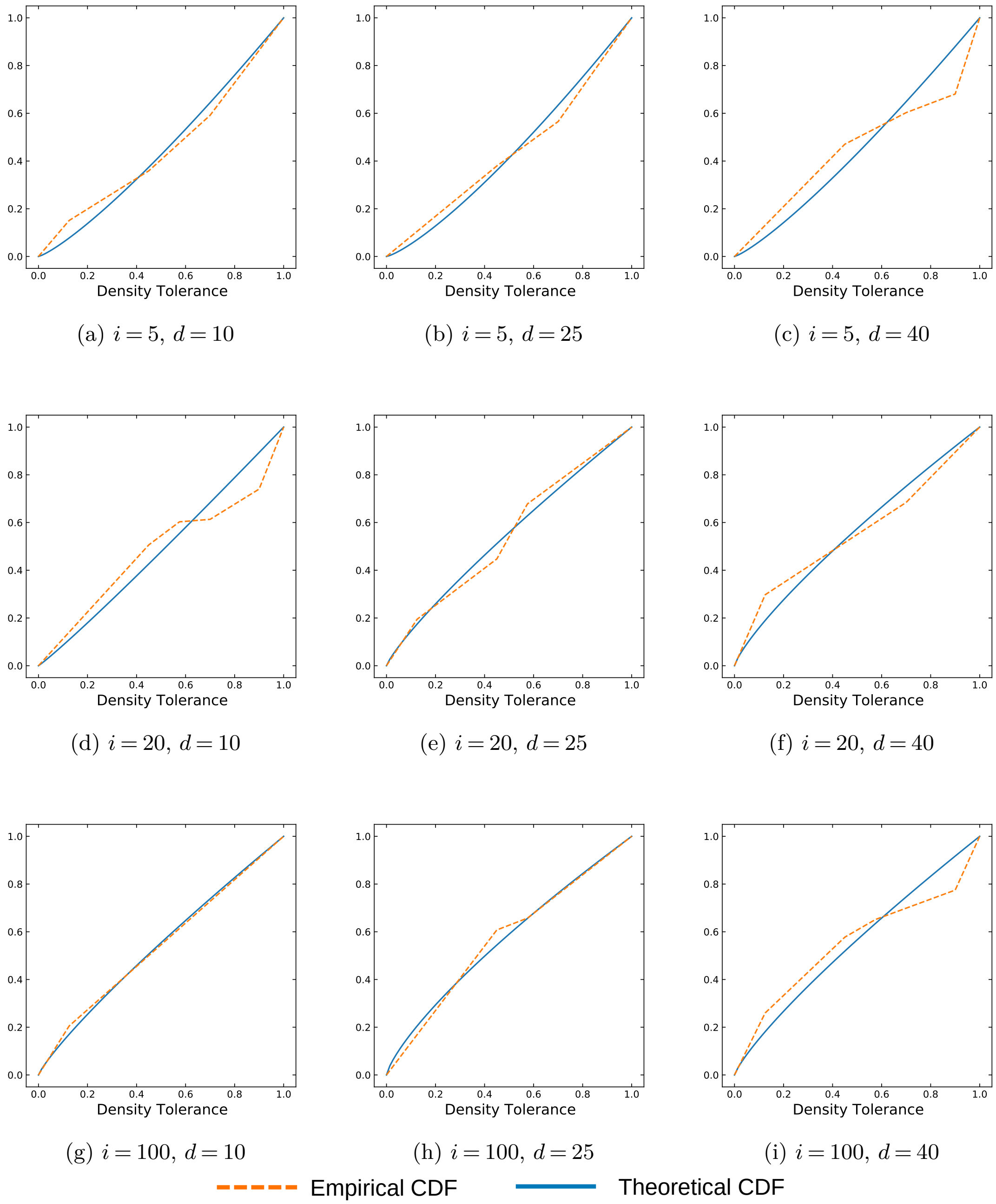
This completes the establishment of the monotonicities of the prices. The remaining steps use the conclusions above to show that $c_{\widetilde{\mathrm{F}}}$ and $c_{\widetilde{\mathrm{P}}}$ are increasing in α.

We will now write $R_{\mathrm{F}}(c,\alpha) = R_{\mathrm{F}}(c)$, $R_{\mathrm{FP}}(c,\alpha) = R_{\mathrm{FP}}(c)$, $R_{\mathrm{FD}}(c,\alpha) = R_{\mathrm{FD}}(c)$, and $c_{\mathrm{FB}}(\alpha) = c_{\mathrm{FB}}$ to make explicit the dependence on α. Note that by the proof of Claim EC.6, we have $R_{\mathrm{FP}}(c,\alpha) \lesseqqgtr R_{\mathrm{FD}}(c,\alpha)$ if and only if $c \lesseqqgtr c_{\mathrm{FB}}$. Hence, by (EC.40), we can write $R_{\mathrm{F}}(c,\alpha) = \min\{R_{\mathrm{FP}}(c,\alpha), R_{\mathrm{FD}}(c,\alpha)\}$. Also note that both $R_{\mathrm{FP}}(c,\alpha)$ and $R_{\mathrm{FD}}(c,\alpha)$ are jointly continuous in c and α by inspection of (EC.40) and (EC.41), respectively. Since the minimum of two continuous functions is continuous, this implies that $R_{\mathrm{F}}(c,\alpha)$ is also jointly continuous in c and α. This joint continuity implies that the optimal density $c_{\widetilde{\mathrm{F}}}$ is continuous in α. Moreover, because $c_{\widetilde{\mathrm{F}}}$ is one of c^*_{FP}, c^*_{FD}, and c_{FB} by Lemma EC.3, it therefore passes continuously among these quantities; since we have shown above that all three are increasing in α, we conclude that $c_{\widetilde{\mathrm{F}}}$ is increasing in α. An analogous argument also shows that $c_{\widetilde{\mathrm{P}}}$ is increasing in α, completing the proof. □

E. Additional Validation of Power Distribution

To justify our focus on the power distribution, we augment our calibrated study in Section 6 with a study based on data from another experimental study (Shelat et al. 2022a), which conducted a series of stated choice experiments to study traveller preferences during the early stages of the COVID-19 pandemic (although Shelat et al. 2022b—used in Section 6—and Shelat et al. 2022a both involve stated choice experiments related to train travel, they are two distinct studies with related but different experimental setups, and data was collected at two different times: December 2020 for Shelat et al. 2022b and May 2020 for Shelat et al. 2022a). In the experiment, respondents participated in a series of choice scenarios where they were informed that they were travelling with the same purpose for which they used the train before the pandemic. They were given two train options with different on-board crowdedness and waiting time (higher waiting time for less crowded train) and asked to rank these trains and the option of not travelling by train. Respondents were also given information about travel duration and prevalent infection rate, which were the same for both train options. Since they might not find it feasible to opt out, we exclude the respondents travelling for the purpose of work. This allows us to focus on those for whom the service (train ride) is discretionary, which aligns with the focus of our study. The on-board crowdedness levels used in choice situations were 5, 18, 23, 28, 36 passengers. Dividing these levels by the car's maximum capacity of 40, we obtain the corresponding service densities of 0.125, 0.45, 0.575, 0.7, and 0.9.

We divide respondents into six groups based on their density tolerance, with each group represented by an interval, i.e. $[0, 0.125)$, $[0.125, 0.45)$, $[0.45, 0.575)$, $[0.575, 0.7)$, $[0.7, 0.9)$ and $[0.9, 1]$. We restrict our study to the cases with



(a) $i = 5,\ d = 10$ (b) $i = 5,\ d = 25$ (c) $i = 5,\ d = 40$

(d) $i = 20,\ d = 10$ (e) $i = 20,\ d = 25$ (f) $i = 20,\ d = 40$

(g) $i = 100,\ d = 10$ (h) $i = 100,\ d = 25$ (i) $i = 100,\ d = 40$

Figure EC.1 Empirical and fitted theoretical CDFs for different values of infection rate i (active cases per 1,000 population) and trip duration d (minutes), when waiting time is 12 minutes

a waiting time of 12 minutes. For each train arrangement—defined by a particular infection prevalence and trip duration—we determine the fraction of respondents with density tolerances in each of the six ranges by calculating the proportion of respondents that ranks not travelling by train higher than travelling with each given density level.[4]

[4] Nearly 30 percent of the respondents made irregular choices in different scenarios of the experiment. That is, a subject ranked the train with a certain density higher than not travelling in one choice situation, but in another choice

Not all train arrangements in the choice experiment were offered to respondents at every density level: for each train arrangement, we group based on the density levels that were offered as a choice in that arrangement (so for some train arrangements, there are fewer than six groups).

Assuming density tolerance is uniformly distributed within each group, we estimate a power distribution using the same method used in Section 6; that is, we match the observed mean with the theoretical mean. For each train arrangement, Figure EC.1 displays the empirical CDF of the data and the fitted theoretical CDF for the estimated power distribution.

We observe from Figure EC.1 that the fitted power distributions are relatively close approximations of the empirical density tolerance distributions. Combining this with our findings in Section 6, we conclude that the power family of distributions is an appropriate choice for our analytical study.

F. Proofs of Supporting Claims

Here, we prove the supporting claims used in the proofs of the main results.

F.1. Proofs of Claims EC.1-EC.5 for Lemma EC.2

Proof of Claim EC.1. It is straightforward to show that $g_{\mathrm{FB}}(0) > 0$ and $g_{\mathrm{FB}}(1) = -1 - \lambda + \alpha < 0$. Differentiating $g_{\mathrm{FB}}(c)$ w.r.t. c, we get

$$g'_{\mathrm{FB}}(c) = -1 - k(2 - \alpha - 2(1-\lambda)c^k)c^{k-1} < 0,$$

so $g_{\mathrm{FB}}(c)$ is strictly decreasing in c. Thus, there exists $c_{\mathrm{FB}} \in (0,1)$ such that $g_{\mathrm{FB}}(c) \gtreqless 0 \iff c \lesseqgtr c_{\mathrm{FB}}$. □

Proof of Claim EC.2. It is straightforward to show that $g_{\mathrm{PB}}(0) = 1 - \lambda + \alpha > 0$, and $g_{\mathrm{PB}}(1) = -1 < 0$. Differentiating $g_{\mathrm{PB}}(c)$ w.r.t c, we get

$$g'_{\mathrm{PB}}(c) = -1 - \alpha c^k - 2k(1-\lambda)c^k(1-c^k) < 0,$$

so $g_{\mathrm{PB}}(c)$ is strictly decreasing in c. Thus, there exists $c_{\mathrm{PB}} \in (0,1)$ such that $g_{\mathrm{PB}} \gtreqless 0 \iff c \lesseqgtr c_{\mathrm{PB}}$. □

Proof of Claim EC.3. It is straightforward to show that $g_{\mathrm{FPP}}(0) = 0$, and $g_{\mathrm{FPP}}(1) = 1 - \alpha > 0$. Direct calculation also reveals that

$$g'_{\mathrm{FPP}}(c) = (c^{1-k} + 2c^k(1-\lambda) - 1 + \lambda - \alpha)c^{k-1}.$$

The sign of the derivative depends on the sign of $c^{1-k} + 2c^k(1-\lambda) - 1 + \lambda - \alpha$, an expression which is strictly increasing in c, i.e., $g_{\mathrm{FPP}}(c)$ is quasiconvex in c. Furthermore, we have

$$\begin{aligned} &\lim_{c\to 0} c^{1-k} + 2c^k(1-\lambda) - 1 + \lambda - \alpha = -1 + \lambda - \alpha < 0, \\ \text{and} \quad &\lim_{c\to 1} c^{1-k} + 2c^k(1-\lambda) - 1 + \lambda - \alpha = 2(1-\lambda) + \lambda - \alpha > 0. \end{aligned}$$

So, $g_{\mathrm{FPP}}(c)$ cannot be monotonically increasing, and we conclude that $g_{\mathrm{FPP}}(c)$ is first decreasing with $g_{\mathrm{FPP}}(0) = 0$ and then increasing with $g_{\mathrm{FPP}}(1) > 0$ for $c \in (0,1]$ (we ignore $c = 0$ in this case since it is never optimal). Thus, there exists $c_{\mathrm{FPP}} \in (0,1)$ such that $g_{\mathrm{FPP}}(c) \gtreqless 0 \iff c \gtreqless c_{\mathrm{FPP}}$. □

Proof of Claim EC.4. It is straightforward to show that $g_{\mathrm{FPD}}(0) = -\alpha < 0$, and $g_{\mathrm{FPD}}(1) = 1 > 0$. Direct calculation also reveals that

$$g'_{\mathrm{FPD}}(c) = (c^{1-k} + 2k(1-\lambda)c^k - k(1-\lambda+\alpha))c^{k-1}.$$

The sign of the derivative depends on the sign of $c^{1-k} + 2k(1-\lambda)c^k - k(1-\lambda+\alpha)$, an expression which is strictly increasing in c, i.e., $g_{\mathrm{FPD}}(c)$ is quasiconvex in c. Together with $g_{\mathrm{FPD}}(0) < 0$ and $g_{\mathrm{FPD}}(1) > 0$, there exists $c_{\mathrm{FPD}} \in (0,1)$ such that $g_{\mathrm{FPD}}(c) \gtreqless 0 \iff c \gtreqless c_{\mathrm{FPD}}$. □

situation with the same train arrangement, the same subject ranked not travelling higher than a train with the same or lower density. We have omitted these respondents from our analysis as outliers.

Proof of Claim EC.5. By the definition of c_{FB} in Claim EC.1, i.e., $g_{\mathrm{FB}}(c_{\mathrm{FB}})=0$, we have

$$g_{\mathrm{FPP}}(c_{\mathrm{FB}})=g_{\mathrm{FPP}}(c_{\mathrm{FB}})+g_{\mathrm{FB}}(c_{\mathrm{FB}})=(1-2c_{\mathrm{FB}}^k)(1-(1-\lambda)c_{\mathrm{FB}}^k) \quad \text{and}$$
$$g_{\mathrm{FPD}}(c_{\mathrm{FB}})=g_{\mathrm{FPD}}(c_{\mathrm{FB}})+g_{\mathrm{FB}}(c_{\mathrm{FB}})=(1-2c_{\mathrm{FB}}^k)(1-\alpha-(1-\lambda)c_{\mathrm{FB}}^k).$$

Since $c_{\mathrm{FB}}\in(0,1)$, we have $1-(1-\lambda)c_{\mathrm{FB}}^k>0$ and $1-\alpha-(1-\lambda)c_{\mathrm{FB}}^k>0$. Therefore, the signs of both $g_{\mathrm{FPP}}(c_{\mathrm{FB}})$ and $g_{\mathrm{FPD}}(c_{\mathrm{FB}})$ depend on the sign of $1-2c_{\mathrm{FB}}^k$, i.e., $g_{\mathrm{FPP}}(c_{\mathrm{FB}})\gtreqqless 0 \iff g_{\mathrm{FPD}}(c_{\mathrm{FB}})\gtreqqless 0$, which implies that $c_{\mathrm{FPP}}\lesseqqgtr c_{\mathrm{FB}} \iff c_{\mathrm{FPD}}\lesseqqgtr c_{\mathrm{FB}}$ by Claims EC.3 and EC.4.

By the definition of c_{PB} in Claim EC.2, i.e., $g_{\mathrm{PB}}(c_{\mathrm{PB}})$, direct calculation also reveals that

$$g_{\mathrm{FPP}}(c_{\mathrm{PB}})=g_{\mathrm{FPP}}(c_{\mathrm{PB}})+g_{\mathrm{PB}}(c_{\mathrm{PB}})=(1-2c_{\mathrm{PB}}^k)(\alpha+(1-\lambda)(1-c_{\mathrm{PB}}^k)) \quad \text{and}$$
$$g_{\mathrm{FPD}}(c_{\mathrm{PB}})=g_{\mathrm{FPD}}(c_{\mathrm{PB}})+g_{\mathrm{PB}}(c_{\mathrm{PB}})=(1-2c_{\mathrm{PB}}^k)(1-\lambda)(1-c_{\mathrm{PB}}^k).$$

Since $c_{\mathrm{PB}}\in(0,1)$, we have $\alpha+(1-\lambda)(1-c_{\mathrm{PB}}^k)>0$ and $(1-\lambda)(1-c_{\mathrm{PB}}^k)>0$. Therefore, the signs of both $g_{\mathrm{FPP}}(c_{\mathrm{PB}})$ and $g_{\mathrm{FPD}}(c_{\mathrm{PB}})$ depend on the sign of $1-2c_{\mathrm{PB}}^k$, i.e., $g_{\mathrm{FPP}}(c_{\mathrm{PB}})\gtreqqless 0 \iff g_{\mathrm{FPD}}(c_{\mathrm{PB}})\gtreqqless 0$, which implies that $c_{\mathrm{FPP}}\lesseqqgtr c_{\mathrm{PB}} \iff c_{\mathrm{FPD}}\lesseqqgtr c_{\mathrm{PB}}$ by Claims EC.3 and EC.4. □

F.2. Proofs of Claims EC.6-EC.10 for Lemma EC.3

Proof of Claim EC.6. Subtracting $R_{\mathrm{FD}}(c)$ from $R_{\mathrm{FP}}(c)$ gives

$$R_{\mathrm{FP}}(c)-R_{\mathrm{FD}}(c)=\frac{-\alpha c g_{\mathrm{FB}}(c)}{(1-(1-\lambda)c^k)\,(1-\alpha-(1-\lambda)c^k)}, \tag{EC.78}$$

where $g_{\mathrm{FB}}(c)$ is given by equation (EC.52) in the proof of Lemma EC.2. Since $\alpha\le\lambda\le 1$, the denominator of equation (EC.78) is always positive, and therefore by Claim EC.1 in the proof of Lemma EC.2, we have $R_{\mathrm{FP}}(c)<R_{\mathrm{FD}}(c)$ for $c<c_{\mathrm{FB}}$, $R_{\mathrm{FD}}(c)<R_{\mathrm{FP}}(c)$ for $c>c_{\mathrm{FB}}$, and $R_{\mathrm{FP}}(c_{\mathrm{FB}})=R_{\mathrm{FD}}(c_{\mathrm{FB}})$. The continuity of $R_{\mathrm{F}}(c)$ then follows because (i) the function is piecewise continuous by the continuity of R_{FP} and R_{FD} and (ii) we have $R_{\mathrm{FP}}(c_{\mathrm{FB}})=R_{\mathrm{FD}}(c_{\mathrm{FB}})$, i.e., the two pieces coincide at the boundary. Furthermore, it is straightforward to show that

$$R'_{\mathrm{F}}(0)=R'_{\mathrm{FP}}(0)=1>0, \quad \text{and} \quad R'_{\mathrm{F}}(1)=R'_{\mathrm{FD}}(1)=\frac{-k(1-\alpha)-2(\lambda-\alpha)}{(\lambda-\alpha)^2}<0.$$

Thus, by continuity the function must have an interior global maximum on $[0,1]$.

We now show by contradiction that $R'_{\mathrm{FP}}(c_{\mathrm{FB}})\le 0$ implies $R'_{\mathrm{FD}}(c_{\mathrm{FB}})\le 0$. Suppose to the contrary that $R'_{\mathrm{FP}}(c_{\mathrm{FB}})\le 0$ and $R'_{\mathrm{FD}}(c_{\mathrm{FB}})>0$. In this case, there must exist $\epsilon>0$ such that $R_{\mathrm{FD}}(c_{\mathrm{FB}}+\epsilon)>R_{\mathrm{FD}}(c_{\mathrm{FB}})=R_{\mathrm{FP}}(c_{\mathrm{FB}})>R_{\mathrm{FP}}(c_{\mathrm{FB}}+\epsilon)$. But this contradicts $R_{\mathrm{FP}}(c)>R_{\mathrm{FD}}(c)$ for $c>c_{\mathrm{FB}}$, which we have just shown. We conclude then that $R'_{\mathrm{FP}}(c_{\mathrm{FB}})\le 0$ implies $R'_{\mathrm{FD}}(c_{\mathrm{FB}})\le 0$.

Consequently, since both $R_{\mathrm{FP}}(c)$ and $R_{\mathrm{FD}}(c)$ are strictly concave by Lemma EC.2, if $R_{\mathrm{F}}(c)$ is decreasing as c approaches c_{FB} from the left, then it cannot increase for $c\ge c_{\mathrm{FB}}$. Note that, if $R'_{\mathrm{FP}}(c_{\mathrm{FB}})>0$, by the strict concavity of $R_{\mathrm{FP}}(c)$ and $R_{\mathrm{FD}}(c)$, the maximizer of $R_{\mathrm{F}}(c)$ will be either c_{FB} or $c^*_{\mathrm{FD}}>c_{\mathrm{FB}}$ depending on whether R_{FD} is decreasing or increasing at $c=c_{\mathrm{FB}}$. Hence, the function cannot have multiple local maxima. We conclude that $R_{\mathrm{F}}(c)$ is unimodal, and its maximizer $c_{\widetilde{\mathrm{F}}}\in(0,1)$. □

Proof of Claim EC.7. Applying the Implicit Function Theorem to equation (EC.59), we compute the derivative of $x(k)$ with respect to k as

$$x'(k)=\frac{x(k)^{\frac{1}{k}+1}\log(x(k))}{k\left(x\left(k\right)^{\frac{1}{k}}+kx\left(k\right)\left(2-\alpha-2\left(1-\lambda\right)x\left(k\right)\right)\right)}<0,$$

which follows from $x(k)\in(0,1)$, $k\in(0,1]$ and $\alpha<\lambda$, and implies that $x(k)$ is decreasing in k. □

Proof of Claim EC.8. The formula for $M_1(k)$ is given in equation (EC.60). First, we show that $M_1(k)>0$ for k sufficiently close to zero, and note that with a slight abuse of notation we write $x(0)$ for $\lim_{k\to 0} x(k)$. We have $x(k)$ decreasing in k (so increasing as $k\downarrow 0$) by Claim EC.7 and also $x(k)$ is bounded above by one, so $\lim_{k\downarrow 0} x(k)$ exists. Since $x(k)$ strictly increases as k decreases, we have $\lim_{k\downarrow 0} x(k)=b$ for some $0<b\le 1$. If $b<1$, then $x(k)^{1/k}\to 0$ as $k\downarrow 0$. If $b=1$, then by equation (EC.59), we have $\alpha-\lambda=\lim_{k\downarrow 0} x(k)^{1/k}$. Since $\alpha\le\lambda$ and $x(k)^{1/k}$ is nonnegative, this is only possible if $\alpha=\lambda$ which implies $\lim_{k\downarrow 0} x(k)^{1/k}=0$. We conclude that $\lim_{k\to 0} x(k)=b<1$, implying that $x(k)^{1/k}\to 0$ as $k\downarrow 0$. Together with equation (EC.59), this implies

$$(1-\lambda)x(0)^2=(2-\alpha)x(0)-1.$$

We then have

$$\begin{aligned}\lim_{k\to 0} M_1(0)&=-1+(3-\alpha)x(0)-2(1-\lambda)x(0)^2=-1+(3-\alpha)x(0)-2\left((2-\alpha)\,x(0)-1\right)\\&=1-(1-\alpha)x(0)>0.\end{aligned}$$

An analogous substitution again using equation (EC.59) yields

$$M_1(1)=-1+2x(1)-(1-\lambda)x(1)^2=-1+2x(1)-(3-\alpha)x(1)+1=-(1-\alpha)x(1)<0.$$

Since $M_1(0)>0$ and $M_1(1)<0$, by the Intermediate Value Theorem there must exist $k_{\mathrm{FP}}\in(0,1)$ such that $M_1(k_{\mathrm{FP}})=0$. Thus, by (EC.60), we can solve for $x(k_{\mathrm{FP}})$, which is given by one of the following two quadratic roots

$$\frac{3-\alpha-k_{\mathrm{FP}}+\alpha k_{\mathrm{FP}}\pm\sqrt{(3-\alpha-k_{\mathrm{FP}}+\alpha k_{\mathrm{FP}})^2-4(2-k_{\mathrm{FP}})(1-\lambda)}}{2(2-k_{\mathrm{FP}})(1-\lambda)}.$$

We claim that the larger root is greater than 1. To see this, note that $0<k_{\mathrm{FP}}<1$ and $0<\alpha<\lambda<1$ imply

$$\begin{aligned}&4(2-k_{\mathrm{FP}})(1-\lambda)((2-k_{\mathrm{FP}})\lambda-(1-k_{\mathrm{FP}})\alpha)>0\\ \implies&\sqrt{(3-\alpha-k_{\mathrm{FP}}+\alpha k_{\mathrm{FP}})^2-4(2-k_{\mathrm{FP}})(1-\lambda)}>1+\alpha(1-k_{\mathrm{FP}})-4\lambda-k_{\mathrm{FP}}(1-2\lambda)\\ \implies&\frac{3-\alpha-k_{\mathrm{FP}}+\alpha k_{\mathrm{FP}}+\sqrt{(3-\alpha-k_{\mathrm{FP}}+\alpha k_{\mathrm{FP}})^2-4(2-k_{\mathrm{FP}})(1-\lambda)}}{2(2-k_{\mathrm{FP}})(1-\lambda)}>1.\end{aligned}$$

Therefore, we must have

$$x(k_{\mathrm{FP}})=\frac{3-\alpha-k_{\mathrm{FP}}+\alpha k_{\mathrm{FP}}-\sqrt{(3-\alpha-k_{\mathrm{FP}}+\alpha k_{\mathrm{FP}})^2-4(2-k_{\mathrm{FP}})(1-\lambda)}}{2(2-k_{\mathrm{FP}})(1-\lambda)}. \tag{EC.79}$$

Direct calculation reveals that

$$M_1'(k)=-x(k)(1-\alpha-(1-\lambda)x(k))+x'(k)(3-k-\alpha+\alpha k-2(2-k)(1-\lambda)x(k)), \tag{EC.80}$$

Using (EC.79) and (EC.80), we now show that $M_1'(k_{\mathrm{FP}})<0$. The quantity $-x(k)(1-\alpha-(1-\lambda)x(k))$ from (EC.80) can be shown to be negative, and $x'(k)$ is negative for all values of k by Claim EC.7. Substituting the value of $x(k_{\mathrm{FP}})$ in (EC.79), we also have

$$3-k_{\mathrm{FP}}-\alpha+\alpha k_{\mathrm{FP}}-2(2-k)(1-\lambda)x(k_{\mathrm{FP}})=\sqrt{(3-\alpha-k_{\mathrm{FP}}+\alpha k_{\mathrm{FP}})^2-4(2-k_{\mathrm{FP}})(1-\lambda)}\ge 0.$$

Therefore, the RHS of equation (EC.80) evaluated at $k=k_{\mathrm{FP}}$ is negative, i.e., $M_1'(k_{\mathrm{FP}})<0$, which immediately implies the uniqueness of k_{FP}. Indeed, suppose that there are multiple values of k with $M_1(k)=0$. In this case, the function must cross zero multiple times, which would require $M_1'(k)\ge 0$ for at least one k with $M_1(k)=0$, leading to a contradiction. The uniqueness of k_{FP} also implies that $M_1(k)\lesseqgtr 0$ for $k\gtreqless k_{\mathrm{FP}}$. □

Proof of Claim EC.9. Starting from the definition of $M_2(k)$ in equation (EC.60), straightforward algebra reveals that $M_2(k) \lesseqqgtr 0$ is equivalent to $g(x(k)) \lesseqqgtr k$, where $g:(0,1]\to\mathbb{R}$ is given by

$$g(x) := -\frac{1-(3-2\alpha)x+2(1-\lambda)x^2}{x\,(1-(1-\lambda)x)}. \tag{EC.81}$$

First, we note that $g(1)=\frac{2(\lambda-\alpha)}{\lambda}>0$ and $\lim_{x\to 0^+} g(x)=-\infty$, implying that $g(x)$ cannot be monotonically decreasing in $x\in(0,1]$. Furthermore, we compute

$$g'(x) = \frac{1-2(1-\lambda)x+(1-2\alpha)(1-\lambda)x^2}{x^2(1-(1-\lambda)x)^2}, \tag{EC.82}$$

whose denominator is positive and whose numerator can be shown to be decreasing for $x\in(0,1]$ by direct calculation as follows:

$$(1-2(1-\lambda)x+(1-2\alpha)(1-\lambda)x^2)' = -2(1-\lambda)(1-(1-2\alpha)x) < 0.$$

Since $g(x)$ cannot be monotonically decreasing in x, $g(x)$ is therefore either monotonically increasing, or first increasing then decreasing, in x. Direct evaluation yields

$$g'(1) = \frac{\lambda-2\alpha(1-\lambda)}{\lambda} \ge 0 \iff \alpha \le \frac{\lambda}{2-2\lambda}.$$

Thus, $g(x)$ is monotonically increasing in $x\in(0,1]$ for $\alpha\le\frac{\lambda}{2-2\lambda}$, and $g(x)$ is first increasing then decreasing in $x\in(0,1]$ for $\alpha>\frac{\lambda}{2-2\lambda}$ because $g'(0+)=+\infty$.

Case 1: $\boldsymbol{\alpha\le\lambda/(2-2\lambda)}$. In this case, $g(x)$ is monotonically increasing in $x\in(0,1]$ and hence $g(x(k))-k$ is strictly decreasing in k by Claim EC.7. Furthermore, by (EC.59) and $\lim_{k\to 0} x^{1/k}=0$, we have

$$x(0) = \frac{2-\alpha-\sqrt{\alpha^2-4\alpha+4\lambda}}{2(1-\lambda)}, \quad \text{and} \quad x(1) = \frac{3-\alpha-\sqrt{\alpha^2-6\alpha+4\lambda+5}}{2(1-\lambda)}. \tag{EC.83}$$

Substitution into equation (EC.81) yields

$$\begin{aligned} g(x(0)) &= \frac{2\left(\sqrt{\alpha^2-4\alpha+4\lambda}+\alpha-2\lambda\right)}{\left(2-\alpha-\sqrt{\alpha^2-4\alpha+4\lambda}\right)\left(\sqrt{\alpha^2-4\alpha+4\lambda}+\alpha\right)}, \quad \text{and} \\ g(x(1)) &= \frac{2\left(7+2\lambda-3\alpha-3\sqrt{\alpha^2-6\alpha+4\lambda+5}\right)}{\left(\sqrt{\alpha^2-6\alpha+4\lambda+5}+\alpha-3\right)\left(\sqrt{\alpha^2-6\alpha+4\lambda+5}+\alpha-1\right)}. \end{aligned}$$

It is straightforward to show that

$$\begin{aligned} &\left.\begin{array}{l}(\alpha^2-4\alpha+4\lambda)-(2\lambda-\alpha)^2 = 4(\lambda-\alpha)(1-\lambda)>0,\\ (2-\alpha)^2-(\alpha^2-4\alpha+4\lambda)=4(1-\lambda)>0\end{array}\right\} \implies g(x(0))>0, \quad \text{and} \\ &\left.\begin{array}{l}(7+2\lambda-3\alpha)^2-9(\alpha^2-6\alpha+4\lambda+5)=4(1-\lambda)(1+3\alpha-\lambda)>0,\\ (\alpha^2-6\alpha+4\lambda+5)-(3-\alpha)^2=-4(1-\lambda)<0,\\ (\alpha^2-6\alpha+4\lambda+5)-(1-\alpha)^2=4(1-\alpha+\lambda)>0\end{array}\right\} \implies g(x(1))<0. \end{aligned} \tag{EC.84}$$

By the Intermediate Value Theorem, there uniquely exists $k_{\rm FD}\in(0,1)$ with $k_{\rm FD}=g(x(k_{\rm FD}))$ and $k \lesseqgtr g(x(k))$ for $k\lesseqgtr k_{\rm FD}$.

Case 2: $\boldsymbol{\alpha>\lambda/(2-2\lambda)}$. In this case, we must have $\lambda<1/2$ and $g(x)$ first increasing and then decreasing in $x\in(0,1]$. More specifically, let $\hat{x}\in(0,1]$ be the solution to $g'(x)=0$; namely

$$1-2(1-\lambda)\hat{x}+(1-2\alpha)(1-\lambda)\hat{x}^2=0, \tag{EC.85}$$

which has two solutions and only the following one falls between zero and one:

$$\hat{x} = \frac{1}{1-2\alpha}-\sqrt{\frac{2\alpha-\lambda}{(1-2\alpha)^2(1-\lambda)}}. \tag{EC.86}$$

Thus, $g(x)$ is increasing in $x\in(0,\hat{x}]$ and then decreasing in $x\in[\hat{x},1]$.

By Claim EC.7, $x(k)$ is decreasing in $k \in [0,1]$, so we have $\underline{x} := x(1) \leq x(k) \leq \bar{x} := x(0)$. By (EC.83), we have

$$\underline{x} = \frac{3-\alpha-\sqrt{\alpha^2-6\alpha+4\lambda+5}}{2(1-\lambda)}, \quad \text{and} \quad \bar{x} = \frac{2-\alpha-\sqrt{4\lambda+\alpha^2-4\alpha}}{2(1-\lambda)}.$$

By (EC.84), we have $g(\bar{x}) > 0$ and $g(\underline{x}) < 0$. Note that we cannot have $\hat{x} \leq \underline{x}$ because $g(\bar{x}) > 0 > g(\underline{x})$ implies that we cannot have $g(x)$ to be monotonically decreasing from $\underline{x}$ to $\bar{x}$.

If $\hat{x} \geq \bar{x}$, then $g(x)$ will be monotonically increasing in $x \in [\underline{x}, \bar{x}]$, and subsequently $g(x(k))$ is decreasing in $k \in (0,1]$. Thus, there exists a unique $k_{\mathrm{FD}} \in (0,1]$ with $k_{\mathrm{FD}} = g(x(k_{\mathrm{FD}}))$ and $k \lesseqgtr g(x(k))$ for $k \lesseqgtr k_{\mathrm{FD}}$.

If $\hat{x} < \bar{x}$, then $g(x)$ is increasing in $x \in [\underline{x}, \hat{x}]$ and decreasing in $x \in (\hat{x}, \bar{x}]$. We show, using the following series of technical results, that there exists a unique $k_{\mathrm{FD}} \in (0,1]$ with $k_{\mathrm{FD}} = g(x(k_{\mathrm{FD}}))$ and $k \lesseqgtr g(x(k))$ for $k \lesseqgtr k_{\mathrm{FD}}$.

- The function $F_2(x) \geq 0$ for $\hat{x} \leq x \leq 1$, where

$$\begin{aligned} F_2(x) := 14 - 4\alpha - 4\lambda + 3(-16 + 4\alpha^2 + \alpha(2-10\lambda) + 19\lambda)x \\ + 6(1-\lambda)(9 - 4\alpha + 4\alpha^2 - 8\lambda)x^2 + 20(1-2\alpha)(1-\lambda)^2 x^3. \end{aligned} \tag{EC.87}$$

To show that $F_2(x)$ is nonnegative on the relevant range, we will show that it is increasing in x and that its value at $x = \hat{x}$ is non-negative. Differentiating $F_2(x)$ w.r.t x, we get

$$F_2'(x) = 3\left(-16 + 4\alpha^2 + \alpha(2-10\lambda) + 19\lambda + 4(1-\lambda)\left(9 - 8\lambda - 4\alpha + 4\alpha^2\right)\right)x - 20(1-2\alpha)(1-\lambda)^2 x^2\right).$$

$F_2'(x)$ is a downward quadratic function of x, so it is concave in x. For λ and α satisfying the hypothesis of this case, evaluating the derivative at $x = 1$ yields

$$F_2'(1) = 3(4\alpha^2(5-4\lambda) - 3\lambda(3-4\lambda) + \alpha(26 - 74\lambda + 40\lambda^2)) \geq 0.$$

Similarly, when $x = \hat{x}$, using $1 - 2(1-\lambda)\hat{x} + (1-2\alpha)(1-\lambda)\hat{x}^2 = 0$, we get

$$F_2'(\hat{x}) = 3\hat{x}(1-\lambda)(4 - 12\alpha + 24\alpha^2 + 6\lambda - 20\alpha\lambda + (-4 + 6\alpha + 8\alpha^3 + \lambda + 8\alpha\lambda - 20\alpha^2\lambda)\hat{x}).$$

For $x \in [0,1]$ and α and λ satisfying this case, we also have

$$4 - 12\alpha + 24\alpha^2 + 6\lambda - 20\alpha\lambda + (-4 + 6\alpha + 8\alpha^3 + \lambda + 8\alpha\lambda - 20\alpha^2\lambda)x \geq 0,$$

which implies that $F_2'(\hat{x}) \geq 0$ as well. Since $F_2'(x)$ is concave in x, and $F_2'(\hat{x}) \geq 0$ and $F_2'(1) \geq 0$, we can conclude that $F_2'(x) \geq 0$ for $x \in [\hat{x}, 1]$, i.e., that $F_2(x)$ is non-decreasing in x for the given interval.

Finally, using $1 - 2(1-\lambda)\hat{x} + (1-2\alpha)(1-\lambda)\hat{x}^2 = 0$, which holds by the hypothesis of this case, we can show that

$$F_2(\hat{x}) = \hat{x}^2(1-\lambda)(2\alpha-\lambda)(2 + 20\alpha - 16\lambda + (1-2\alpha)(1-6\alpha+8\lambda)\hat{x}). \tag{EC.88}$$

We also have

$$2 + 20\alpha - 16\lambda + (1-2\alpha)(1-6\alpha+8\lambda)\hat{x} \geq 2 + 20\alpha - 16\lambda \geq \frac{2(1-4\lambda+8\lambda^2)}{1-\lambda} \geq 0. \tag{EC.89}$$

The first inequality holds because the far LHS is increasing in $\hat{x}$, and the second two inequalities follow from the case hypothesis. Equations (EC.88) and (EC.89) together imply that $F_2(\hat{x}) \geq 0$, and we conclude that $F_2(x) \geq 0$ for all $x \in [\hat{x}, 1]$ because we have shown that $F_2(x)$ is non-decreasing in this range.

- The function $F_1(x) \leq 0$ for $\hat{x} \leq x \leq 1$, where

$$\begin{aligned} F_1(x) := 1 - (6 - 2\alpha - 3\lambda)x + 2(1-\lambda)(7 - 2\alpha - 2\lambda) + (1-\lambda)^2(9 - 4\alpha + 4\alpha^2 - 8\lambda) \\ - (1-\lambda)\left(16 - 4\alpha^2 - 19\lambda - 2\alpha(1-5\lambda)\right)x^3 - 2(1-\lambda)^3(1-2\alpha)x^5. \end{aligned} \tag{EC.90}$$

Differentiating $F_1(x)$ twice, we get

$$F_1''(x) = 2(1-\lambda)F_2(x). \tag{EC.91}$$

As we have shown in the previous step that $F_2(x) \geq 0$ for $\hat{x} \leq x \leq 1$, equation (EC.91) therefore implies that $F_1''(x) \geq 0$ in the same range, i.e., $F_1(x)$ is convex for $\hat{x} \leq x \leq 1$.

For $x=1$, we have

$$F_1(1)=2((2-3\lambda)\lambda^2+\alpha\lambda(-6+9\lambda-2\lambda^2)+2\alpha^2(2-3\lambda+\lambda^2)). \tag{EC.92}$$

The RHS of equation (EC.92) is a convex quadratic function of α for a given λ, and for $\lambda \le 1/2$, it is non-positive both for $\alpha=\lambda$ and $\alpha=\lambda/(2-2\lambda)$. Together with equation (EC.92), this implies that $F_1(1)\le 0$. Similarly, for $x=\hat{x}$, again using equation (EC.85), we have

$$F_1(\hat{x})=2\hat{x}^4(1-\lambda)^2(2\alpha-\lambda)(-1+6\alpha-4\lambda+(1-2\alpha)(1-2\alpha+2\lambda)\hat{x}). \tag{EC.93}$$

We also have

$$\begin{aligned}-1+6\alpha-4\lambda+(1-2\alpha)(1-2\alpha+2\lambda)\hat{x} &\le -1+6\alpha-4\lambda+(1-2\alpha)(1-2\alpha+2\lambda)\\ &=-2(1+2\alpha)(\lambda-\alpha)\\ &\le 0,\end{aligned}$$

which by equation (EC.93) implies that $F_1(\hat{x})\le 0$ (note that $\alpha\ge\lambda/(2-2\lambda)$ implies $2\alpha-\lambda>0$). Since $F_1(x)$ is convex in x for $x\in[\hat{x},1]$, and $F_1(\hat{x})\le 0$ and $F_1(1)\le 0$, we can conclude that $F_1(x)\le 0$ for $x\in[\hat{x},1]$.

- For $\hat{x}\le x\le 1$, we have $\Upsilon(x)\ge 0$, where

$$\Upsilon(x):=\log(x)-\frac{x(1-(1-\lambda)\left(-1+(3-2\alpha)\,x-2\,(1-\lambda)\,x^2\right))}{(1-2(1-\lambda)+(1-2\alpha)(1-\lambda))}. \tag{EC.94}$$

Differentiating $\Upsilon(x)$ w.r.t. x gives

$$\Upsilon'(x)=\frac{F_1(x)}{x(1-2(1-\lambda)x+(1-2\alpha)(1-\lambda)x^2)^2}. \tag{EC.95}$$

The denominator of equation (EC.95) is positive, so $\Upsilon'(x)$ has the same sign as $F_1'(x)$. As we have just shown, $F_1'(x)\le 0$ in the relevant range, and we thus have that $\Upsilon'(x)\le 0$—i.e., $\Upsilon(x)$ is decreasing—for $\hat{x}\le x\le 1$. Evaluating $\Upsilon(x)$ at $x=1$, we get

$$\Upsilon(1)=\frac{2(\lambda-\alpha)\lambda}{2\alpha(1-\lambda)-\lambda}\ge 0,$$

where the inequality holds by the case hypothesis. We conclude that $\Upsilon(x)\ge 0$ for $x\in(\hat{x},1)$.

- The equation

$$\frac{1}{\log(1-(2-\alpha)x+(1-\lambda)x^2)}=-\frac{1-(3-2\alpha)x+2(1-\lambda)x^2}{x(1-(1-\lambda)x)\log(x)}. \tag{EC.96}$$

has at most one solution for $x\in[\hat{x},\bar{x}]$.

The expression $1-(2-\alpha)x+(1-\lambda)x^2$ is decreasing in x, and the result of the expression is also between 0 and 1 for the relevant range. This implies that the LHS of (EC.96) is increasing in x. We can also show that the right hand side is non-increasing in x. Differentiating the right hand side w.r.t x we get

$$\left(-\frac{1-(3-2\alpha)x+2(1-\lambda)x^2}{x(1-(1-\lambda))\log(x)}\right)'=\frac{(1-2(1-\lambda)+(1-2\alpha)(1-\lambda))}{x^2(1-(1-\lambda)x)^2\log^2(x)}\Upsilon(x). \tag{EC.97}$$

The numerator is negative and the denominator is positive in the fraction on the RHS of (EC.97), so the sign of the LHS is the opposite of the sign of $\Upsilon(x)$. We showed in the previous step that $\Upsilon(x)\ge 0$ for $\hat{x}\le x\le 1$, which therefore implies that the RHS of (EC.97) is non-positive and, in turn, that the RHS of (EC.96) is non-increasing in x. Therefore, on the interval $(\hat{x},\bar{x}]$, there is at most one solution in x to (EC.96) because the LHS is increasing and the RHS is non-increasing.

- There exists a unique $k_{\mathrm{FD}}\in(0,1]$ such that $k_{\mathrm{FD}}=g(x(k_{\mathrm{FD}}))$, implying $k\lesseqgtr g(x(k))$ for $k\lesseqgtr k_{\mathrm{FD}}$.

Rewriting equation (EC.59) yields

$$k = \frac{\log x(k)}{\log(1-(2-\alpha)x(k)+(1-\lambda)x^2(k))}. \tag{EC.98}$$

Thus, by the definition of $g(\cdot)$ in (EC.81), the equation $k = g(x(k))$ can be equivalently expressed as

$$\frac{1}{\log(1-(2-\alpha)x(k)+(1-\lambda)x^2(k))} = -\frac{1-(3-2\alpha)x(k)+2(1-\lambda)x^2(k)}{x(k)(1-(1-\lambda)x(k))\log x(k)}.$$

Since there is at most one solution to equation (EC.96) on $(\hat{x}, \bar{x}]$, by Claim EC.7, there is also at most one solution to $k = g(x(k))$ on $(0, \hat{k})$, where $\hat{k} = x^{-1}(\hat{x})$ (the inverse function x^{-1} exists because $x(k)$ is strictly decreasing in k). For $k \in [\hat{k}, 1]$, we have $g(x(k)) - k$ decreasing in k, so there is at most one solution to $k = g(x(k))$ on $[\hat{k}, 1]$. Thus, there are at most two solutions to $k = g(x(k))$ on the interval $(0,1]$. Recall that by (EC.84), we have $g(x(1)) - 1 < 0 < g(x(0)) - 0$, so by the Intermediate Value Theorem so there is at least one solution to $k = g(x(k))$. Moreover, there cannot be exactly two solutions; otherwise, $g(x(0)) - 0$ and $g(x(1)) - 1$ must be of the same sign, a contradiction. □

Proof of Claim EC.10. We can verify that $k_{\mathrm{FD}} \le k_{\mathrm{FP}}$ as follows. We have $M_2(k) = M_1(k) - \alpha(1+k)x(k)$, and also $M_1(k_{\mathrm{FP}}) = 0$ by Claim EC.8. We can then write

$$M_2(k_{\mathrm{FP}}) = -\alpha(1+k_{\mathrm{FP}})x(k_{\mathrm{FP}}) < 0 \implies k_{\mathrm{FP}} > k_{\mathrm{FD}},$$

where the implication follows because $M_2(k) < 0$ if and only if $k > k_{\mathrm{FD}}$ by Claim EC.9. □

F.3. Proofs of Claims EC.11-EC.15 for Lemma EC.4

Proof of Claim EC.11. Subtracting $R_{\mathrm{PD}}(c)$ from $R_{\mathrm{PP}}(c)$ gives

$$R_{\mathrm{PP}}(c) - R_{\mathrm{PD}}(c) = \frac{-\alpha c g_{\mathrm{PB}}(c)}{(1-\lambda)(1-c^k)(a+(1-\lambda)(1-c^k))}, \tag{EC.99}$$

where $g_{\mathrm{PB}}(c)$ is given by equation (EC.54) in the proof of Lemma EC.2. Since $\alpha \le \lambda < 1$, the denominator of equation (EC.99) is always positive, and therefore by Claim EC.2 in the proof of Lemma EC.2, we have $R_{\mathrm{PP}}(c) < R_{\mathrm{PD}}(c)$ for $c < c_{\mathrm{PB}}$, $R_{\mathrm{PD}}(c) < R_{\mathrm{PP}}(c)$ for $c > c_{\mathrm{PB}}$, and $R_{\mathrm{PP}}(c_{\mathrm{PB}}) = R_{\mathrm{PD}}(c_{\mathrm{PB}})$. The continuity of $R_{\mathrm{P}}(c)$ then follows because (i) the function is piecewise continuous by the continuity of R_{PP} and R_{PD} and (ii) we have $R_{\mathrm{PP}}(c_{\mathrm{PB}}) = R_{\mathrm{PD}}(c_{\mathrm{PB}})$, i.e., the two pieces coincide at the boundary. Furthermore, it is straightforward to show that

$$R'_{\mathrm{P}}(0) = R'_{\mathrm{PP}}(0) = 1 > 0, \quad \text{and} \quad R'_{\mathrm{P}} = R'_{\mathrm{PD}}(1) = -\infty < 0.$$

Thus, by continuity the function must have an interior global maximum on $[0,1]$.

We now show by contradiction that $R'_{\mathrm{PP}}(c_{\mathrm{PB}}) \le 0$ implies $R'_{\mathrm{PD}}(c_{\mathrm{PB}}) \le 0$. Suppose to the contrary that $R'_{\mathrm{PP}}(c_{\mathrm{PB}}) \le 0$ and $R'_{\mathrm{PD}}(c_{\mathrm{PB}}) > 0$. In this case, there must exist $\epsilon > 0$ such that $R_{\mathrm{PD}}(c_{\mathrm{PB}}+\epsilon) > R_{\mathrm{PD}}(c_{\mathrm{PB}}) = R_{\mathrm{PP}}(c_{\mathrm{PB}}) > R_{\mathrm{PP}}(c_{\mathrm{PB}}+\epsilon)$. But this contradicts $R_{\mathrm{PP}}(c) > R_{\mathrm{PD}}(c)$ for $c > c_{\mathrm{PB}}$, which we have just shown. We conclude then that $R'_{\mathrm{PP}}(c_{\mathrm{PB}}) \le 0$ implies $R'_{\mathrm{PD}}(c_{\mathrm{PB}}) \le 0$.

Consequently, since both $R_{\mathrm{PP}}(c)$ and $R_{\mathrm{PD}}(c)$ are strictly concave by Lemma EC.2, if $R_{\mathrm{P}}(c)$ is decreasing as c approaches c_{PB} from the left, then it cannot increase for $c \ge c_{\mathrm{PB}}$. Note that, if $R'_{\mathrm{PP}}(c_{\mathrm{PB}}) > 0$, by the strict concavity of $R_{\mathrm{PP}}(c)$ and $R_{\mathrm{PD}}(c)$, the maximizer of $R_{\mathrm{P}}(c)$ will be either c_{PB} or $c^*_{\mathrm{PD}} > c_{\mathrm{PB}}$ depending on whether R_{PD} is decreasing or increasing at $c = c_{\mathrm{PB}}$. Hence, the function cannot have multiple local maxima. We conclude that $R_{\mathrm{P}}(c)$ is unimodal, and its maximizer $c_{\widetilde{\mathrm{P}}} \in (0,1)$. □

Proof of Claim EC.12. **Part 1: y is strictly decreasing in k:** Applying the Implicit Function Theorem to equation (EC.62), we compute the derivative of $y(k)$ with respect to k as

$$y'(k) = \frac{y(k)^{1+\frac{1}{k}}\log(y(k))}{k\left(y(k)^{\frac{1}{k}} + k y(k)(\alpha + 2(1-\lambda)(1-y(k)))\right)} < 0,$$

which follows from $y(k) \in (0,1]$, $k \in [0,1]$ and $\alpha < \lambda < 1$, and implies that $y(k)$ is strictly decreasing in k.

Part 2: y is convex in k: Temporarily suppressing the dependence of $y(k)$ on k, we can rewrite equation (EC.62) as

$$y^{1/k} = 1-\lambda+\alpha-2(1-\lambda+\alpha/2)y+(1-\lambda)y^2. \tag{EC.100}$$

Let us denote the right hand side of the above equation as $\Psi(y)$, i.e,

$$\Psi(y) = 1-\lambda+\alpha-2(1-\lambda+\alpha/2)y+(1-\lambda)y^2, \tag{EC.101}$$

which can be shown to be between zero and one for $y\in[0,1]$ because

$$\Psi(y) = (1-\lambda)(1-y)^2+\alpha(1-y)\in[0,1).$$

Direct calculation reveals that

$$\Psi'(y) = -2(1-\lambda)(1-y)-\alpha<0.$$

Differentiating both sides of (EC.100) with respect to k gives

$$\frac{y^{1/k-1}y'}{k}-\frac{y^{1/k}\log(y^{1/k})}{k} = \Psi'(y)y'$$
$$\implies \frac{y'}{y}-\log(\Psi(y)) = \frac{ky'\Psi'(y)}{\Psi(y)} \implies y' = \frac{\log(\Psi(y))}{\frac{1}{y}-k\frac{\Psi'(y)}{\Psi(y)}}.$$

Furthermore, taking the logarithm of the both sides of (EC.100) reveals that

$$\log(y^{1/k}) = \log(\Psi(y)) \implies k = \frac{\log(y)}{\log(\Psi(y))},$$

which allows us to rewrite y' as

$$y' = \frac{\log(\Psi(y))}{\frac{1}{y}-\frac{\Psi'(y)\log(y)}{\Psi(y)\log(\Psi(y))}}. \tag{EC.102}$$

The numerator of the RHS of (EC.102) is negative and increasing in k, because $\Psi(y)\in[0,1]$ is decreasing in y and y is decreasing in k. Since $\Psi'(y)<0$ and $\log y<0$, $\log\Psi(y)$, the denominator of the RHS (EC.102) is always positive, thus showing that the denominator is increasing in k or equivalently decreasing in y will be sufficient for the RHS to be increasing in k. Since $1/y$ is decreasing in y, it suffices to show $\frac{\Psi'(y)\log(y)}{\Psi(y)\log(\Psi(y))}$ is increasing in $y\in[0,1]$. Direct differentiation reveals that

$$\left(\frac{\Psi'(y)\log(y)}{\Psi(y)\log(\Psi(y))}\right)' = -\frac{\log(y)\left(\alpha+2\left(1-\lambda\right)\left(1-y\right)\right)^2}{\Psi(y)^2\log(\Psi(y))^2}-\frac{\left(\alpha^2+2\left(1-\lambda\right)\left(1-y\right)+2\left(1-\lambda\right)^2\left(1-y\right)^2\right)}{\Psi(y)^2\log(\Psi(y))}\Gamma(y), \tag{EC.103}$$

where

$$\Gamma(y) = \frac{\Psi(y)\left(\alpha+2\left(1-\lambda\right)\left(1-y\right)\right)}{y\left(\alpha^2+2\left(1-\lambda\right)\left(1-y\right)+2\left(1-\lambda\right)^2\left(1-y\right)^2\right)}-\log(y). \tag{EC.104}$$

Since both y and $\Psi(y)$ are within $(0,1]$, we have

$$-\frac{\log(y)(\alpha+2(1-\lambda)(1-y))^2}{\Psi(y)^2\log(\Psi(y))^2}\geq 0 \quad\text{and}\quad -\frac{\left(\alpha^2+2\left(1-\lambda\right)\left(1-y\right)+2\left(1-\lambda\right)^2\left(1-y\right)^2\right)}{\Psi(y)^2\log(\Psi(y))}\geq 0. \tag{EC.105}$$

We now show that $\Gamma(y)\geq 0$. Direct calculation reveals that

$$\Gamma'(y) = -\frac{(1-y)\left(\alpha^4+10\alpha\left(1-\lambda\right)\left(1-y\right)^3+4\left(1-\lambda\right)^4\left(1-y\right)^4\right)}{y^2\left(\alpha^2+2\left(1-\lambda\right)\left(1-y\right)+2\left(1-\lambda\right)^2\left(1-y\right)^2\right)^2}$$
$$-\frac{(1-y)\left(\alpha^3\left(1-\lambda\right)\left(5-3y\right)+2\alpha^2\left(1-\lambda\right)^2\left(1-y\right)\left(5-4y\right)\right)}{y^2\left(\alpha^2+2\left(1-\lambda\right)\left(1-y\right)+2\left(1-\lambda\right)^2\left(1-y\right)^2\right)^2}\leq 0,$$

and $\Gamma(1)=0$ by (EC.104) which implies that $\Gamma(y)\geq 0$ for $y\in[0,1]$. Together with (EC.103) and (EC.105), this implies

$$\left(\frac{\Psi'(y)\log(y)}{\Psi(y)\log(\Psi(y))}\right)'\geq 0,$$

which means that it is increasing in y. Since $\Psi'(y) \leq 0$, it is straightforward to show that

$$\frac{1}{y} - \frac{\Psi'(y)\log(y)}{\Psi(y)\log(\Psi(y))} \geq 0 \quad \text{and} \quad \left(\frac{1}{y} - \frac{\Psi'(y)\log(y)}{\Psi(y)\log(\Psi(y))}\right)' \leq 0.$$

Together with the fact that y is decreasing in k, this result implies that the denominator of the RHS of (EC.102) is positive and increasing in k. Since the numerator and denominator of the RHS of (EC.102) are, respectively, negative increasing and positive increasing in k, RHS of (EC.102) is increasing in k, that is, $\frac{\mathrm{d}y}{\mathrm{d}k}$ is increasing in k, or equivalently y is convex in k. □

Proof of Claim EC.13. The formula for $M_3(k)$ is given in equation (EC.63). First, we show that $M_3(k) > 0$ for k sufficiently close to zero, and note that with a slight abuse of notation we write $y(0)$ for $\lim_{k\to 0} y(k)$. We have $y(k)$ decreasing in k (so increasing as $k \downarrow 0$) by Claim EC.12 and also $y(k)$ is bounded above by one, so $\lim_{k\downarrow 0} y(k)$ exists. Since $y(k)$ strictly increases as k decreases, we have $\lim_{k\downarrow 0} y(k) = b$ for some $0 < b \leq 1$. If $b < 1$, then $y(k)^{1/k} \to 0$ as $k \downarrow 0$. If $b = 1$, then by equation (EC.62), we have $\lim_{k\downarrow 0} y(k)^{1/k} = 0$. Furthermore, note that (EC.62) can be rewritten as $y(k)^{1/k} = (1-\lambda)(1-y(k))^2 + \alpha(1-y(k))$, hence $\lim_{k\downarrow 0} y^{1/k} = 0 \implies y(0) = 1$.

Together with equation (EC.62), this implies

$$\lim_{k\to 0} M_3(k) = -1 + \lambda - \alpha + y(0)(2\alpha + (1-\lambda)(3-k)) - y(0)^2(1-\lambda)(2-k) = \alpha > 0.$$

The equation (EC.62) also reveals that

$$-(1-\lambda)y(1)^2 = 1 - \lambda + \alpha - (3 - 2\lambda + \alpha)y(1).$$

We then have

$$M_3(1) = -1 + \lambda - \alpha + y(1)(2\alpha + 2(1-\lambda)) - y(1)^2(1-\lambda) = -y(1-\alpha) < 0.$$

Since $M_3(0) > 0$ and $M_3(1) < 0$, by the Intermediate Value Theorem there must exist $k_{\mathrm{PP}} \in (0,1)$ such that $M_3(k_{\mathrm{PP}}) = 0$. Thus, by (EC.63), we can solve for quantity $y(k_{\mathrm{PP}})$, which is given by one of the following two quadratic roots:

$$\frac{2\alpha + (1-\lambda)(3-k) \pm \sqrt{4\alpha^2 + 4\alpha(1-\lambda+(1-k)^2)(1-\lambda)^2}}{2(2-k)(1-l)}.$$

We ignore the larger root, because it can be shown to be greater than 1, thus we have

$$y(k_{\mathrm{PP}}) = \frac{2\alpha + (1-\lambda)(3-k) - \sqrt{4\alpha^2 + 4\alpha(1-\lambda+(1-k)^2)(1-\lambda)^2}}{2(2-k)(1-l)}. \tag{EC.106}$$

Furthermore, differentiating $M_3(k)$ w.r.t k, we get

$$M_3'(k) = -y(k)(1-\lambda)(1-y(k)) + y(k)'(2\alpha + (1-\lambda)(3-k) - 2(2-k)(1-\lambda)y(k)). \tag{EC.107}$$

Using (EC.106) and (EC.107), we now show that $M_3'(k_{\mathrm{PP}}) < 0$. The quantity $-y(k)(1-\lambda)(1-y(k))$ from (EC.107) can be shown to be negative, and $y'(k)$ is negative for all values of k by Claim EC.12. Substituting the value of $y(k_{\mathrm{PP}})$ in (EC.106), we also have

$$2\alpha + (1-\lambda)(3-k) - 2(2-k)(1-l)y^{(}k_{\mathrm{PP}}) = \sqrt{4\alpha^2 + 4\alpha(1-\lambda+(1-k)^2)(1-\lambda)^2} \geq 0.$$

Therefore, the RHS of equation (EC.107) evaluated at $k = k_{\mathrm{PP}}$ is negative, i.e., $M_3'(k_{\mathrm{PP}}) < 0$, which immediately implies the uniqueness of k_{PP}. Indeed, suppose that there are multiple values of k with $M_3(k) = 0$. In this case, the function must cross zero multiple times, which would require $M_3'(k) \geq 0$ for at least one k with $M_3(k) = 0$, leading to a contradiction. The uniqueness of k_{PP} also implies that $M_3(k) \lesseqgtr 0$ for $k \gtreqless k_{\mathrm{PP}}$. □

Proof of Claim EC.14. Starting from the definition of $M_4(k)$ in equation (EC.63), straightforward algebra reveals that $M_4(k) \gtreqless 0$ is equivalent to $h(y) \gtreqless k$, where $h:(0,1] \to \mathbb{R}$ is given by

$$h(y) := \frac{(-1+y)(1-\lambda+\alpha-2y+2\lambda y)}{y(1-\lambda+\alpha-(1-\lambda)y)}. \tag{EC.108}$$

First, we note that $h(1)=0$ and $\lim_{y\to 0^+} h(y) = -\infty$, implying that $h(y)$ cannot be monotonically decreasing in $y\in(0,1]$. Furthermore, we compute

$$h'(y) = \frac{\alpha^2+\alpha(1-\lambda)(2-2y-y^2)+(1-\lambda)^2(1-y)^2}{y^2(\alpha+(1-\lambda)(1-y))^2}., \tag{EC.109}$$

whose denominator is positive and whose numerator can be shown to be decreasing for $y\in(0,1]$ by direct calculation as follows:

$$(\alpha^2+\alpha(1-\lambda)(2-2y-y^2)+(1-\lambda)^2(1-y)^2)' = -2(1-\lambda)(\alpha(1+y)+(1-\lambda)(1-y)) < 0.$$

Since $h(y)$ cannot be monotonically decreasing in y, $h(y)$ is therefore either monotonically increasing, or first increasing then decreasing, in y. Direct evaluation yields

$$h'(1) = \frac{\lambda+\alpha-1}{\alpha} \geq 0 \iff \alpha \geq 1-\lambda.$$

Thus, $h(y)$ is monotonically increasing in $y\in(0,1]$ for $\alpha\geq 1-\lambda$, and $h(y)$ is first increasing then decreasing in $y\in(0,1]$ for $\alpha<1-\lambda$ because $h'(0+)=+\infty$.

Case 1: $\boldsymbol{\alpha \geq 1-\lambda}$. In this case, $h(y)$ is monotonically increasing in $y\in(0,1]$ and hence $h(y(k))-k$ is strictly decreasing in k by Claim EC.12. Furthermore, by (EC.62) we have found that $y(0)=1$ (refer to the proof of the Claim EC.13), thus $h(y(0))=0$. Thus, there does not exist $k\in(0,1]$ satisfying $h(y(k))-k=0$.

Case 2: $\boldsymbol{\alpha < 1-\lambda}$. In this case, $h(y)$ is first increasing and then decreasing in $y\in(0,1]$. More specifically, let $\hat{y}\in(0,1]$ be the solution to $h'(y)=0$; thus $h'(y)\geq 0$ for $y\in(0,\hat{y}]$, and $h'(y)<0$ for $y\in(\hat{y},1]$. More specifically, by (EC.109), when $y\in(\hat{y},1]$ we must have

$$h'(y)\leq 0 \iff \alpha^2+\alpha(1-\lambda)(2-2y-y^2)+(1-\lambda)^2(1-y)^2 \leq 0. \tag{EC.110}$$

Since $y(k)$ is decreasing in $k\in[0,1]$ by Claim EC.12, we have $\underline{y} := y(1) \leq y(k) \leq \overline{y} := y(0)$. We have already shown that $y(0)=1$, thus $\overline{y}=1$ and $h(\overline{y})=0$. Similarly, when $k=1$, using the equation (EC.62), we find

$$\underline{y} = \frac{3+\alpha-2\lambda-\sqrt{5-4\lambda+\alpha^2+2\alpha}}{2(1-\lambda)},$$

and

$$\begin{aligned} h(\underline{y}) &= \frac{\left(-1+\underline{y}\right)\left(1-\lambda+\alpha-2\underline{y}+2\lambda\underline{y}\right)}{\underline{y}\left(1-\lambda+\alpha-(1-\lambda)\underline{y}\right)} \\ &= \frac{\left(-1+\underline{y}\right)\left(1-\lambda+\alpha-2\underline{y}+2\lambda\underline{y}\right)+(1-\lambda+\alpha-(3-2\lambda+\alpha)\underline{y}+(1-\lambda)\underline{y}^2)}{\underline{y}\left(1-\lambda+\alpha-(1-\lambda)\underline{y}\right)} \\ &= \frac{-\underline{y}-\lambda(1-\underline{y})}{\alpha+(1-\lambda)(1-\underline{y})} < 0. \end{aligned}$$

Note that by this result, we cannot have $\hat{y}\leq\underline{y}$ because $h(\overline{y})=0>h(\underline{y})$ implies that we cannot have $h(y)$ to be monotonically decreasing from $\underline{y}$ to $\overline{y}$. Also not that we cannot have $\hat{y}\geq\overline{y}$ because $\overline{y}=1$.

• We now show that $h(y(k))$ is concave in k when $y(k)\in(\hat{y},1]$: Direct calculation reveals that

$$h''(y) = \frac{-2(1-\lambda+\alpha)(\alpha+(1-\lambda)(1-y))^2+2(1-\lambda)y(\alpha^2+\alpha(1-\lambda)(2-2y-y^2)+(1-\lambda)^2(1-y)^2)}{(\alpha+(1-\lambda)(1-y))^3y^3} \leq 0,$$

for $y\in(\hat{y},1]$ by (EC.110). Combined with the convexity of $y(k)$ in k by Claim EC.12 and the fact that $h'(y)<0$ for $y\in(\hat{y},1]$, this result implies

$$\frac{\mathrm{d}^2h(y(k))}{\mathrm{d}k} = \frac{\mathrm{d}^2h(y(k))}{\mathrm{d}y(k)^2}\left(\frac{\mathrm{d}y(k)}{\mathrm{d}k}\right)^2 + \frac{\mathrm{d}h(y(k))}{\mathrm{d}y(k)}\frac{\mathrm{d}^2y(k))}{\mathrm{d}k^2} \leq 0,$$

which implies that $h(y(k))$ is concave in $k \in [0, \hat{k})$ where $\hat{k} = y^{-1}(\hat{y})$ because $y(k)$ is strictly decreasing in k.

• Now, we show the uniqueness of k_{PD} when $\alpha < 1-\lambda$. Since $\lim_{k\to 0} h(y(k)) = 0$, $h(y(1)) < 0$, and $h(y(k))$ is increasing concave in $k \in [0, \hat{k})$, and decreasing in $k \in [\hat{k}, 1]$, $k = h(y(k))$ can only happen at a unique value of $k = k_{\text{PD}}$ for $0 < k \le 1$ if the $\lim_{k\to 0} \frac{\mathrm{d}h(y(k))}{\mathrm{d}k} > 1$. To check this, using the equation (EC.102) and $\lim_{k\to 0} y(0) = 1$, we first write the limit of $d(y(k))/dk$ when k goes to 0 as

$$y'(0) = \lim_{k\to 0} \frac{\mathrm{d}y(k)}{\mathrm{d}k} = \lim_{y\to 1} \frac{\log(\Psi(y))}{\frac{1}{y} - \frac{\Psi'(y)\log(y)}{\Psi(y)\log(\Psi(y))}}.$$

The numerator of the above expression goes to $-\infty$ as $\Psi(y)$ (defined by (EC.101)) goes to zero. Using L'hopital rule, we also have

$$\lim_{y\to 1} \frac{\Psi'(y)\log(y)}{\Psi(y)\log(\Psi(y))} = \lim_{y\to 1} \frac{\Psi''(y)\log(y) + \Psi'(y)/y}{\Psi'(y)\log(\Psi(y)) + \Psi'(y)} = 0,$$

which implies that

$$y'(0) = \frac{-\infty}{1+0} = -\infty.$$

This allows us to compute

$$\begin{aligned}\lim_{k\to 0} \frac{\mathrm{d}h(y(k))}{\mathrm{d}k} &= \lim_{y\to 1} \frac{\alpha^2 + \alpha(1-\lambda)(2-2y-y^2) + (1-\lambda)^2(1-y)^2}{y^2(\alpha + (1-\lambda)(1-y))^2} y' \\ &= \frac{\alpha+\lambda-1}{\alpha} y'(0) > 1 \iff \alpha < 1-\lambda.\end{aligned}$$

Hence, when $\alpha < 1-\lambda$, there exists a unique $k = k_{\text{PD}} > 0$ satisfying $k = h(y(k))$, and $h(y(k)) \gtreqless k \iff k \lesseqgtr k_{\text{PD}}$ which is equivalent to $M_4(k) \gtreqless 0 \iff k \lesseqgtr k_{\text{PD}}$. □

Proof of Claim EC.15. We can verify that $k_{\text{PD}} < k_{\text{PP}}$ as follows. We have $M_4(k) = M_3(k) - \alpha(1+k)y(k)$, and also $M_3(k_{\text{PP}}) = 0$ by Claim EC.13. We can then write

$$M_4(k_{\text{PP}}) = -\alpha(1+k_{\text{PP}})y(k_{\text{PP}}) < 0 \implies k_{\text{PD}} < k_{\text{PP}},$$

where the implication follows because $M_4(k) < 0$ if and only if $k > k_{\text{PD}}$ by Claim EC.14. □

F.4. Proofs of Claims EC.16-EC.19 for Proposition EC.2

Proof of Claim EC.16. Suppose $G(c_{\widetilde{\text{F}}}) + F(p_{\text{F}}(c_{\widetilde{\text{F}}})) > 1$, i.e., that $(c_{\widetilde{\text{F}}})^k + p_{\text{F}}(c_{\widetilde{\text{F}}}) > 1$. We proceed by cases.

Case 1: $c_{\widetilde{\text{F}}} \le c_{\text{FB}}$. In this case, we have $p_{\text{F}}(c_{\widetilde{\text{F}}}) = p_{\text{FP}}(c_{\widetilde{\text{F}}})$ by (EC.40). We thus have $(c_{\widetilde{\text{F}}})^k + p_{\text{FP}}(c_{\widetilde{\text{F}}}) > 1$, which in turn implies that $c_{\widetilde{\text{F}}} < c_{\text{FPP}}$ by (EC.38). By (EC.37) and (EC.38), we have $c^k \le p_{\text{FP}}(c)$ and $c^k + p_{\text{FP}}(c) \le 1$ if and only if $c \in [c_{\text{FPP}}, c_{\text{FB}}]$. Similarly, we have $c^k \ge p_{\text{FD}}(c)$ and $c^k + p_{\text{FD}}(c) \le 1$ if and only if $c \in [\max\{c_{\text{FPD}}, c_{\text{FB}}\}, 1]$. If $c_{\text{FPP}} \le c_{\text{FB}}$, then by Claim EC.5, we also have $c_{\text{FPD}} \le c_{\text{FB}}$, so $[\max\{c_{\text{FPD}}, c_{\text{FB}}\}, 1] = [c_{\text{FB}}, 1]$. Moreover, for $c \le c_{\text{FB}}$, by (EC.40) we have $p_{\text{F}}(c) = p_{\text{FP}}(c)$, and for $c \ge c_{\text{FB}}$, we have $p_{\text{F}}(c) = p_{\text{FD}}(c)$. Altogether, these arguments imply that $c^k + p_{\text{F}}(c) \le 1$ if and only if $c \in [c_{\text{FPP}}, 1]$, i.e., the feasible region for problem (F) is $[c_{\text{FPP}}, 1]$. Given $c_{\widetilde{\text{F}}} < c_{\text{FPP}}$, Claim EC.6 implies that $R_{\text{F}}(c)$ is decreasing in c for $c \in [c_{\text{FPP}}, 1]$, thus $c^*_{\text{F}} = c_{\text{FPP}}$ and $p^*_{\text{F}} = p_{\text{FP}}(c_{\text{FPP}})$. Hence, by Lemma EC.2 (see equation EC.38), we have $(c^*_{\text{F}})^k + p_{\text{F}}(c^*_{\text{F}}) = 1$.

If $c_{\text{FPP}} > c_{\text{FB}}$, then by Claim EC.5, $c_{\text{FPD}} > c_{\text{FB}}$ also holds. By similar reasoning to above, this implies that the feasible region of (F) is $[c_{\text{FPD}}, 1]$. Given $c_{\widetilde{\text{F}}} \le c_{\text{FB}} < c_{\text{FPD}}$, Claim EC.6 implies that $R_{\text{F}}(c)$ is decreasing in c for $c \in [c_{\text{FPD}}, 1]$, hence $c^*_{\text{F}} = c_{\text{FPD}}$ and $p^*_{\text{F}} = p_{\text{FD}}(c_{\text{FPD}})$. By Claim EC.4, we have $c^k_{\text{FPD}} + p_{\text{FD}}(c_{\text{FPD}}) = 1$, which thus implies that $(c^*_{\text{F}})^k + p_{\text{F}}(c^*_{\text{F}}) = 1$.

Case 2: $c_{\widetilde{\text{F}}} > c_{\text{FB}}$. In this case, we have $p_{\text{F}}(c_{\widetilde{\text{F}}}) = p_{\text{FD}}(c_{\widetilde{\text{F}}})$ by (EC.40). We thus have $(c_{\widetilde{\text{F}}})^k + p_{\text{FD}}(c_{\widetilde{\text{F}}}) > 1$, which implies $c_{\widetilde{\text{F}}} < c_{\text{FPD}}$ by Lemma EC.2, which in turn implies $c_{\text{FPD}} > c_{\text{FB}}$. Furthermore, by Claim EC.5, $c_{\text{FPD}} > c_{\text{FB}} \implies c_{\text{FPP}} > c_{\text{FB}}$, hence by analogous logic to that used for Case 1, the feasible region of (F) becomes $[c_{\text{FPD}}, 1]$. Since $c_{\widetilde{\text{F}}} < c_{\text{FPD}}$, Claim EC.6 implies that R_{F} is decreasing in $c \in [c_{\text{FPD}}, 1]$, hence $c^*_{\text{F}} = c_{\text{FPD}}$ and $p^*_{\text{F}} = p_{\text{FD}}(c_{\text{FPD}})$. By Claim EC.4, we also have $c^k_{\text{FPD}} + p_{\text{FD}}(c_{\text{FPD}}) = 1$, which means that the optimal solution of the problem (F) satisfies $(c^*_{\text{F}})^k + p^*_{\text{F}} = 1$. □

Proof of Claim EC.17. Suppose $G(c_{\widetilde{P}}) + F(p_P(c_{\widetilde{P}})) < 1$, i.e., that $(c_{\widetilde{P}})^k + p_P(c_{\widetilde{P}}) < 1$. We proceed by cases.
Case 1: $c_{\widetilde{P}} \le c_{PB}$. In this case, we have $p_P(c) = p_{PP}(c)$ by (EC.41). We thus have $(c_{\widetilde{P}})^k + p_{PP}(c_{\widetilde{P}}) < 1$, which implies that $c_{\widetilde{P}} > c_{FPP}$ by Lemma EC.2, which, in turn, implies that $c_{FPP} < c_{PB}$. By Lemma EC.2, we have $c^k \le p_{PP}(c)$ and $c^k + p_{PP}(c) \ge 1$ if and only if $c \in [0, \min\{c_{FPP}, c_{PB}\}]$. Similarly, we have $c^k \ge p_{PD}(c)$ and $c^k + p_{PD}(c) \ge 0$ if and only if $c \in [c_{PB}, c_{FPD}]$. Since $c_{FPP} < c_{PB}$, by Claim EC.5, we also have $c_{FPD} < c_{PB}$, hence $[0, \min\{c_{FPP}, c_{PB}\}] = [0, c_{FPP}]$. Moreover, for $c \le c_{PB}$, by (EC.41) we have $p_P(c) = p_{PP}(c)$, and for $c \ge c_{PB}$, we have $p_P(c) = p_{PD}(c)$. Altogether, these arguments imply that $c^k + p_P(c) \ge 1$ if and only if $c \in [0, c_{FPP}]$, i.e., the feasible region for problem (F) is $[0, c_{FPP}]$. Since $c_{FPP} < c_{\widetilde{P}}$, Claim EC.11 implies that $R_P(c)$ is increasing in $c \in [0, c_{FPP}]$. Hence, the optimal solution to (P) is $c_P^* = c_{FPP}$ and $p_P^* = p_{PP}(c_{FPP})$, which, in turn, implies that $(c_P^*)^k + p_P(c_P^*) = 1$ by Lemma EC.2 (see equation EC.38).
Case 2: $c_{\widetilde{P}} > c_{PB}$. In this case, we have $p_P(c_{\widetilde{P}}) = p_{PD}(c_{\widetilde{P}})$. We thus have $(c_{\widetilde{P}})^k + p_{PD}(c_{\widetilde{P}}) < 1$, which implies $c_{\widetilde{P}} > c_{FPD}$ by Lemma EC.2. If $c_{FPD} > c_{PB}$, then $c_{FPP} > c_{PB}$ by Claim EC.5, so by similar logic to that used for Case 1, the feasible region of (P) is $[0, c_{FPD}]$. Since Claim EC.11 implies that $R_P(c)$ is increasing in $c \in [0, c_{FPD}]$. Hence, the optimal solution to (P) is $c_P^* = c_{FPD}$ and $p_P = p_{PD}(c_{FPD})$, which implies that $(c_P^*)^k + p_P(c_P^*) = 1$.

Similarly, if $c_{FPD} \le c_{PB}$, then $c_{FPP} \le c_{PB}$ by Claim EC.5. The feasible region of the problem (P) then becomes $[0, c_{FPP}]$ by similar logic to that used for Case 1. Since $c_{FPP} \le c_{PB} < c_{\widetilde{P}}$, Claim EC.11 implies that $R_F(c)$ is increasing for $c \in [0, c_{FPP}]$. Hence, the solution to the problem (P) is $c_P^* = c_{FPP}$ and $p_P^* = p_{PP}(c_{FPP})$, which implies that $(c_P^*)^k + p_P(c_P^*) = 1$ by Lemma EC.2 (see equation EC.38). □

Proof of Claim EC.18. We prove the claim by contradiction. There are two possible cases for $F(p^*) + G(c^*) = 1$ and $F(p^*) \ne G(c^*)$ to happen, that is, either $F(p^*) + G(c^*) = 1$ and $G(c^*) < F(p^*)$, or $F(p^*) + G(c^*) = 1$ and $F(p^*) < G(c^*)$.
Case 1: $(c^*)^k + p^* = 1$ and $(c^*)^k < p^*$. In this case, the global optimal solution is in the feasible region of sub-problem (FP) and (FD), therefore, by Lemma 1, it must satisfy $c^k + p_{FP}(c) = 1$ and $c^k + p_{PP}(c) = 1$. By Lemma EC.2, this can only happen at $c^* = c_{FPP}$ and $p^* = p_{FP}(c_{FPP}) = p_{PP}(c_{FPP})$, where c_{FPP} is defined in Claim EC.3. Note that since $c_{FPP}^k < p_{FP}(c_{FPP})$ and $c_{FPP}^k < p_{PP}(c_{FPP})$, by Lemma EC.2, we have $c_{FPP} < c_{FB}$ and $c_{FPP} < c_{PB}$. Hence, the feasible region of sub-problem FP is $[c_{FPP}, c_{FB}]$, and the feasible region of sub-problem PP is $[0, c_{FPP}]$ by Lemma EC.2. Given c_{FPP} is the optimal solution, we must have $R'_{FP}(c_{FPP}) \le 0$. Otherwise, $R'_{FP}(c_{FPP}) > 0$ would imply that another feasible solution $c = c_{FPP} + \epsilon \in [c_{FPP}, c_{FB}]$ generates a higher revenue, leading to a contradiction. By the similar argument, we must also have $R'_{PP}(c_{FPP}) \ge 0$. Combining these two results, in this case, we have $R'_{FP}(c_{FPP}) \le 0 \le R'_{PP}(c_{FPP})$.

We now show that

$$R'_{FP}(c_{FPP}) \le 0 \implies R'_{PP}(c_{FPP}) \le 0 \quad \text{and} \quad R'_{PP}(c_{FPP}) \ge 0 \implies R'_{FP}(c_{FPP}) \ge 0. \tag{EC.111}$$

Note that subtracting $R_{PP}(c)$ from $R_{FP}(c)$ defined in (FP) gives

$$R_{FP}(c) - R_{PP}(c) = \frac{c(\lambda - \alpha) g_{FPP}(c)}{(1 - (1-\lambda)c^k)(1 - \lambda + \alpha + (1-\lambda)c^k)},$$

where $g_{FPP}(c)$ is given by (EC.56). We can rewrite the difference defined above as $R_{FP}(c) - R_{PP}(c) = P_1(c) g_{FPP}(c)$ where $P_1(c)$ is defined as

$$P_1(c) = \frac{c(\lambda - \alpha)}{(1 - (1-\lambda)c^k)(1 - \lambda + \alpha + (1-\lambda)c^k)}.$$

Since $\alpha < \lambda < 1$, $P_1(c) > 0$, and therefore by Claim EC.3 in the proof of Lemma EC.2, we have $R_{FP}(c) \lesseqqgtr R_{PP}(c)$ for $c \lesseqqgtr c_{FPP}$. Using this, if $R'_{FP}(c_{FPP}) \le 0$, then $R'_{PP}(c_{FPP}) \le 0$ must be satisfied as well. Otherwise, $R'_{PP}(c_{FPP}) > 0$ would imply $R_{PP}(c_{FPP} + \epsilon) > R_{PP}(c_{FPP}) = R_{FP}(c_{FPP}) \ge R_{FP}(c_{FPP} + \epsilon)$ for some $\epsilon > 0$, leading to a contradiction to $R_{FP}(c) > R_{PP}(c)$ for $c > c_{FPP}$. Similarly, if $R'_{PP}(c_{FPP}) \ge 0$, $R'_{FP}(c_{FPP}) \ge 0$ must also be satisfied. Otherwise, $R'_{FP}(c_{FPP}) < 0$ would imply $R_{FP}(c_{FPP} - \epsilon) > R_{FP}(c_{FPP}) = R_{PP}(c_{FPP}) > R_{PP}(c_{FPP} - \epsilon)$, which is a contradiction to $R_{FP}(c) < R_{PP}(c)$ for $c < c_{FPP}$. Combining these two results, we establish (EC.111). This, together with $R'_{FP}(c_{FPP}) \le 0 \le R'_{PP}(c_{FPP})$, implies $R'_{FP}(c_{FPP}) = R'_{PP}(c_{FPP}) = 0$. If $R'_{FP}(c_{FPP}) = R'_{PP}(c_{FPP}) = 0$, using $g_{FPP}(c_{FPP}) = 0$, direct calculation reveals that

$$\begin{aligned} R'_{FP}(c_{FPP}) - R'_{PP}(c_{FPP}) &= P'_1(c_{FPP}) g_{FPP}(c_{FPP}) + P_1(c_{FPP}) g'_{FPP}(c_{FPP}) \\ &= P_1(c_{FPP}) g'_{FPP}(c_{FPP}) \implies g'_{FPP}(c_{FPP}) = 0, \end{aligned}$$

which is a contradiction to $g'_{\text{FPP}}(c_{\text{FPP}}) > 0$ by Claim EC.3 in the proof of Lemma EC.2. We conclude that $(c^*)^k + p^* = 1$ and $c^* < p^*$ cannot hold simultaneously.

Case 2: $(c^*)^k + p^* = 1$ and $(c^*)^k > p^*$. In this case, the global optimal solution is in the feasible region of sub-problem (FD) and (PD), therefore, by Lemma 1, it must satisfy $c^k + p_{\text{FD}}(c) = 1$ and $c^k + p_{\text{PD}}(c) = 1$. By Lemma EC.2, this can only happen at $c^* = c_{\text{FPD}}$ and $p^* = p_{\text{FD}}(c_{\text{FPD}}) = p_{\text{PD}}(c_{\text{FPD}})$ where c_{FPD} is defined in Claim EC.4. Note that since $c^k_{\text{FPD}} > p_{\text{FD}}(c_{\text{FPD}})$ and $c^k_{\text{FPD}} > p_{\text{PD}}(c_{\text{FPD}})$, by Lemma EC.2, we have $c_{\text{FPD}} > c_{\text{FB}}$ and $c_{\text{FPD}} > c_{\text{PB}}$. Hence, by (FD), (PD) and Lemma EC.2, the feasible region of the sub-problem FD is $[c_{\text{FPD}}, 1]$, and the feasible region of the sub-problem PD is $[c_{\text{PB}}, c_{\text{FPD}}]$. Given c_{FPD} is the optimal solution, we must have $R'_{\text{FD}}(c_{\text{FPD}}) \le 0$. Otherwise, $R'_{\text{FD}}(c_{\text{FPD}}) > 0$ would imply that another feasible solution $c = c_{\text{FPD}} + \epsilon \in [c_{\text{FPD}}, 1]$ generates a higher revenue, leading to a contradiction. By the similar argument, we must also have $R'_{\text{PD}}(c_{\text{FPD}}) \ge 0$. Combining these two results, in this case, we have $R'_{\text{FD}}(c_{\text{FPD}}) \le 0 \le R'_{\text{PD}}(c_{\text{FPD}})$.

We now show that

$$R'_{\text{FD}}(c_{\text{FPD}}) \le 0 \implies R'_{\text{PD}}(c_{\text{FPD}}) \le 0 \quad \text{and} \quad R'_{\text{PD}}(c_{\text{FPD}}) \ge 0 \implies R'_{\text{FD}}(c_{\text{FPD}}) \ge 0. \tag{EC.112}$$

Subtracting $R_{\text{PD}}(c)$ from $R_{\text{FD}}(c)$ defined in (FD) and (PD) gives

$$R_{\text{FD}}(c) - R_{\text{PD}}(c) = \frac{c(\lambda - \alpha) g_{\text{FPD}}(c)}{(1-\lambda)(1-c^k)(1-\alpha-(1-\lambda)c^k)},$$

where $g_{\text{FPD}}(c)$ is given by (EC.58) We can rewrite the difference defined above as $R_{\text{FD}}(c) - R_{\text{PD}}(c) = P_2(c) g_{\text{FPD}}(c)$ where $P_2(c) =$ is defined as

$$P_2(c) = \frac{c(\lambda - \alpha)}{(1-\lambda)(1-c^k)(1-\alpha-(1-\lambda)c^k)} > 0.$$

Since $\alpha < \lambda < 1$, $P_2(c) > 0$, and therefore by Claim EC.4 in the proof of Lemma EC.2, we have $R_{\text{FD}}(c) \lesseqgtr R_{\text{PD}}(c)$ for $c \lesseqgtr c_{\text{FPD}}$. Using this, it is straightforward to show that if $R'_{\text{FD}}(c_{\text{FPD}}) \le 0$, then $R'_{\text{PD}}(c_{\text{FPD}}) \le 0$ must hold. Otherwise, $R'_{\text{PD}}(c_{\text{FPD}}) > 0$ would imply $R_{\text{PD}}(c_{\text{FPD}} - \epsilon) < R_{\text{PD}}(c_{\text{FPD}}) = R_{\text{FD}}(c_{\text{FPD}}) \le R_{\text{FD}}(c_{\text{FPD}} - \epsilon)$, leading to a contradiction. Similarly, if $R'_{\text{PD}}(c_{\text{FPD}}) \ge 0$, then $R'_{\text{FD}}(c_{\text{FPD}}) \ge 0$. Otherwise, $R'_{\text{FD}}(c_{\text{FPD}}) < 0$ would imply $R_{\text{FD}}(c_{\text{FPD}} + \epsilon) < R_{\text{FD}}(c_{\text{FPD}}) = R_{\text{PD}}(c_{\text{FPD}}) \le R_{\text{PD}}(c_{\text{FPD}} + \epsilon)$, which is also a contradiction. Combining these two results, we establish (EC.112). This, together with $R'_{\text{FD}}(c_{\text{FPD}}) \le 0 \le R'_{\text{PD}}(c_{\text{FPD}})$ imply $R'_{\text{FD}}(c_{\text{FPD}}) = R'_{\text{PD}}(c_{\text{FPD}}) = 0$ when c_{FPD} is the optimal solution. Using $g_{\text{FPD}}(c_{\text{FPD}}) = 0$, direct calculation reveals that

$$\begin{aligned}(R_{\text{FD}}(c_{\text{FPD}}) - R_{\text{PD}}(c_{\text{FPD}}))' &= P'_2(c_{\text{FPD}}) g_{\text{FPD}}(c_{\text{FPD}}) + P_2(c_{\text{FPD}}) g'_{\text{FPD}}(c_{\text{FPD}}) \\ R'_{\text{FD}}(c_{\text{FPD}}) - R'_{\text{PD}}(c_{\text{FPD}}) &= P_2(c_{\text{FPD}}) g'_{\text{FPD}}(c_{\text{FPD}}) \implies 0 = g'_{\text{FPD}}(c_{\text{FPD}}),\end{aligned}$$

which is a contradiction to $g'_{\text{FPD}}(c_{\text{FPD}}) > 0$ by Claim EC.4 in the proof of Lemma EC.2. We conclude that $(c^*)^k + p^* = 1$ and $c^* > p^*$ cannot hold simultaneously. □

Proof of Claim EC.19. If the optimal solution satisfies $(c^*)^k + p^* = 1$ and $(c^*)^k = p^*$, i.e., $p^* = (c^*)^k = 1/2$, then by (EC.37) and (EC.38), $p^* = (c^*)^k = 1/2$ implies that $c_{\text{FB}} = c_{\text{FPP}} = c_{\text{FPD}} = c_{\text{PB}} = c^*$. In this case, it is straightforward to show that the feasible regions of the sub-problems (FP), (FD), (PP), (PD) are $\{c^*\}$, $[c^*, 1]$, $[0, c^*]$ and $\{c^*\}$, respectively by Lemma EC.2 (see equations EC.37 and EC.38). Therefore, since c^* is the global optimal solution, we must have

$$R'_{\text{FD}}(c^*) \le 0 \quad \text{and} \quad R'_{\text{PP}}(c^*) \ge 0.$$

Otherwise, there would be a larger c in the feasible region of sub-problem FD, or smaller c in the feasible region of sub-problem PP generating a higher revenue, leading to a contradiction.

By (EC.111), (EC.112) and $c_{\text{FPP}} = c_{\text{FPD}} = c_{\text{FB}} = c_{\text{PB}} = c^*$, $R'_{\text{FD}}(c^*) \le 0 \implies R'_{\text{PD}}(c^*) \le 0$ and $R'_{\text{PP}}(c^*) \ge 0 \implies R'_{\text{FP}}(c^*) \ge 0$, and we get

$$\begin{aligned}R'_{\text{FP}}(c_{\text{FB}}) \ge 0 \text{ and } R_{\text{FD}}(c_{\text{FB}}) \le 0 &\implies c_{\widetilde{\text{F}}} = c_{\text{FB}} = c^* \quad \text{and} \\ R'_{\text{PP}}(c_{\text{PB}}) \ge 0 \text{ and } R'_{\text{PD}}(c_{\text{PB}}) \le 0 &\implies c_{\widetilde{\text{P}}} = c_{\text{PB}} = c^*,\end{aligned}$$

which follows from Claims EC.6 and EC.11. Finally, using equations EC.37 and EC.38, it is straightforward to show that $p_{\text{FP}}(c^*) = p_{\text{FD}}(c^*) = p_{\text{PP}}(c^*) = p_{\text{PD}}(c^*) = p^*$, hence (c^*, p^*) is also the solution of the relaxed problems $(\widetilde{\text{F}})$ and $(\widetilde{\text{P}})$. □