ON NONABELIAN REPRESENTATIONS OF TWIST KNOTS

JAMES C. DEAN AND ANH T. TRAN

Abstract. We study representations of the knot groups of twist knots into $SL_2(\mathbb{C})$. The set of nonabelian $SL_2(\mathbb{C})$ representations of a twist knot K is described as the zero set in $\mathbb{C} \times \mathbb{C}$ of a polynomial $P_K(x, y) = Q_K(y) + x^2 R_K(y) \in \mathbb{Z}[x, y]$, where x is the trace of a meridian. We prove some properties of $P_K(x, y)$. In particular, we prove that $P_K(2, y) \in \mathbb{Z}[y]$ is irreducible over $\mathbb{Q}$. As a consequence, we obtain an alternative proof of a result of Hoste and Shanahan that the degree of the trace field is precisely two less than the minimal crossing number of a twist knot.

1. Introduction

Let $J(k, l)$ be the two-bridge knot/link in Figure 1, where $k, l \neq 0$ denote the numbers of half twists in the boxes. Positive (resp. negative) numbers correspond to right-handed (resp. left-handed) twists. Note that $J(k, l)$ is a knot if and only if kl is even. The knots $J(2, 2n)$, where $n \neq 0$, are known as twist knots. Moreover, $J(2, 2)$ is the trefoil knot and $J(2, -2)$ is the figure eight knot. For more information about $J(k, l)$, see [HS1].

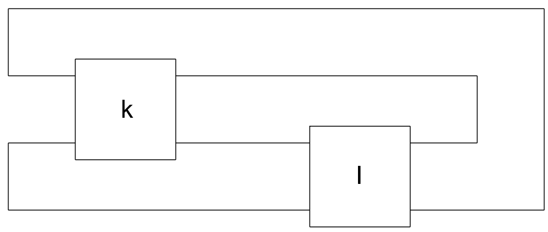

Figure 1. The two-bridge knot/link $J(k, l)$.

We study representations of the knot groups of twist knots into $SL_2(\mathbb{C})$, where $SL_2(\mathbb{C})$ denotes the set of all 2×2 matrices with determinant one. From now on we fix a twist knot $J(2, 2n)$. By [HS2] the knot group of $J(2, 2n)$ has a presentation $\pi_1(J(2, 2n)) = \langle c, d \mid cu = ud \rangle$, where c, d are meridians and $u = (cd^{-1}c^{-1}d)^n$. This presentation is closely related to the standard presentation of the knot group of a two-bridge knot. Note that $J(2, 2n)$ is the twist knot K_{2n} in [HS2]. In this note we will follow [Tr2, Lemma 1.1] and use a different presentation

$$\pi_1(J(2, 2n)) = \langle a, b \mid aw = wb \rangle$$

where a, b are meridians and $w = (ab^{-1})^{-n} a (ab^{-1})^n$. This presentation has shown to be useful for studying invariants of twist knots, see [NT, Tr1, Tr2, Tr3].

2010 *Mathematics Classification:* Primary 57N10. Secondary 57M25.
Key words and phrases: Chebychev polynomial, nonabelian representation, parabolic representation, trace field, twist knot.

A representation $\rho : \pi_1(J(2,2n)) \to SL_2(\mathbb{C})$ is called nonabelian if the image of ρ is a nonabelian subgroup of $SL_2(\mathbb{C})$. Suppose $\rho : \pi_1(J(2,2n)) \to SL_2(\mathbb{C})$ is a nonabelian representation. Up to conjugation, we may assume that

$$\rho(a) = \begin{bmatrix} s & 1 \\ 0 & s^{-1} \end{bmatrix} \quad \text{and} \quad \rho(b) = \begin{bmatrix} s & 0 \\ 2-y & s^{-1} \end{bmatrix}$$

where $s \neq 0$ and $y \neq 2$ satisfy a polynomial equation $P_n(s,y) = 0$. The polynomial P_n can be chosen so that $P_n(s,y) = P_n(s^{-1},y)$, and hence it can be considered as a polynomial in the variables $x := s + s^{-1}$ and y. Note that $x = \operatorname{tr}\rho(a) = \operatorname{tr}\rho(b)$ and $y = \operatorname{tr}\rho(ab^{-1})$. An explicit formula for $P_n(x,y)$ will be derived in Section 2.2 and it is given by

$$P_n(x,y) = 1 - (y + 2 - x^2)S_{n-1}(y)(S_{n-1}(y) - S_{n-2}(y)),$$

where $S_k(z)$'s are the Chebychev polynomials of the second kind defined by $S_0(z) = 1$, $S_1(z) = z$ and $S_k(z) = zS_{k-1}(z) - S_{k-2}(z)$ for all integers k. Note that $P_n(x,y)$ is different from the Riley polynomial [Ri] of the two-bridge knot $J(2,2n)$, see e.g. [NT]. Moreover, $P_n(2,y)$ is also different from the polynomial $\Phi_{-n}(y)$ studied in [HS2].

In this note we prove the following two properties of $P_n(x,y)$.

Theorem 1. *Suppose $x_0^2 \in \mathbb{R}$ such that $4 - \frac{1}{|n|} < x_0^2 \leq 4$. Then the polynomial $P_n(x_0,y)$ has no real roots y if $n < 0$, and has exactly one real root y if $n > 0$.*

Theorem 2. *The polynomial $P_n(2,y) \in \mathbb{Z}[y]$ is irreducible over $\mathbb{Q}$.*

A nonabelian representation $\rho : \pi_1(J(2,2n)) \to SL_2(\mathbb{C})$ is called parabolic if the trace of a meridian is equal to 2. The zero set in $\mathbb{C}$ of the polynomial $P_n(2,y)$ describes the set of all parabolic representations of the knot group of $J(2,2n)$ into $SL_2(\mathbb{C})$. Theorem 1 is related to the problem of determining the existence of real parabolic representations in the study of the left-orderability of the fundamental groups of cyclic branched covers of two-bridge knots, see [Hu, Tr1].

As in the proof of [HS2, Theorem 1], Theorem 2 gives an alternative proof of a result of Hoste and Shanahan that the degree of the trace field is precisely two less than the minimal crossing number of a twist knot. Indeed, by definition the trace field of a hyperbolic knot K is the extension field $\mathbb{Q}(\operatorname{tr}\rho_0(g) : g \in \pi_1(K))$, where $\rho_0 : \pi_1(K) \to SL_2(\mathbb{C})$ is a discrete faithful representation. The representation ρ_0 is a parabolic representation. Since $P_n(2,y)$ is irreducible over $\mathbb{Q}$, the trace field of the twist knot $J(2,2n)$ is $\mathbb{Q}(y_0)$, where y_0 is a certain complex root of $P_n(2,y)$ corresponding to the presentation ρ_0. Consequently, the degree of $P_n(2,y)$ gives the degree of the trace field. The conclusion follows, since the minimal crossing number of $J(2,2n)$ is $2n+1$ if $n > 0$ and is $2 - 2n$ if $n < 0$.

The rest of this note is devoted to the proofs of Theorems 1 and 2.

2. Proofs of Theorems 1 and 2

In this section we first recall some properties of the Chebychev polynomials $S_k(z)$. We then compute the polynomial $P_n(x,y)$. Finally, we prove Theorems 1 and 2.

2.1. **Chebychev polynomials.** Recall that $S_k(z)$'s are the Chebychev polynomials defined by $S_0(z) = 1$, $S_1(z) = z$ and $S_k(z) = zS_{k-1}(z) - S_{k-2}(z)$ for all integers k. Note that $S_k(2) = k+1$ and $S_k(-2) = (-1)^k(k+1)$. Moreover if $z = t + t^{-1}$, where $t \neq \pm 1$, then $S_k(z) = \frac{t^{k+1} - t^{-(k+1)}}{t - t^{-1}}$. It is easy to see that $S_{-k}(z) = -S_{k-2}(z)$ for all integers k.

The following lemma is elementary, see e.g. [Tr4, Lemma 1.4].

Lemma 2.1. *One has*

$$S_k^2(z) - zS_k(z)S_{k-1}(z) + S_{k-1}^2(z) = 1$$

for all integers k.

Lemma 2.2. *For all $k \geq 1$ one has*

$$\begin{aligned} S_k(z) &= \prod_{j=1}^{k} \Big(z - 2\cos\frac{j\pi}{k+1} \Big), \\ S_k(z) - S_{k-1}(z) &= \prod_{j=1}^{k} \Big(z - 2\cos\frac{(2j-1)\pi}{2k+1} \Big). \end{aligned}$$

Proof. We prove the second formula. The first one can be proved similarly.

Since $S_k(z) - S_{k-1}(z)$ is a polynomial of degree k, it suffices to show that its roots are $2\cos\frac{(2j-1)\pi}{2k+1}$, where $1 \leq j \leq k$. Let $\theta_j = \frac{(2j-1)\pi}{2k+1}$. Then $e^{i(2k+1)\theta_j} = -1$. Hence, if $z = 2\cos\theta_j = e^{i\theta_j} + e^{-i\theta_j}$ then we have

$$S_k(z) = \frac{e^{i(k+1)\theta_j} - e^{-i(k+1)\theta_j}}{e^{i\theta_j} - e^{-i\theta_j}} = \frac{-e^{-ik\theta_j} + e^{ik\theta_j}}{e^{i\theta_j} - e^{-i\theta_j}} = S_{k-1}(z).$$

This means that $z = 2\cos\theta_j$ is a root of $S_k(z) - S_{k-1}(z)$. □

Lemma 2.3. *Suppose $z \in \mathbb{R}$ such that $-2 \leq z \leq 2$. Then*

$$|S_{k-1}(z)| \leq |k|$$

for all integers k.

Proof. See [Tr1, Lemma 2.6]. □

Lemma 2.4. *Suppose $M \in SL_2(\mathbb{C})$. Then*

$$M^k = S_{k-1}(z)M - S_{k-2}(z)I$$

for all integers k, where I is the identity 2×2 matrix and $z := \operatorname{tr} M$.

Proof. Since $\det M = 1$, by the Cayley-Hamilton theorem we have $M^2 - zM + I = 0$. This implies that $M^k - zM^{k-1} + M^{k-2} = 0$ for all integers k. Then, by induction on k we have $M^k = S_{k-1}(z)M - S_{k-2}(z)I$ for all $k \geq 0$.

For $k < 0$, since $\operatorname{tr} M^{-1} = \operatorname{tr} M = z$ we have

$$\begin{aligned} M^k = (M^{-1})^{-k} &= S_{-k-1}(z)M^{-1} - S_{-k-2}(z)I \\ &= -S_{k-1}(z)(zI - M) + S_k(z)I. \end{aligned}$$

The lemma follows, since $zS_{k-1}(z) - S_k(z) = S_{k-2}(z)$. □

2.2. **The polynomial P_n.** Recall that we use the following presentation of the knot group of $J(2, 2n)$:

$$\pi_1(J(2,2n)) = \langle a, b \mid aw = wb \rangle$$

where a, b are meridians and $w = (ab^{-1})^{-n}a(ab^{-1})^n$. See [Tr2, Lemma 1.1].

Suppose $\rho : \pi_1(J(2,2n)) \to SL_2(\mathbb{C})$ is a nonabelian representation. Up to conjugation, we may assume that

$$\rho(a) = \begin{bmatrix} s & 1 \\ 0 & s^{-1} \end{bmatrix} \quad \text{and} \quad \rho(b) = \begin{bmatrix} s & 0 \\ 2-y & s^{-1} \end{bmatrix}$$

where $s \neq 0$ and $y \neq 2$ satisfy a polynomial equation $P_n(s,y) = 0$. We now compute the polynomial P_n from the matrix equation $\rho(aw) = \rho(wb)$.

Since $\rho(ab^{-1}) = \begin{bmatrix} y-1 & s \\ s^{-1}(y-2) & 1 \end{bmatrix}$, by Lemma 2.4 we have

$$\begin{aligned} \rho((ab^{-1})^n) &= S_{n-1}(y)\rho(ab^{-1}) - S_{n-2}(y)I \\ &= \begin{bmatrix} (y-1)S_{n-1}(y) - S_{n-2}(y) & sS_{n-1}(y) \\ s^{-1}(y-2)S_{n-1}(y) & S_{n-1}(y) - S_{n-2}(y) \end{bmatrix}. \end{aligned}$$

Hence, by a direct (but lengthy) calculation we have

$$\begin{aligned} \rho(aw) - \rho(wb) &= \rho\big(a(ab^{-1})^{-n}a(ab^{-1})^n\big) - \rho\big((ab^{-1})^{-n}a(ab^{-1})^n b\big) \\ &= \begin{bmatrix} (y-2)P_n(s,y) & sP_n(s,y) \\ -s^{-1}(y-2)P_n(s,y) & 0 \end{bmatrix} \end{aligned}$$

where $P_n(s,y) = (s^2 + s^{-2} + 1 - y)S_{n-1}^2(y) - (s^2 + s^{-2})S_{n-1}(y)S_{n-2}(y) + S_{n-2}^2(y)$.

By Lemma 2.1 we have $S_{n-1}^2(y) - yS_{n-1}(y)S_{n-2}(y) + S_{n-2}^2(y) = 1$. Hence

$$P_n(s,y) = 1 - (y - s^2 - s^{-2})S_{n-1}(y)(S_{n-1}(y) - S_{n-2}(y)).$$

Since $P_n(s,y) = P_n(s^{-1},y)$, from now on we consider P_n as a polynomial in the variables $x = s + s^{-1}$ and y. With these new variables we have

$$P_n(x,y) = 1 - (y + 2 - x^2)S_{n-1}(y)(S_{n-1}(y) - S_{n-2}(y)).$$

2.3. **Proof of Theorem 1.** We first prove the following lemma.

Lemma 2.5. *Suppose $x_0^2 \in \mathbb{R}$ such that $4 - \frac{1}{|n|} < x_0^2 \leq 4$. If $y \in \mathbb{R}$ satisfying $P_n(x_0,y) = 0$, then $y > 2$.*

Proof. Since $P_n(x_0,y) = 0$ we have $S_{n-1}(y)(S_{n-1}(y) - S_{n-2}(y)) = (y + 2 - x_0^2)^{-1}$. Hence

$$\begin{aligned} \big((y + 2 - x_0^2)S_{n-1}(y)\big)^{-2} &= \big(S_{n-1}(y) - S_{n-2}(y)\big)^2 \\ &= 1 + (y-2)S_{n-1}(y)S_{n-2}(y) \\ &= 1 + (y-2)\big(S_{n-1}^2(y) - (y + 2 - x_0^2)^{-1}\big), \end{aligned}$$

which implies that

$$1 = (y + 2 - x_0^2)(4 - x_0^2)S_{n-1}^2(y) + (y-2)(y + 2 - x_0^2)^2 S_{n-1}^4(y).$$

Assume $y \leq 2$. Then it follows from the above equation that

$$1 \leq (y + 2 - x_0^2)(4 - x_0^2)S_{n-1}^2(y). \tag{2.1}$$

In particular, $y > x_0^2 - 2 > -2$. Since $-2 < y \leq 2$, by Lemma 2.3 we have $S_{n-1}^2(y) \leq n^2$. Hence $(y + 2 - x_0^2)(4 - x_0^2)S_{n-1}^2(y) \leq (4 - x_0^2)^2 n^2 < 1$. This contradicts (2.1). □

We now complete the proof of Theorem 1. Suppose $x_0^2 \in \mathbb{R}$ and $4 - \frac{1}{|n|} < x_0^2 \leq 4$. By Lemma 2.5, it suffices to consider $P_n(x_0,y)$ where y is a real number greater than 2. The equation $P(x_0,y) = 0$ is equivalent to

$$x_0^2 - 4 = y - 2 - \frac{1}{S_{n-1}(y)(S_{n-1}(y) - S_{n-2}(y))}. \tag{2.2}$$

Denote by $f_n(y)$ the right hand side of (2.2), where $y > 2$. We now use the factorizations of $S_{n-1}(y)$ and $S_{n-1}(y) - S_{n-2}(y)$ in Lemma 2.2.

If $n=-1$ then $f_n(y) = y-2+\frac{1}{y-1} > 0 \ge x_0^2-4$. Hence $f_n(y)=x_0^2-4$ has no solutions on $(2,\infty)$.

If $n<-1$ then, by letting $m=-n>1$, we have

$$\begin{aligned} f_n(y) &= y-2+\frac{1}{S_{m-1}(y)(S_m(y))-S_{m-1}(y))} \\ &= y-2+\frac{1}{\prod_{k=1}^{m-1}\left(y-2\cos\frac{k\pi}{m}\right)\prod_{l=1}^{m}\left(y-2\cos\frac{(2l-1)\pi}{2m+1}\right)} > 0 \ge x_0^2-4. \end{aligned}$$

Hence $f_n(y)=x_0^2-4$ has no solutions on $(2,\infty)$.

If $n=1$ then $f_n(y)=y-3$. Since $x_0^2>3$, the equation $f_n(y)=x_0^2-4$ has a unique solution $y=x_0^2-1$ on $(2,\infty)$.

If $n>1$ then we have

$$f_n(y) = y-2-\frac{1}{\prod_{k=1}^{n-1}\left(y-2\cos\frac{k\pi}{n}\right)\prod_{l=1}^{n-1}\left(y-2\cos\frac{(2l-1)\pi}{2n-1}\right)}.$$

It is easy to see that $f_n(y)$ is an increasing function on $(2,\infty)$. Moreover $\lim_{y\to\infty} f_n(y)=\infty$ and $\lim_{y\to 2} f_n(y) = -\frac{1}{n} < x_0^2-4$. Hence $f_n(y)=x_0^2-4$ has a unique solution on $(2,\infty)$.

The proof of Theorem 1 is complete.

2.4. **Proof of Theorem 2.** We write $P_n(y)$ for $P_n(2,y)$. Let $y=t^2+t^{-2}$. Then

$$\begin{aligned} P_n(y) &= \big(S_{n-1}(y)-S_{n-2}(y)\big)^2-(y-2)S_{n-1}^2(y) \\ &= \frac{(t^{2n}+t^{2-2n})^2-t^2(t^{2n}-t^{-2n})^2}{(t^2+1)^2} \\ &= \frac{(t^{2n}+t^{2-2n}+t^{2n+1}-t^{1-2n})(t^{2n}+t^{2-2n}-t^{2n+1}+t^{1-2n})}{(t^2+1)^2}. \end{aligned}$$

Up to a factor t^k, each of the polynomials $t^{2n}+t^{2-2n}+t^{2n+1}-t^{1-2n}$ and $t^{2n}+t^{2-2n}-t^{2n+1}+t^{1-2n}$ is obtained from the other by replacing t by t^{-1}. To show that $P_n(y)$ is irreducible over $\mathbb{Q}$, it suffices to show that

$$t^{4n}+t^{4n-1}+t-1=(t^2+1)Q_n(t) \tag{2.3}$$

where $Q_n(t)\in\mathbb{Z}[t]$ is irreducible over $\mathbb{Q}$.

As in the proof of [BP, Lemma 6.8], we will use the following theorem of Ljunggren [Lj]. Consider a polynomial of the form $R(t)=t^{k_1}+\varepsilon_1 t^{k_2}+\varepsilon_2 t^{k_3}+\varepsilon_3$ where $\varepsilon_j=\pm 1$ for $j=1,2,3$. Then, if R has $r>0$ roots of unity as roots then R can be decomposed into two factors, one of degree r which has these roots of unity as zeros and the other which is irreducible over $\mathbb{Q}$. Hence, to prove (2.3) it suffices to show that $\pm i$ are the only roots of unity which are roots of $t^{4n}+t^{4n-1}+t-1$ and these occur with multiplicity one.

Let t be a root of unity such that $t^{4n}+t^{4n-1}+t-1=0$. Write $t=e^{i\theta}$ where $\theta\in\mathbb{R}$. Since $t^{2n-1}+t^{1-2n}+t^{2n}-t^{-2n}=0$ we have

$$2\cos(2n-1)\theta+2i\sin 2n\theta=0,$$

which implies that both $\cos(2n-1)\theta$ and $\sin 2n\theta$ are equal to zero. There exist integers k,l such that $(2n-1)\theta=(k+\frac{1}{2})\pi$ and $2n\theta=l\pi$. This implies that $\frac{2k+1}{l}=\frac{2n-1}{n}$. Since $\frac{2n-1}{n}$ is a reduced fraction, there exists an *odd* integer m such that $2k+1=m(2n-1)$ and $l=mn$. Hence $\theta=\frac{m}{2}\pi$, which implies that $t=e^{i\theta}=\pm i$. It is easy to verify that $\pm i$ are roots of $t^{4n}+t^{4n-1}+t-1=0$ with multiplicity one.

Ljunggren's theorem then completes the proof of Theorem 2.

Acknowledgements

This research was supported by the Summer 2015 Pioneer REU in the Department of Mathematical Sciences at the University of Texas at Dallas. We would like thank the referee for carefully reading our paper and for giving helpful comments and suggestions.

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University of Dallas, Irving, TX 75062, USA
E-mail address: jdean@udallas.edu

Department of Mathematical Sciences, The University of Texas at Dallas, Richardson, TX 75080, USA
E-mail address: att140830@utdallas.edu