The effect of base flow uncertainty on transitional channel flows

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Joint work with:

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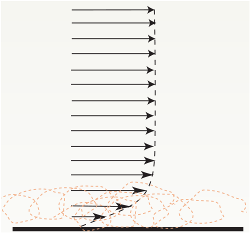
Reynolds decomposition

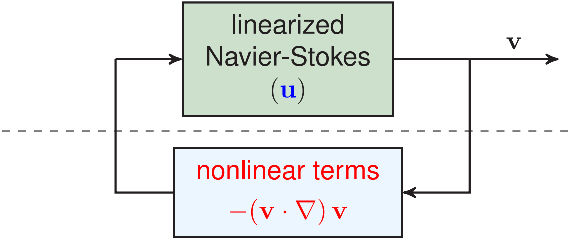
$$\tilde{\mathbf{u}} = \mathbf{u} + \mathbf{v}$$



$$\mathbf{u} = \mathbf{E}(\tilde{\mathbf{u}}) \qquad \mathbf{E}(\mathbf{v}) = 0$$

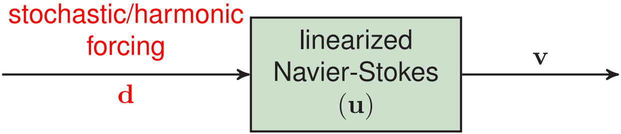
Low-complexity modeling

$$\begin{aligned} \mathbf{v}_t &= -(\nabla\cdot\mathbf{u})\,\mathbf{v} - (\nabla\cdot\mathbf{v})\,\mathbf{u} - \nabla p + \frac{1}{Re}\Delta\mathbf{v} - (\mathbf{v}\cdot\nabla)\,\mathbf{v} \\ 0 &= \nabla\cdot\mathbf{v}. \end{aligned}$$



Low-complexity modeling

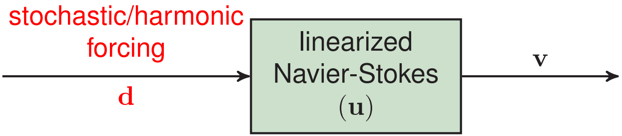
$$\begin{aligned} \mathbf{v}_t &= -(\nabla \cdot \mathbf{u})\,\mathbf{v} - (\nabla \cdot \mathbf{v})\,\mathbf{u} - \nabla p + \frac{1}{Re}\Delta \mathbf{v} + \mathbf{d} \\ 0 &= \nabla \cdot \mathbf{v}. \end{aligned}$$



Low-complexity modeling

uncertainty

unmodeled dynamics, exogenous disturbances, base flow, . . .



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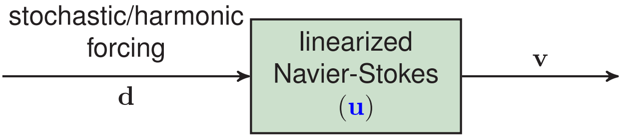
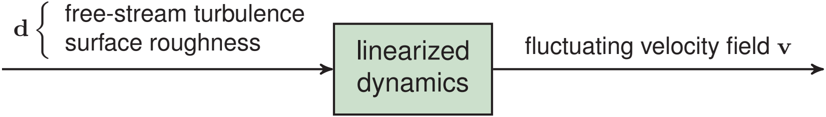


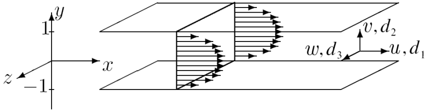
- Ensemble average of velocity fields from experiments or simulations
- Steady-state solution to the NS equations
- Reynolds-averaged NS: turbulence modeling
- . . .

Input-output analysis

Quantifying flow sensitivity via spatio-temporal frequency responses



$$\underbrace{\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}}_{\mathbf{d}} \xrightarrow{\text{amplification}} \underbrace{\begin{bmatrix} u \\ v \\ w \end{bmatrix}}_{\mathbf{v}}$$



Implications:

- insights into underlying physical mechanisms

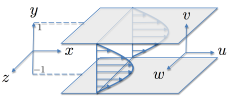
Trefethen et al., *Science* '93

Farrell & Ioannou, *Phys. Fluids* '93

Bamieh & Dahleh, *Phys. Fluids* '01

Jovanović & Bamieh, *J. Fluid Mech.* '05

Energy amplification

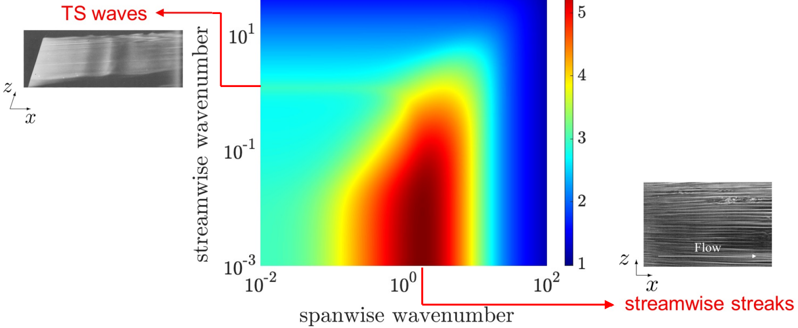


Flow subject to white-in-time forcing $\mathbf{d}$

⋆ solution to the Lyapunov equation at various (k_x, k_z) pairs

$$A\,X \;+\; X A^* \;+\; \Omega \;=\; 0$$

energy spectrum: $\mathrm{trace}(X)$

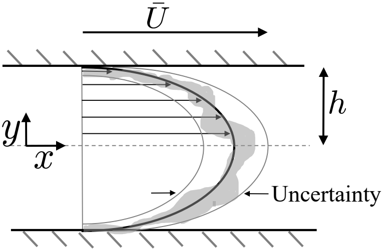


Jovanović & Bamieh, *J. Fluid Mech.* '05

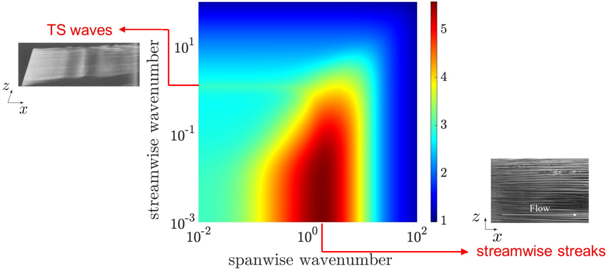
Uncertain base flow

$\gamma_u(y,t)$ zero-mean white-in-time stochastic process

$$\mathbf{u}(y,t) \;=\; \bar{\mathbf{u}}(y) \;+\; \alpha\,\gamma_u(y,t)$$



Effect of base flow uncertainty $\gamma(t)$ on $\begin{cases} \textbf{stability} \\ \textbf{frequency response} \end{cases}$



Fluctuation dynamics

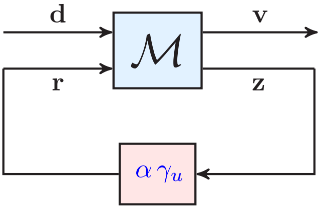
Linearization around *uncertain* base flow

$$\begin{aligned}\dot{\boldsymbol{\psi}}(t) &= A(t)\,\boldsymbol{\psi}(t) + \mathbf{d}(t)\\ \mathbf{v}(t) &= C\,\boldsymbol{\psi}(t)\end{aligned}$$

$$A(t) \;=\; \underset{\substack{\downarrow\\ \textbf{nominal dynamics}}}{\bar{A}} \;+\; \underbrace{\alpha\,\gamma_u\,A_u}_{\textbf{stochastic}}$$

Input-output reformulation

Stochastic feedback interconnection



$$\begin{bmatrix} \mathbf{v} \\ \mathbf{z} \end{bmatrix} = \mathcal{M} \begin{bmatrix} \mathbf{f} \\ \mathbf{r} \end{bmatrix}$$

$$\mathcal{M} : \begin{cases} \dot{\boldsymbol{\psi}}(t) \;=\; \bar{A}\,\boldsymbol{\psi}(t) \,+\, \mathbf{r}(t) \,+\, \mathbf{d}(t) \\ \mathbf{z}(t) \;=\; A_u\,\boldsymbol{\psi}(t) \\ \mathbf{v}(t) \;=\; C\,\boldsymbol{\psi}(t) \end{cases}$$

$$\mathbf{r}(t) \;=\; \alpha\,\gamma_u(t)\,\mathbf{z}(t)$$

Filo & Bamieh, *Trans. Automat. Control* '20

Mean-square stability

- Stability of the mean and boundedness of output variances
- Stochastic multiplicative uncertainty *can* lead to a loss of stability
- Conditions for mean-square stability:
  - ✓ $\bar{A}$ Hurwitz
  - ✓ spectral radius $\rho\,(\mathbb{L}) \,<\, 1$

Loop gain operator $\mathbb{L}$:

$$\begin{cases} \mathbb{L}(\bar{\mathbf{R}}) \,=\, \Gamma \circ (A_u\, X\, A_u^\dagger) \\ \bar{A}\, X \,+\, X\, \bar{A}^\dagger \,=\, -\bar{\mathbf{R}} \end{cases}$$

Filo & Bamieh, *Trans. Automat. Control* '20

Second-order statistics

- Generalized Lyapunov equation

$$\bar{A}\,X \;+\; X\,\bar{A}^* \;+\; \alpha^2\,\Gamma \circ (A_u\,X\,A_u^*) \;=\; -\,\Omega$$

Computationally expensive!

Second-order statistics

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Computationally expensive!

✓ Perturbation analysis; small amplitudes α

$$X \;=\; X_0 \;+\; \alpha^2\,X_2 \;+\; O(\alpha^4)$$

$$\Downarrow$$

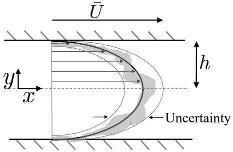
$$E \;=\; E_0 \;+\; \alpha^2\,E_2 \;+\; O(\alpha^4)$$

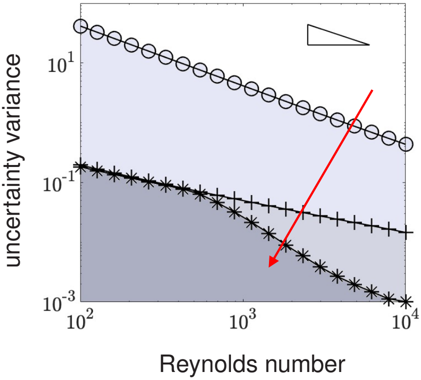
Hewawaduge & Zare, *Phys. Rev. Fluids* '21

Results

Mean-square stability

$(k_x, k_z) = (1, 1)$





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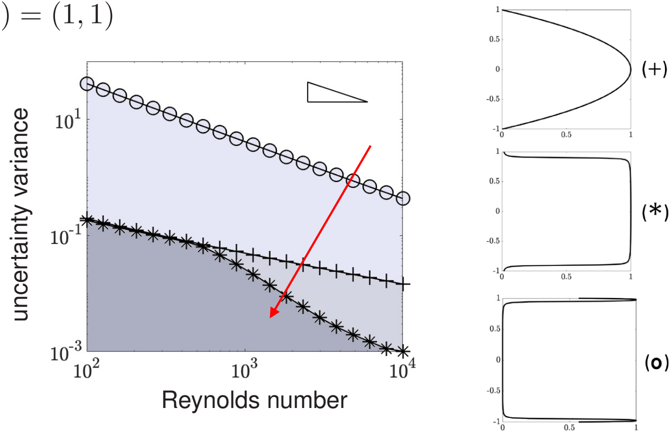


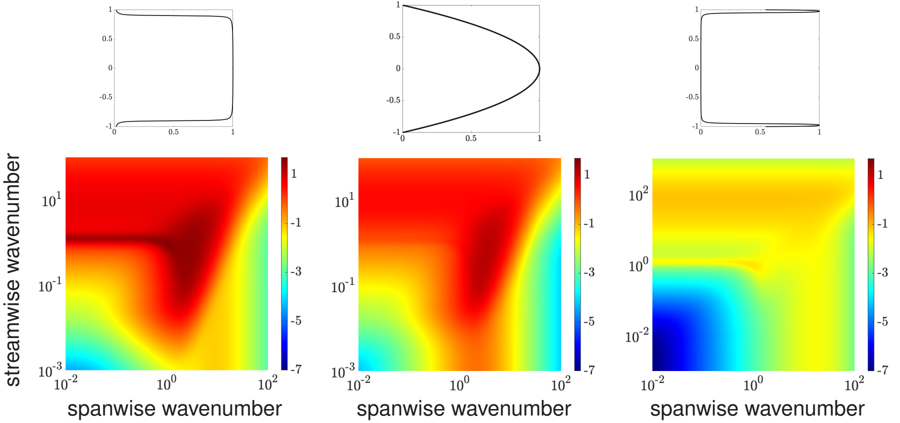
⋆ $O(R^{-1})$ scaling in agreement with (deterministic) worst-case analysis

Bottaro, Corbett, Luchini, *J. Fluid Mech.* '03

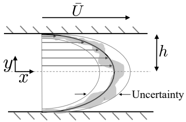
Correction to energy spectrum

$$X = X_0 + \alpha^2 X_2 + O(\alpha^4)$$

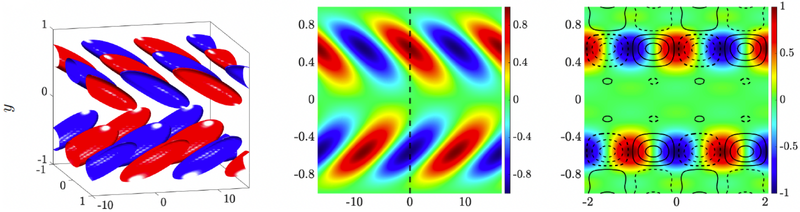
$$E_c = \alpha^2 \, \mathrm{trace}(X_2)$$



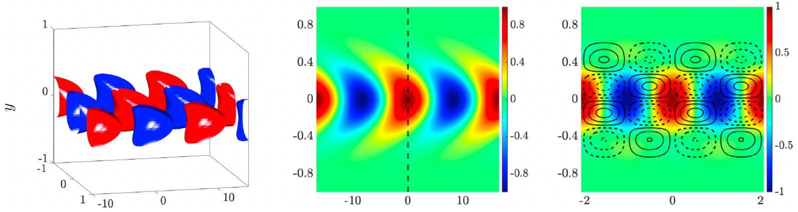
Flow structures



Nominal



With perturbed base flow; $\alpha = 1$, $\sigma_u^2 = 0.21$



Summary

Effect of stochastic base flow perturbations:

- mean-square stability
- frequency response

Reynolds number scaling of critically stable variance levels

Robustness of streaks ($k_x = 0$); R-dependance of variance amplification

$$E(k_z) \; = \; f(k_z)R \, + \, g(k_z)R^2 \, + \, h(k_z)R^3$$

Hewawaduge & Zare, *Phys. Rev. Fluids* '21