# Endogenous Option Pricing*

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**Abstract**

We show that a dynamic model of investment and capital structure choices, where the firm faces real and financial frictions, can generate option prices and implied volatilities that are in line with those of the average optionable stock. As the balance between the fundamental economic forces that are responsible for the way options are priced is state-dependent, the model is also able to generate a wide cross-sectional dispersion in implied volatility surfaces that matches what we observe in the data.

*JEL Classifications: G12, G32*

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# 1. Introduction

Stocks' implied volatility surfaces exhibit a large degree of cross-sectional and time-series variability, reflecting heterogeneous market's expectations of forward looking (risk-neutral) distributions. At any point in time the implied volatility surface of a stock for a given maturity can assume many shapes. At three months maturity, 73% of stocks exhibit an implied volatility smirk (i.e., implied volatility is monotonically decreasing with strikes), while about 18% present a smile (i.e., implied volatility decreases and then increases with strikes). In the remaining cases, the surface is either concave (i.e., an inverted smile that we call frown), or increasing with strikes. Over more than 20 years of data, the percentage of stocks that exhibit a smirk varies considerably from 41.3% to 87.7%. Substantial variation also occurs through contracts' maturities: Implied volatilities are on average decreasing with options maturities; the percentage of stocks that exhibit a smirk increases to over 80% when the implied surface is extracted from contracts that are approximately one year to maturity. Analogues to implied volatility surfaces, option prices imply large variation in the moments of risk-neutral distribution. On average risk-neutral skewness is negative and kurtosis is in excess of three, but there is considerable cross-sectional and time-series dispersion.

We show that quantitatively equivalent patterns can be produced in the context of a dynamic capital structure model where firms revisit their investment and financing decisions in each period. We follow in many others' footsteps and adopt what, following the lead of Hennessy and Whited (2005, 2007) and Zhang (2005), has become the workhorse model of dynamic corporate and investment based asset pricing studies. The application of a dynamic model of the firm to option pricing follows directly from the seminal work of Merton (1973, 1974) and later addressed by Geske (1979) in his compound option pricing formula (i.e., the compound option model), and by Toft and Prucyk (1997), who price an equity option on a firm that faces taxes and bankruptcy costs as in the Leland (1994) model. Both, Toft and Prucyk (1997) and Geske, Subrahmanyam, and Zhou (2016) show that accounting for the leverage effect reduces pricing errors relative to the traditional model of Black and Scholes (1973).

When it comes to option pricing a dynamic model of the firm allows leverage to play a role and, as is the case for the compound option model, produce skewness in the risk-neutral distribution (e.g., implied volatility surface sloping down). However, differently from the compound option model of Geske (1979), the finite stream of growth options embedded in the firm's endogenous investment policy allows for an economic force that interacts with the effect of leverage. The resulting equilibrium implied volatility surface is therefore state-contingent and can assume any form: upward sloping, downward sloping, u-shaped, or even inverted u-shape. In the model, growth options accumulate because of the future investment prospects of the firm and the presence of capital adjustment costs that make the investment process lumpy. Future investment prospects are predictable in the model as the basic source of uncertainty in the economy (i.e., productivity shock) is modeled as a persistent stochastic process. Persistent productivity shocks also deliver a downward sloping implied volatility curve relative to the option's maturity.

The basic intuition for the main mechanism in the model resides in the famous relationship derived by Merton (1974) that links the equity volatility to the asset volatility through the equity elasticity. It is possible to prove, at least in the context of Geske (1979) model, that such relationship implies a downward sloping and convex implied volatility curve (see Rathgeber, Stadler, and Stockl, 2020). In the Merton model, the equity value of a levered firm can be seen as a call option on some physical asset with a strike price equal to the value of the debt, and the compound option model of Geske (1979) models a financial option on such equity. Our model differs in many aspects, but essentially because the equity of our firm can be seen as a call option on the asset in place plus a stream of growth and contraction options. The contribution of such growth and contraction options to the equity elasticity makes the relationship between asset volatility and equity volatility (and hence implied volatility curve) non trivial. In particular, the part of the sensitivity of the firm's equity to the asset that is due to the growth/contraction options can take many forms (as opposed to be strictly decreasing with the value of the equity as in the case of the Merton model), thus delivering many possible shapes for the implied volatility curve.

We calibrate the model to match firm characteristics, implied volatilities, and moments of the return distribution of the average optionable stock. On the one hand, our simulated firms make investment and capital structure choices in line with those observed in the data. On the other, the physical distribution of equity returns shows positive skewness and positive excess kurtosis. The risk-neutral distribution instead displays negative skewness and a larger excess-kurtosis. This translates into implied volatility surfaces that are remarkably close to the data, in the shape and the frequency with which they are observed. The average implied volatility surface is downward sloping along the moneyness and along maturity levels. However the frequency of times that the curve assumes another shape is in line with what we see in the data, and in more than 20% of the cases assumes another shape (at 90 days maturity).

We use the simulated economy to validate relationships between properties of option prices (i.e., of implied volatilities) and firm characteristics that we observe in the data. We confirm that the level of the implied volatility is an increasing function of leverage and a decreasing function of market-to-book ratio. The steepness of the left slope of the implied volatility curve (i.e., the left smirk) is greater for firms with higher leverage and lower market-to-book ratio. On the other side of the distribution, firms with low leverage and high market-to-book values tend to exhibit positive right slopes of the implied volatility curve.

This paper follows an established literature that aims at measuring and understanding the impact of corporate policies on asset prices (see for example, Kuehn and Schmid, 2014, who use a similar model to analyze the pricing of corporate debt.) Similar to Toft and Prucyk (1997) and Geske, Subrahmanyam, and Zhou (2016), we offer an alternative approach to option pricing studies that rely on exogenous specifications of stochastic properties of equity prices. While we do not believe that our approach could be as successful in delivering small pricing errors for each security as this last class of models, our calibration is remarkably close in pricing options on the average firm, and in producing, with a single set of parameters, a widespread cross-section that is entirely produced by optimal investment and capital structure decisions.

## 2. Related literature

This paper is primarily related to the strand of literature that aims at explaining equity option prices in the cross-section of stocks. Starting from the seminal work of Merton (1974), there have been a few attempts at incorporating option pricing into a structural model of the firm. Geske (1979) offers a first attempt by producing a double compound option that allows one to price a call option on the equity of a levered firm. Toft and Prucyk (1997) extends this approach to the Leland (1994) economy, thus allowing for taxes and bankruptcy costs to determine the optimal leverage policy of the firm. Geske, Subrahmanyam, and Zhou (2016) show that accounting for the leverage effect greatly reduces option pricing errors relative to the Black and Scholes (1973) model. Bai, Goldstein, and Yang (2019) show that the leverage effect is essential to explain the spread between index and individual banks equity options. Following Hennessy and Whited (2005, 2007), we introduce a fully dynamic model where shareholders endogenously choose production capacity, financial leverage, and default. We show that these ingredients are essential to reproduce the heterogeneity in option prices present in the data.

The leverage effect introduced by Merton (1974) has been considered in a number of applications that link volatility to stock prices/returns. For example, Engle and Siriwardane

(2017) propose a structural GARCH model that embeds the leverage effect into equity volatility forecasting models.

Because we introduce a model where firms are exposed both to systematic and idiosyncratic risk, our work is also related to studies such as Duan and Wei (2009). Similar to Duan and Wei (2009), our model also implies that large variation in the prices of individual equity options will be produced by realizations of aggregate risk. Because two key ingredient in our model are operating leverage and growth options, our paper is also related to Morellec and Zhdanov (2019) who study the link between equity risk and product market competition.

Because we share many model features and because we rely on some of the same intuition our paper is also related to the rather large literature that studies corporate credit risk: from Leland (1994) to more recent contributions such as Kuehn and Schmid (2014).

# 3. Data

We construct our sample of optionable stocks by combining CRSP and COMPUSTAT with OptionMetrics. To increase the frequency of observations we obtain quarterly balance sheet observations and match them to stock returns data using common filters. We construct stock returns and accounting ratios (leverage, profitability, market-to-book) using standard definitions.

We then match the resulting sample with OpionMetrics. In particular, for each firm and quarterly reporting date we extract option prices that at that point in time have maturities closest to 90 days (one quarter), 180 days, and 360 days. This allows us to construct a term structure of option prices. We record implied volatilities as reported by OptionMetrics and use call and put option prices to obtain model-free risk neutral moments as in Bakshi, Kapadia, and Madan (2003), Dennis and Mayhew (2002), and Hansis, Schlag, and Vilkov (2010).

Finally, we eliminate all observations for which CRSP reports a dividend payment in the next 12 months. The final sample is composed of 3,536 stocks and includes quarterly observations between the years 1996 and 2019.

# 4. Empirical evidence about the cross-section of equity option prices

The option pricing literature has mainly focused on two different ways to organize option prices for different maturities and moneyness: implied volatility surfaces and implied risk-neutral moments. Mots of these efforts have been concentrated on index options, which offer a great way to understand aggregate risk premia and investor attitudes towards risk.

We organize the data along the same lines but we focus on the individual equity options. We present in this section some empirical regularities that we deem important in thinking about what an option pricing model should address.

*4.1. Implied volatility surfaces*

We construct implied volatility surfaces from option prices recorded on the last three days of the month. To lessen the impact of microstructure biases due to thin trading and stale prices, we average over the 3 months that correspond to the earning reporting quarter. So for example, the implied volatility of a stock that ends its first earning quarter on March 31 will be the average of the last three trading days of January, February, and March.

The average implied volatility surface is downward sloping with both moneyness and maturity (see Figure 2), although less pronouncedly than the index option surface. At the shortest maturity of 90 days, the average difference between OTM (i.e., moneyness of 0.8) and ATM is 3.5%, while the average difference between ATM and ITM (i.e., moneyness of 1.2) is about 1.2%. Along maturities, the average difference between 90 and 360 days varies between 5.3% for the OTM strike, to 4% for the ATM strike.

**Figure 1: Implied volatility surface**

The figure plots the average implied volatility surface extracted from the data. The sample contains all industrial firms with options trading on their equity between 1996 and 2019. Data is sampled at quarterly frequency, and observations are removed if companies pay dividends in the next 12 months. A total of 3,536 firms are included.

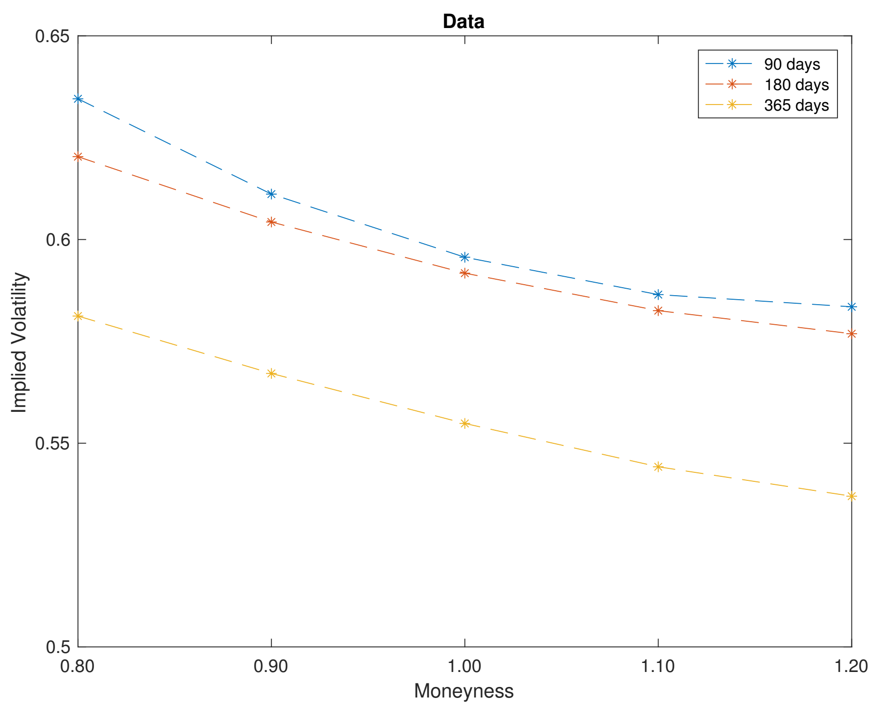
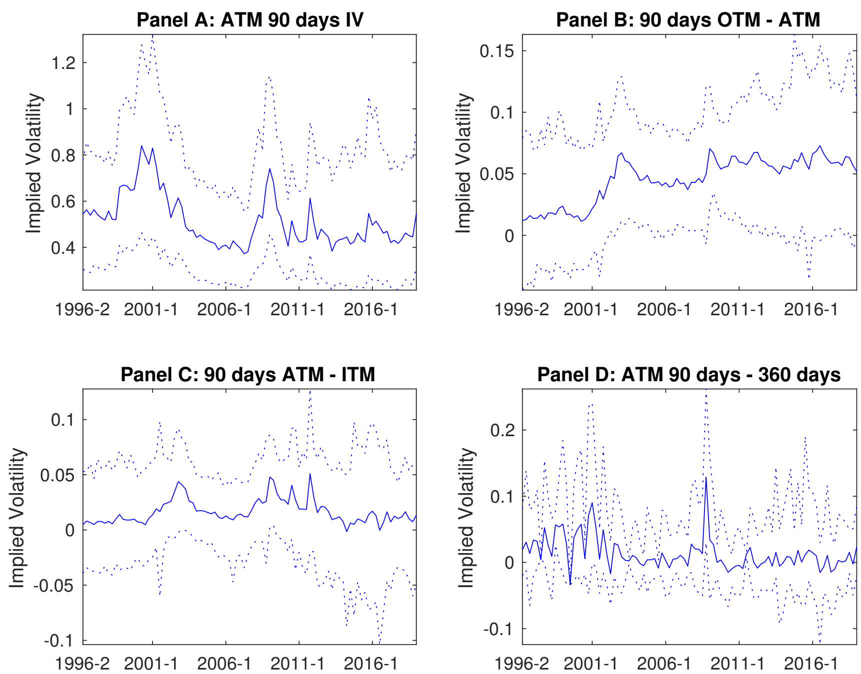


**Figure 2: Implied volatility surface – time series**

The figure plots the time series of cross sectional averages, as well fifth and ninety-fifth percentiles, of implied volatility surface extracted from the data. The sample contains all industrial firms with options trading on their equity between 1996 and 2019. Data is sampled at quarterly frequency, and observations are removed if companies pay dividends in the next 12 months. A total of 3,536 firms are included.

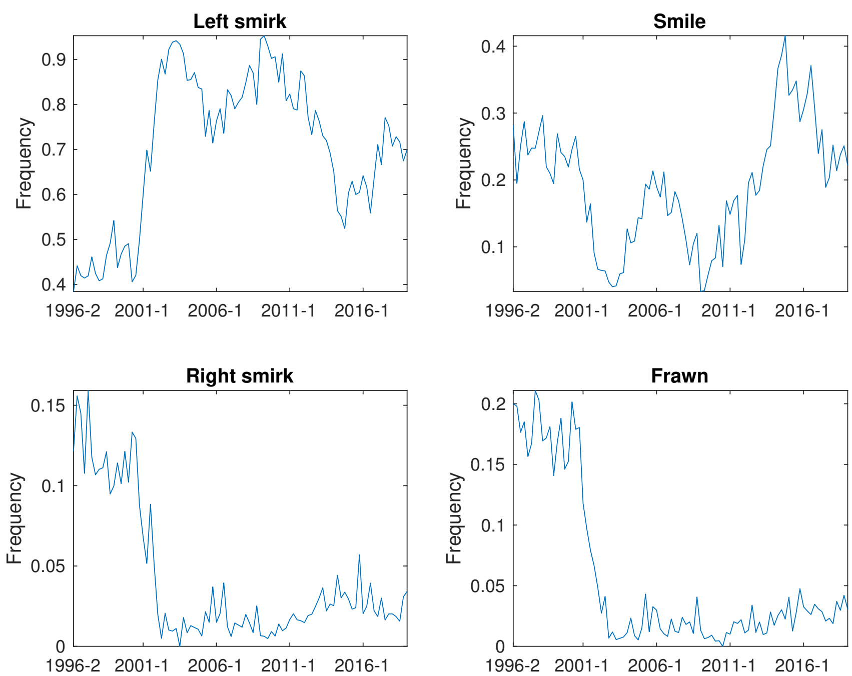



There is a considerable amount of cross-sectional and time-series variability in implied volatility surfaces. For example, the cross-sectional average ATM implied volatility varies between 90% at the height of the internet bubble crash to 35% in the middle of 2005 (see Panel A of Figure 2). At the same time, there is a fair amount of cross-sectional dispersion: for example at the hight of the financial crisis, the $95^{th}$ percentile of implied volatility is higher than 100%. Similarly, the left tail of the implied volatility curve can be as high as 15% and as low as -5% (see Panel B of Figure 2). The right tail varies even more from 10% to -10% (see Panel C). Similar variation can be seen even across maturities (see Panel D), where the slope of the volatility surface hovers around 4% but can be even negative for some stocks at particular points in time.

Variation in implied volatilities through time and across stocks produces also a very rich cross-section of different "shapes". We categorize the shape of the implied volatility curve into four types: left smirk (i.e., implied volatility decreasing with moneyness), smile, right

**Figure 3: Surface types – time series**

The figure plots the time series of frequencies of different implied volatility surface types extracted from the data. The sample contains all industrial firms with options trading on their equity between 1996 and 2019. Data is sampled at quarterly frequency, and observations are removed if companies pay dividends in the next 12 months. A total of 3,536 firms are included.



smirk (i.e., implied volatility rising with moneyness), and frown (i.e., inverted smile). We plot the cross-sectional frequency of each surface type for 90 days options in Figure 3.

The most predominant surface type is a left smirk, which is observed on average 73% of the times, with a large time-series variation between 40% and 95% (see Panel A). The second most frequent surface is a smile (Panel B), which is observed on average in 18% of the cases. Right smirks and frowns are less frequent on average; they however manifest in a significant number of stocks during the years of the internet bubble (Panels C and D).

*4.2. Risk-neutral moments*

We follow Hansis, Schlag, and Vilkov (2010) and calculate risk-neutral model free moments for each stock and for each option maturity.[1] For each stock we select options that at a particular point in time have approximate maturity 90, 180, or 360 days, that have non-zero bid prices and open interest, have moneyness (strike divided by stock price) between 0.7 and 1.3. We interpolate their implied volatilities in order to obtain a dense grid of prices relative to moneyness. We then compute implied moments. On average, 8 option contracts enter the calculation of risk-neutral moments.

We plot time-series of the cross-sectional averages, as well as the $5^{th}$ and $95^{th}$ percentiles, of risk neutral skewness and kurtosis in Figure 4. At each maturity, risk neutral skewness is negative, with a sharp decrease around the internet bubble. Risk-neutral kurtosis is in excess of three and also increasing through the period. Large cross-sectional variation is observed for each series.

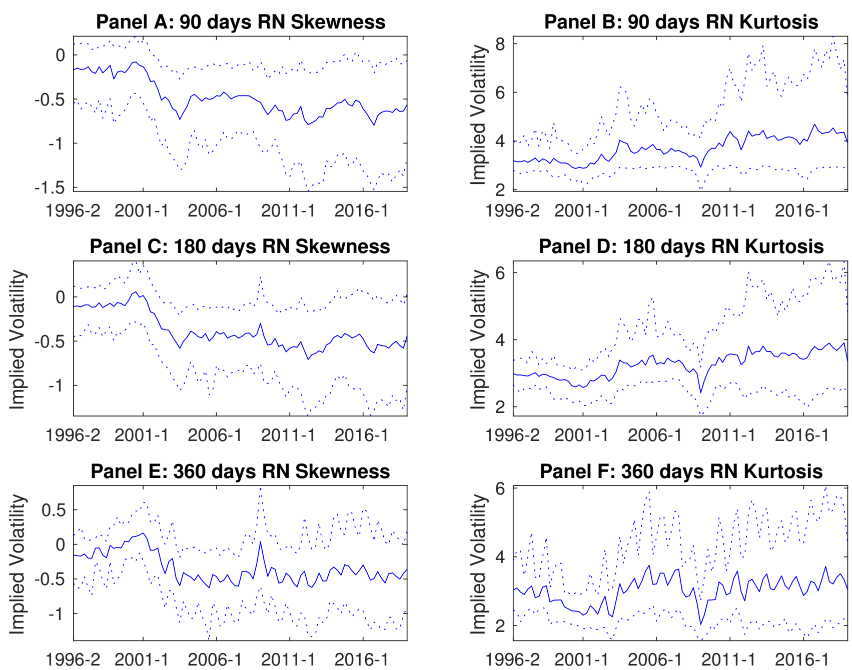
Overall, implied volatility surfaces or at risk-neutral moments present a pretty consistent picture of the cross-section of option prices. There is large time-series and cross-sectional variation in the variables that we can construct to summarize the information contained in option prices. While it is entirely possible that such variation can be explained by exogenously specifying the equity and volatility process, we propose a structural approach based on the idea that optimal firm decisions shape the physical and risk-neutral distributions of equity returns.

Thus in the next section we develop a structural model similar in spirit to Merton (1974), which is however fully dynamic and incorporate many realistic features. We show that the sensitivity of equity volatility to changes in leverage and investment policies allows one to recover a simulated economy that presents many of the features present in the data.

[1] Many thanks go to Grigory Vilkov for making his code freely available at https://www.vilkov.net.

**Figure 4: Risk neutral moments – time series**

The figure plots the time series of cross sectional averages, as well fifth and ninety-fifth percentiles, of risk-neutral skewness and kurtosis extracted from the data. The sample contains all industrial firms with options trading on their equity between 1996 and 2019. Data is sampled at quarterly frequency, and observations are removed if companies pay dividends in the next 12 months. A total of 3,536 firms are included.



# 5. The equity elasticity of growth options

We can take a preliminary look at understanding what impact growth options have on the equity volatility (and hence on the implied volatility surface) by framing the problem in terms of the firm studied by Merton (1974). We analyze a simple firm that invests in an option to acquire a project. The project current value is $X$ but grows following a Geometric Brownian motion with volatility $\sigma$. To take the project, an investment cost $K$ must be sunk at a future date $T$. Hence, the real option is European, has payoff equal to $\max\{X_T - K, 0\}$, and its current value equals $A$.

The firm finances the purchase of the real option with a mix of equity $E$ and debt $B$, which is due at a future date $t < T$, thus ruling out the possibility that the firm defaults on the debt payment if the real option expires unexercised. Since the firm does not hold any real physical asset, we assume that the value of the real option at the time the debt matures, $A_t$, can be monetized to generate the financial resources to pay the debt. Because the debt is defaulted upon if $A_t < B$ at $t$ and the equity contract is protected by limited liability, then the value of equity at that date is $\max\{A_t - B, 0\}$.

The current value of the real option can be obtained directly from the Black and Scholes (1973) model:

$$A = X\mathcal{N}_1(d_1) - Ke^{-rT}\mathcal{N}_1(d_2), \tag{1}$$

where

$$d_1 = \frac{\log(e^{rT}X/K)}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}, \qquad d_2 = d_1 - \frac{1}{2}\sigma\sqrt{T},$$

and $\mathcal{N}_1$ is the cumulative of the standard normal distribution. The value of the firm's equity can be obtained instead by applying the compound pricing formula of Geske (1979) as

$$E = X\mathcal{N}_2(a_1, d_1) - Ke^{-rT}\mathcal{N}_2(a_2, d_2) - Be^{-rt}\mathcal{N}_1(a_2), \tag{2}$$

where

$$a_1 = \frac{\log(e^{rT}X/\overline{X}_t)}{\sigma\sqrt{t}} + \frac{1}{2}\sigma\sqrt{t}, \qquad a_2 = a_1 - \frac{1}{2}\sigma\sqrt{t},$$

where $\mathcal{N}_2$ is the cumulative of the bivariate standard normal distribution, with correlation coefficient $\sqrt{t/T}$, and $\overline{X}_t$ is the value of the project at the maturity of the debt below which the firm defaults. In other words, $\overline{X}_t$ is the only root of the equation $A(X_t) = B$.

The intuitive relation proposed by Merton (1974) to link the volatility of the asset to the volatility of the equity still remains but changes to:

$$\sigma_E = \frac{\partial E}{\partial X}\frac{X}{E}\sigma = \mathcal{N}_2(a_1, d_1)\frac{X}{E}\sigma = \mathcal{E}\sigma. \tag{3}$$

Contrary to the Merton case, where the only force affecting the equity elasticity is leverage, here there is another economic force at play: the moneyness of the real option. The two effect combine to produce some interesting effect.

Similarly to the Merton case, the elasticity is always increasing with firm's leverage. Assume for the moment that $K = 0$, which makes $A = X$, and the value of the real option coincides with the value of the project. In this case

$$\sigma_E = \frac{\partial E}{\partial X}\frac{X}{E}\sigma = \mathcal{N}_1(\tilde{a}_1)\frac{X}{E}\sigma = \mathcal{E}\sigma, \tag{4}$$

where

$$\tilde{a}_1 = \frac{\log(e^{rT}X/B)}{\sigma\sqrt{t}} + \frac{1}{2}\sigma\sqrt{t}$$

The volatility of the underlying asset (in this case, of the project) is amplified by the elasticity of the equity with respect to the value of the asset, $\mathcal{E}$. In particular, the elasticity is higher the lower $E$, which occurs when $B$ is high relative to $X$. However, a low $e^{rT}X/B$ reduces $\tilde{a}_1$, which in turn lowers $\mathcal{E}$. The first effect prevails on the second so that a high leverage increases $\mathcal{E}$ and makes $\sigma_E$ higher than $\sigma$.

If $K > 0$, the impact of leverage is directionally unaffected (higher leverage implies higher $\mathcal{E}$), but the total effect has to take into account the economics of the underlying real option

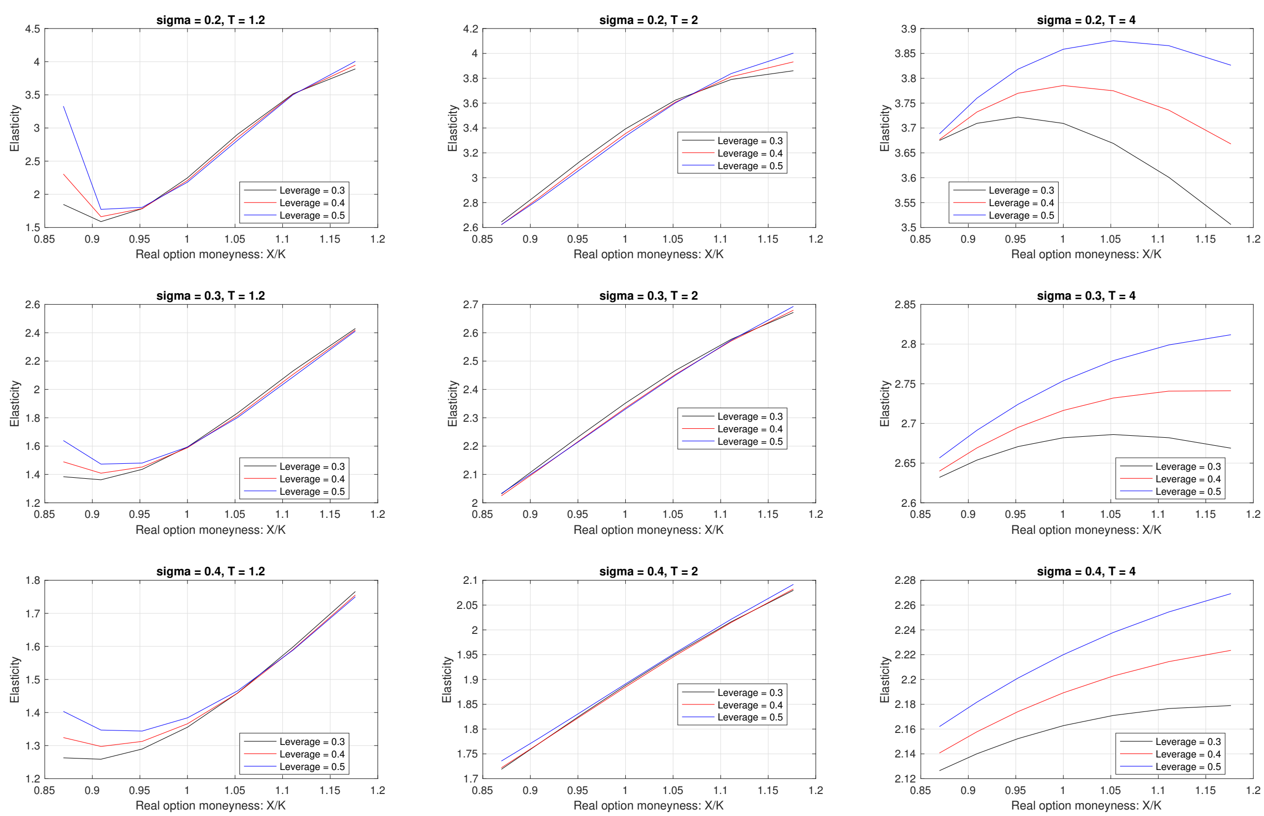
$$\mathcal{E} = \mathcal{N}_2(a_1, d_1)\frac{X}{E}. \tag{5}$$

where here $a_1$ is now different than $\tilde{a}_1$ because it has to take into account that the equity is an option on another option. Notice that $\mathcal{N}_2(a_1, d_1)$ is increasing in both arguments. In particular, because $A$ in (1) is a strictly increasing function of $X$, the root $\overline{X}$ is strictly increasing in $B$. Hence, the higher $B$ the lower $a_1$, everything else equal, which lowers $\mathcal{N}_2(a_1, d_1)$. On the other hand, a higher leverage reduces $E$, everything else equal, which is the prevailing force that produces the leverage effect on the equity volatility.

Low moneyness of the real option, which occurs when $e^{rT}X/K$ is low (either because $X$ is low or because $K$ is high), has also two effects. On the one hand, it is less likely that the firm will pick up the investment project, $d_1$ decreases, and that reduces $\mathcal{N}_2(a_1, d_1)$. On the other hand, the equity value is lower because the real option is the underlying asset of the firm and its value, $A$, is an increasing function of $e^{rT}X/K$. Therefore the two terms of the elasticity Equation 5 move in opposite direction relative to the real option moneyness: as moneyness increases $\mathcal{N}_2(a_1, d_1)$ increases but $X/E$ decreases. Unfortunately, which of the two effects prevails is function of the volatility of the project value, $\sigma$, and of how far apart the debt maturity is from the maturity of the real option, $T - t$. In most cases the cumulative effect of moneyness is to increase the equity elasticity, but there are some parameter combinations for which we obtain a decreasing relationship (see Figure 5 and 7).

**Figure 5: Equity elasticity relative to growth option moneyness**

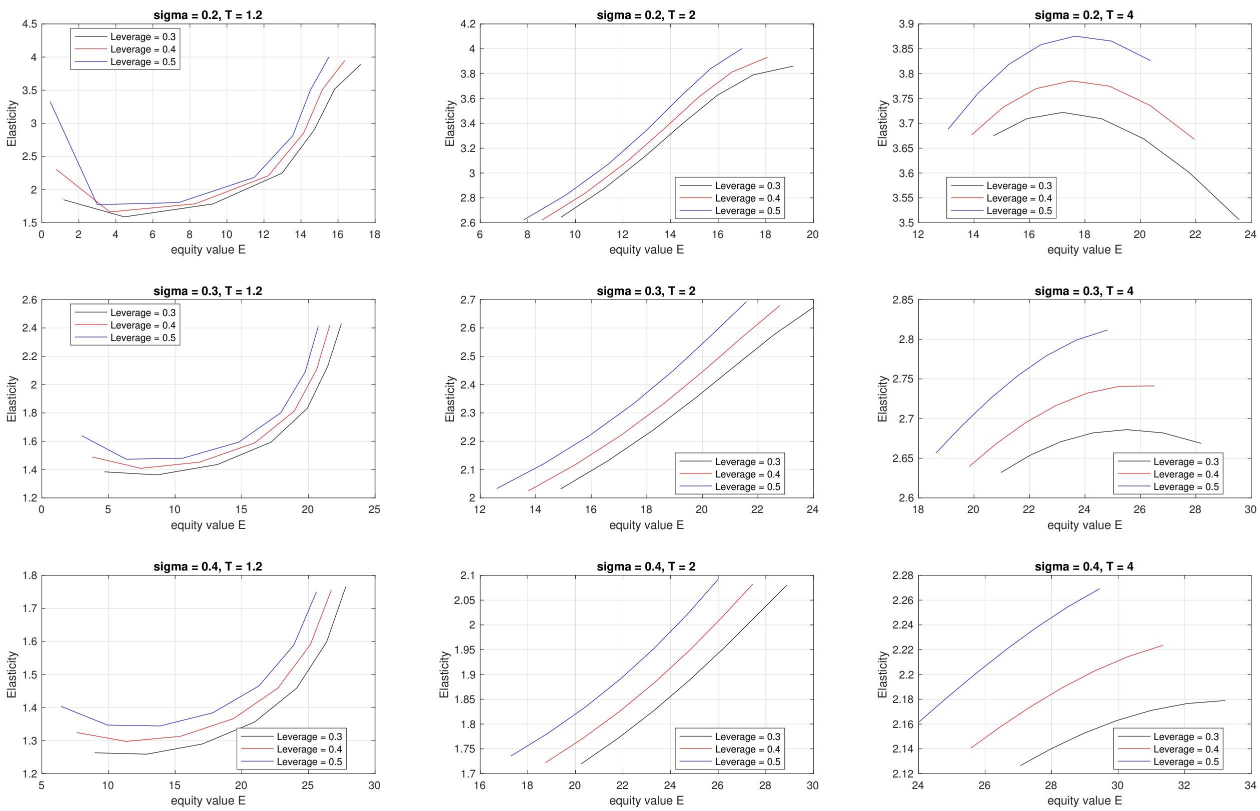
The figure plots equity elasticity from Equation 5 as function of the moneyness of the real option. The current value of the project $X$ is set to 100. The investment costs varies from 85 to 115. The firm leverage, $(A-E)/A$ varies from 0.3 to 0.5. We set the maturity of the debt to $t = 1$ period, and vary the maturity of the real option from 1.2 to 4 periods. We vary the volatility of the underlying project value from 0.2 to 0.4. The risk free rate is 5%.




In a similar way to the option to invest we can also derive the equity elasticity in case the firm acquires an option to disinvest a project, in which case the firm's equity can be thought

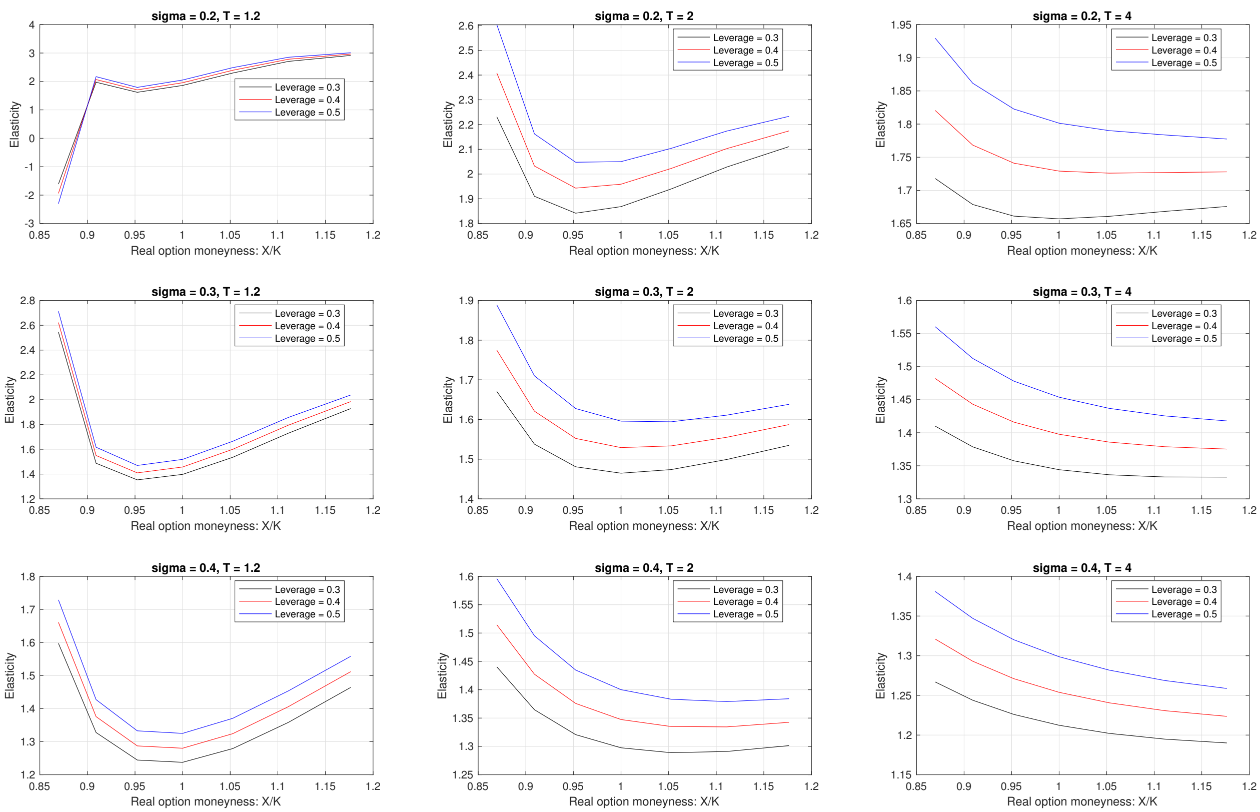
**Figure 6: Equity elasticity relative to equity value**

The figure plots equity elasticity from Equation 5 as function of the firm's equity $E$. The current value of the project $X$ is set to 100. The investment costs varies from 85 to 115. The firm leverage, $(A-E)/A$ varies from 0.3 to 0.5. We set the maturity of the debt to $t=1$ period, and vary the maturity of the real option from 1.2 to 4 periods. We vary the volatility of the underlying project value from 0.2 to 0.4. The risk free rate is 5%.



as a call on a put option. In this case, the equity elasticity is high for low moneyness of the real option (in-the-money put) and is lower when the moneyness is high. The relationship is however not monotonic, and exhibit some convexity, which is more or less pronounced depending on the combination of $\sigma$ and $T - t$.

**Figure 7: Equity elasticity relative to equity value for an option to disinvest**
The figure plots equity elasticity from Equation 5 as function of the firm's equity $E$. The current value of the project $X$ is set to 100. The cash received in case of disinvestment varies from 85 to 115. The firm leverage, $(A - E)/A$ varies from 0.3 to 0.5. We set the maturity of the debt to $t = 1$ period, and vary the maturity of the real option from 1.2 to 4 periods. We vary the volatility of the underlying project value from 0.2 to 0.4. The risk free rate is 5%.




# 6. Model

We propose a dynamic model of corporate decisions that is characterized by firm heterogeneity and endogenous default. We include corporate taxes, real adjustment costs, external equity financing frictions, debt adjustment costs, operating leverage, financial distress costs, and consider countercyclical risk premia. The model is therefore similar, in spirit, to that of Hennessy and Whited (2007) in the description of the firm's decisions, and to those of Berk, Green, and Naik (1999), Zhang (2005) and Gomes and Schmid (2010) in the choice of a reasonably simple (exogenously specified) pricing kernel.

*6.1. The economy*

Information is revealed and decisions are made at a set of discrete dates $\{0, 1, \ldots, t, \ldots\}$. The time horizon is infinite. The economy is composed by a utility maximizing representative agent and a fixed number of heterogenous firms ($j = 1, \ldots, J$) that produce the same good. Firms make dynamic investment and financing decisions and are allowed to default on their obligations. Defaulted firms are restructured and then continue operations, so as to guarantee a constant number of firms in the economy. The agent consumes the dividends paid by the firms and saves by investing in the financial market. We do not close the economy and derive the equilibrium, but instead choose an exogenously specified stochastic discount factor.

There are two sources of risk that capture variation in the firm's productivity. The first, $z_j$, captures variations in productivity caused by firms' specific events. Idiosyncratic shocks are independent across firms, and have a common transition function $Q_z(z_j, z'_j)$. $z_j$ denotes the current (or time–$t$) value of the variable, and $z'_j$ denotes the next period (or time–$(t+1)$) value.

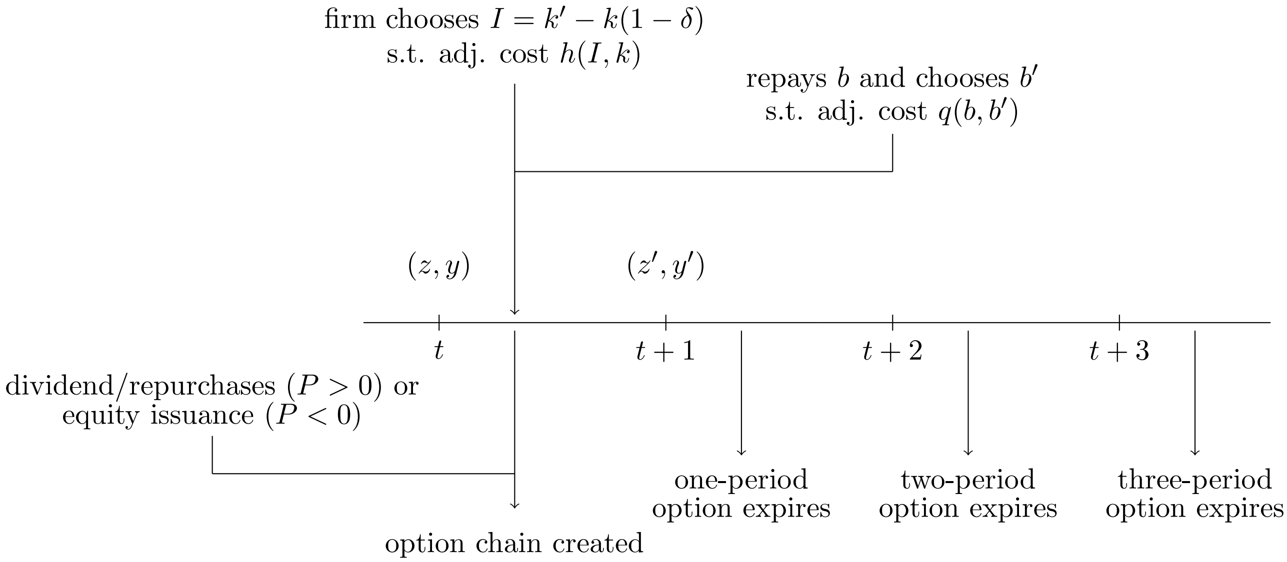
The second source of risk, $y$, captures variations in productivity caused by macroeconomics events. The aggregate risk is independent of the idiosyncratic shocks and has transition function $Q_y(y, y')$. $Q_z$ and $Q_y$ are stationary and monotonic Markov transition functions that satisfy the Feller property. $z$ and $y$ have compact support. For convenience of exposition, we define the state variable $x = (y, z)$, whose transition function, $Q(x, x')$, is the product of $Q_y$ and $Q_z$. As there is no risk of confusion, we drop the index $j$ in the rest of the section.

*6.2. Firm policies*

We assume that firm's decisions are made to maximize shareholders' value. An intuitive description of the chronology of the firm's decision problem is presented in Figure 8. At $t$, the two shocks $x = (y, z)$ are realized and the firm cash flow is determined based on current capital stock, $k$, and total face value of debt, $b$. Immediately after that, the firm simultaneously chooses the new set of capital, $k'$, and debt, $b'$ for the period $]t, t+1]$. This decision determines $P$, the payout to shareholders, which can be positive (dividends and/or share repurchases) or negative (an injection of equity capital by issuing new shares).

**Figure 8: Model time line**

This figure offers a description of the chronology of the firm's recursive decision problem. At $t$, the shocks $x = (y, z)$ are realized, and the firm's cash flow is determined based on the capital stock $k$ and the debt $b$, or $a = (k, b)$. Immediately after $t$, the firm chooses the new set of capital and debt, as the combination $a' = (k', b')$ that maximizes the value of the equity, given by the sum of the current cash flow plus the continuation value.




At $t$, the cash flow from operations (EBITDA) depends on the idiosyncratic and aggregate shocks, and on the current level of asset in place, $\pi = \pi(y, z, k) = e^{y+z}k^{\alpha} - f$, where $\alpha < 1$ models decreasing returns to scale and $f \geq 0$ is a operating cost parameter that summarizes all operating expenses excluding interest on debt.

The capital stock of the firm might change over time. The asset depreciates both economically and for accounting purposes at a constant rate $\delta > 0$. After observing the realization of the shocks at time $t$, the firm chooses the new capital stock $k'$, which will be in operation during the period $]t, t+1]$. The firm can either increase or decrease the capital stock, and the net investment equals to $I = k' - k(1-\delta)$. Similar to Abel and Eberly (1994) and many others after them, we assume that the change in capital entails an asymmetric and quadratic adjustment cost $h(I, k) = \left(\lambda_1 \mathbf{1}_{\{I>0\}} + \lambda_2 \mathbf{1}_{\{I<0\}}\right) I^2/\delta k$, where $0 < \lambda_1 < \lambda_2$ model costly reversibility, and $\mathbf{1}_{\{.\}}$ is the indicator function. The economic interpretation of $\lambda_i$, $i = 1, 2$, is straightforward: it is the per cent cost of a (dis)investment $I = \delta k$.

The debt level might also change over time. At any date, the firm can issue a one–period zero–coupon unsecured debt. As is shown in Figure 8, at time $t$ the firm chooses the face value of the debt, $b'$, that will be repaid at $t+1$. If the firm is solvent, the market value of the debt, $B(x, a')$, depends on the current state $x$ and on the choices of the face value and the capital stock, $a' = (k', b')$, that are made after observing the shocks.

Changing the debt level entails a proportional adjustment cost, $\theta|b' - b|$, with $\theta \geq 0$. Since the issuance decision is contemporaneous to repayment of the nominal value of old debt $b$, the debt decision generates a net cash flow equal to $B(x, a') - b - \theta|b' - b|$.

We assume a linear corporate tax function with rate $\tau$. The tax code allows deduction from the taxable income of the depreciation of assets in place, $\delta k$, and of interest expenses. Modeling deduction of the interest at maturity of the bond would entail keeping track of the value of the debt at issuance, therefore increasing the number of state variables. For the sake of numerical tractability, we assume that the expected present value of the end-of-period interest payment $b' - B(x, a')$, which we denote $H(x, a')$, can be expensed when the new debt is issued at time $t$. In case of linear corporate tax, and assuming knowledge of the equilibrium conditional default probability, this is equivalent to the standard case of deduction at $t + 1$. The after–tax cash flow from operations plus the net proceeds from the debt decision is

$$v = v(x, a, a') = (1 - \tau)\pi + \tau\delta k + \tau H(x, a') + B(x, a') - b - \theta|b' - b|. \tag{6}$$

The cash flow to equity is therefore equal to $w = w(x, a, a') = v - I - h(I, k)$ where, on the right-hand side, the first term is the after–tax cash flow from operations and the other terms are the net proceeds from (dis)investment. If the cash flow to equity is positive, the firm pays dividends and/or repurchases shares from the current shareholders; if the cash flow to equity is negative the firm issues new shares. In the latter case, the company incurs a proportional issuance cost $\zeta \geq 0$, as only $w$ is the actual inflow to the corporation

$$P = P(x, a, a') = w \cdot (1 + \zeta \mathbf{1}_{\{w<0\}}). \tag{7}$$

*6.3. The value of corporate securities*

Following Berk, Green, and Naik (1999), Zhang (2005), and Gomes and Schmid (2010), we exogenously define a pricing kernel that depends on the aggregate source of risk, $y$. The associated one–period stochastic discount factor $M(x, x')$ defines the risk-adjustment corresponding to a transition from the current state $y$ to state $y'$. We assume that $M$ is a continuous function of both arguments.

The firm can issue two types of securities, debt and equity, whose equilibrium prices are determined under rational expectations in a competitive market. The cum–dividend price of equity, $S(x,a)$, is the sum of current payout, $P$, and the present value of the expected future optimal distributions, which is equal to the next period price $S(x',a')$. Since this sum can be negative, a limited liability provision is also included (i.e., default on a value basis), in which case the firm's equity is worthless:

$$S(x,a) = \max\left\{0, \max_{a'}\left\{P(x,a,a') + \mathbb{E}_x\left[M(x,x')S(x',a')\right]\right\}\right\}. \tag{8}$$

The value function, $S$, is the solution of functional equation (8). We define $\omega = \omega(x,a)$ as in indicator function that captures the event of default. Note that, if $\omega = 0$, the optimal investment and financing decision is $\varphi(x,a) = a^*$, where $a^* = (k^*, b^*)$ is the optimal choice of the second argument in the max in (8). The optimal policy is therefore summarized by $(\omega, \varphi)$.

As for the debt contract, the end-of-period payoff to debt holders, $u(x',a')$, depends on the current policy, $a' = (k', b')$, the new realization of the shocks $x'$, and on whether the firm is in default:

$$u(x',a') = b'(1-\omega(x',a')) + \left[\pi' + \tau\delta k' + k'(1-\delta)\right](1-\eta)\omega(x',a'). \tag{9}$$

In case of default, similarly to Hennessy and Whited (2007), the bondholders receive the sum of the cash flow from operations, the depreciated book value of the asset, and the tax shield from depreciation, all net of a proportional bankruptcy cost, $\eta$. Hence, at issuance the debt value is

$$B(x,a') = \mathbb{E}_x\left[M(x,x')u(x',a')\right]. \tag{10}$$

One final item that needs to be evaluated is the expected present value of the interest payment, $H(x,a')$, which enters the determination of the after tax cash flow in (6):

$$H(x,a') = \left[b' - B(x,a')\right]\mathbb{E}_x\left[M(x,x')(1-\omega(x',a'))\right]. \tag{11}$$

Because the interest is deductible only if the firm is not in default, the expectation term is the conditional price of a default contingent claim.

*6.4. Option prices*

We derive the option prices from the stock price, under the assumption that distribution to equity holders do not happen in the form of a cash dividend but are either a share repurchase or an equity issuance (when negative).[2]

Denote with $n(x, a)$ the number of outstanding shares before the current payout decision is made. The stock price of one share is

$$s(x, a) = \frac{S(x, a)}{n(x, a)}.$$

Define $S'(x, a) = S(x, a) - P(x, a, a')$ the equity value after the payout, where $a' = \varphi(x, a)$ is the optimal policy from (8).

After a payout, the firm changes the number of shares for next period to $n'(x, a)$. In particular, if $P(x, a, a') > 0$, some shares are repurchased; if $P(x, a, a') < 0$ new share are issued. The new number of shares is

$$n'(x, a) = \frac{S'(x, a)}{s(x, a)} = \frac{S'(x, a)}{S(x, a)}\; n(x, a). \tag{12}$$

While $n$ and $n'$ are integer numbers in real life, we assume here that $n, n' \in \mathbb{R}$.

The evolution of the number of shares is given by the application of the current optimal policy, $a' = \varphi(x, a)$, and the state transition from $x$ to $x'$, so that at the new state $(x', a')$ following from $(x, a)$,

$$n(x', a') = n'(x, \varphi(x, a)), \tag{13}$$

with $n'(x, a)$ from (12).

We assume options are on a single share of equity. For definiteness, we consider a European call option with strike $k$, with payoff at maturity $\max\{s(x, a) - k, 0\}$, which is based on the convention that the dividend has been paid before the option expires, and therefore the payoff is based on the ex dividend price.

Because the shares number is endogenous (i.e., it depends on the payout policy), option pricing by straightforward backward induction is numerically intractable. The drawback

[2]It is possible to solve the model and compute prices even when the firm pays an exogenous dividend. In that case, we are also able to price an American option.

introduced by path dependency is due to the fact that the option price at the current state, and the stock price $s(x,a)$, is

$$c(x,a;k) = \beta \mathbb{E}_x \left[ \max\{s(x',a') - k, c(x',a';k)\} \right],$$

in which $a' = \varphi(x,a)$. To determine $s(x',a')$, the underlying asset of the option in state $(x',a')$, from $S(x',a')$ we need $n(x',a')$. However, as one can see from equation (13), $c(x',a';k)$ also depends on $n(x,a)$.

We avoid the issue of path dependency by observing that

$$\begin{aligned} c(x,a;k) &= \beta\ \mathbb{E}_x \left[ \max\left\{ \frac{S(x',a')}{n(x',a')} - k, c(x',a';k) \right\} \right] \\ &= \frac{1}{n(x',a')} \beta\ \mathbb{E}_x \left[ \max\left\{ S(x',a') - k\ n(x',a'), c(x',a';k) n(x',a') \right\} \right]. \end{aligned}$$

From the expression above, defining the sum of prices of all options with strike $k$ written on the firm's stock, $C(x,a;k\ n(x',a')) = c(x,a;k)n(x',a')$, we can write

$$C(x,a;k\ n(x',a')) = \beta\ \mathbb{E}_x \left[ \max\{S(x',a') - k\ n(x',a'), C(x',a';k\ n(x',a'))\} \right],$$

which shows that we use backward induction to price total equity options on a predetermined set of strike prices $\mathcal{K} = \{K_1, \ldots, K_N\}$, such that for each $K \in \mathcal{K}$ we solve

$$C(x,a;K) = \beta\ \mathbb{E}_x \left[ \max\{S(x',a') - K, C(x',a';K)\} \right],$$

working backward from the option maturity to the current period. Given these prices, we can determine the current price of a European call option with strike price $k$, by interpolating $\widehat{C}(x,a;k\ n'(x,a))$ on the grid $\mathcal{K}$, and then

$$\widehat{c}(x,a;k) = \frac{1}{n'(x,a)}\ \widehat{C}(x,a;k\ n'(x,a)).$$

Using (12), the previous equation becomes

$$\widehat{c}(x,a;k) = \frac{1}{n(x,a)} \frac{S^{ex}(x,a)}{S'(x,a)}\ \widehat{C}\left( x,a;k\ n(x,a) \frac{S'(x,a)}{S^{ex}(x,a)} \right). \tag{14}$$

Given the current equity value, $S(x,a)$, our goal is to calculate the price of options on equity value at $t=0$ with maturity $T$ and moneyness $m \in \{m_1, m_2, \ldots, m_N\}$. Where the strikes are $\mathcal{K} = \{S(x,a)m_i, i = 1,\ldots,N\}$. Because the current number of shares is arbitrary, we choose $n(x,a) = S(x,a)$, which is equivalent to assuming that the current (ex dividend) stock price is \$1. Then our goal is met by solving the pricing problem

$$\widehat{c}(x,a;m) = \frac{1}{S'(x,a)}\ \widehat{C}\left(x,a;m\ S'(x,a)\right),$$

where $\widehat{c}(x,a;m)$ is the price of an European call option on a stock with current price \$1 and strike $m$.

*6.5. Stochastic discount factor*

We assume that the idiosyncratic shock $z$ and the aggregate shock, $y$, follow autoregressive processes of first order, $z' = (1-\rho_z)\overline{z} + \rho_z z + \sigma_z \varepsilon'_z$ and $y' = (1-\rho_y)\overline{y} + \rho_y y + \sigma_y \varepsilon'_y$, respectively. In the above equations, for $i = y, z$, $|\rho_i| < 1$ and $\varepsilon_i$ are i.i.d. and obtained from a truncated standard normal distribution, so that the actual support is compact around the unconditional average. We assume that $\varepsilon_z$ are uncorrelated across firms and time and are also uncorrelated with the aggregate shock, $\varepsilon_y$. The parameters $\rho_z$, $\sigma_z$, and $\overline{z}$ are the same for all the firms in the economy, $\overline{z}$ and $\overline{y}$ denote the long term mean of idiosyncratic risk and of macroeconomic risk, respectively, $(1-\rho_i)$ is the speed of mean reversion, and $\sigma_i$ is the conditional standard deviation. With this specification, the transition function $Q$ satisfies all the assumptions required for the existence of the value function.

Finally, we adopt the stochastic discount factor proposed by Jones and Tuzel (2013):

$$M(y,y') = \beta e^{-g(y)\varepsilon'_y - \frac{1}{2}g(y)^2\sigma_y^2},$$

with $\beta \in (0,1)$, and where the state-dependent coefficient of risk-aversion is $g(y) = \exp(\gamma_1 + \gamma_2 y)$, with $\gamma_1 > 0$ and $\gamma_2 < 1$. With this choice, the coupon is equal to the state-independent real risk-free rate, $r = 1/\beta - 1$.

Following the literature, the aggregate risk parameters are taken from Cooley and Prescott (1995) and converted to quarterly frequency. We obtain a value for the persistence of the systematic risk ($\rho_x$) and the aggregate volatility ($\sigma_x$) of 0.979 and 0.0072, respectively. The personal discount factor ($\beta$) is set to 0.9851, and the SDF parameters ($\gamma_1$ and $\gamma_2$) to 3.22 and -15.3, respectively. These parameters produce an annualized average real interest rate of 6.1%.

## 6.6. Calibration

We fix the five parameters that that describe the aggregate source of risk and the SDF, equity and debt floatation costs, and the depreciation rate (as for example, Warusawitharana and Whited, 2016). We calibrate the remaining parameters by minimizing the sum of square deviations of a set of quantities that are observable in the data and in the simulated economy.

Important objectives of the calibration exercise are that the model captures the outcomes of the decisions that firms make and that affect the relationship between the asset and the equity volatility. The model should therefore match the average (book and market) leverage ratio and the average investment as the real economy. As the relevant sources of total risk match up with the economy, firms should exhibit similar market to book ratios, and similar equity distributions in the physical measure (i.e., average, standard deviation, skewness and kurtosis of equity returns). We also calibrate the model to fit the average ATM 90 days implied volatility, as well the frequency of each implied volatility surface (i.e., left smirk, smile, right smirk, frown).

We report parameter values and quantities used for calibration in Panel A of Table 1. The estimated marginal corporate tax rate, $\tau$, is 0.129, close to the estimates produced by Graham (1996a) and Graham (1996b) (i.e., approximately 14% for our sample). The estimate for the production function parameter $\alpha$ is 0.216. There are large bounds around figures reported in the literature, which are largely affected by the frequency at which models are calibrated and what type of fixed costs (proportional or not) are considered. Our value is close to the 0.3 figure used in Zhang (2005) and Gomes (2001). We estimate the operating cost to 1.175 (unit of capital), which translates to approximately 40% per year of the average capital. The calibrated value of the bankruptcy cost parameter, $\eta$, is 0.432, which is almost exactly equal to the firm's average default cost parameter estimated by Glover (2016) (i.e., 0.432). There is not a direct benchmark for the two capital adjustment costs. Our estimate imply that the cost of divesting is almost twice as large as the cost of increasing the size of the productive capital.

In Panel B of Table 1, we compare the simulated economy to the real data along the dimensions used to calibrate the mode. The investment and financing choices of the average simulated firm reflects well those of real firms (investment and leverage are really close). Valuations are also appropriately close, as well the physical distribution of equity returns. Average option prices are also relatively well matched as is the frequency of implied volatility shapes. We discuss in greater length the property of option prices in the model in the next sections.

**Table 1: Model calibration**

This table presents the calibration results of the firm model. In Panel A, we report the list of model parameters. In Panel B, we compare the quantities that are weighted to calibrate the model. In the left column (*Data*) we report the value of the moment conditions computed from the observed empirical sample, while in the right column (*Model*) we report the moment conditions computed from the simulated sample. Data is from various sources and spans the period between January 1996 throughout December 2019.

| Panel A: Parameters | | |
|---|---|---|
| *Aggregate* | | |
| Systematic Productivity Autocorrelation | $\rho_x$ | 0.979 |
| Systematic Productivity Volatility | $\sigma_x$ | 0.007 |
| Discount Factor | $\beta$ | 0.985 |
| Constant Price of Risk Parameter | $g_0$ | 3.220 |
| Time-varying Price of Risk Parameter | $g_1$ | -15.300 |
| *Firm Specific* | | |
| Depreciation | $\delta$ | 0.040 |
| Equity Issuance Cost | $\zeta$ | 0.060 |
| Debt Adjustment Cost | $\theta$ | 0.020 |
| *Calibrated* | | |
| Idiosyncratic Productivity Autocorrelation | $\rho_y$ | 0.737 |
| Idiosyncratic Productivity Volatility | $\sigma_y$ | 0.432 |
| Production Function | $\alpha$ | 0.216 |
| Fix Cost | $f$ | 1.175 |
| Cost of Expansion | $\lambda_1$ | 0.255 |
| Cost of Contraction | $\lambda_2$ | 0.459 |
| Corporate Taxes | $\tau$ | 0.129 |
| Bankruptcy Cost | $\eta$ | 0.432 |

| Panel B: Calibrated quantities | | |
|---|---|---|
| | Data | Model |
| *Option Prices (90 days to maturity)* | | |
| IV OTM | 0.647 | 0.639 |
| IV ATM | 0.609 | 0.611 |
| IV ITM | 0.585 | 0.598 |
| Percentage Left Smirk | 0.731 | 0.766 |
| Percentage Smile | 0.177 | 0.172 |
| Percentage Right Smirk | 0.038 | 0.038 |
| Percentage Frown | 0.056 | 0.052 |
| *Stock Return* | | |
| Average | 0.029 | 0.026 |
| Standard Deviation | 0.344 | 0.327 |
| Skewness | 0.649 | 0.739 |
| Kurtosis | 4.187 | 4.268 |
| *Firm characteristics* | | |
| Market-to-Book | 2.580 | 2.446 |
| Leverage | 0.495 | 0.493 |
| Investments | 0.044 | 0.045 |

# 7. Comparison of simulated and observed option prices

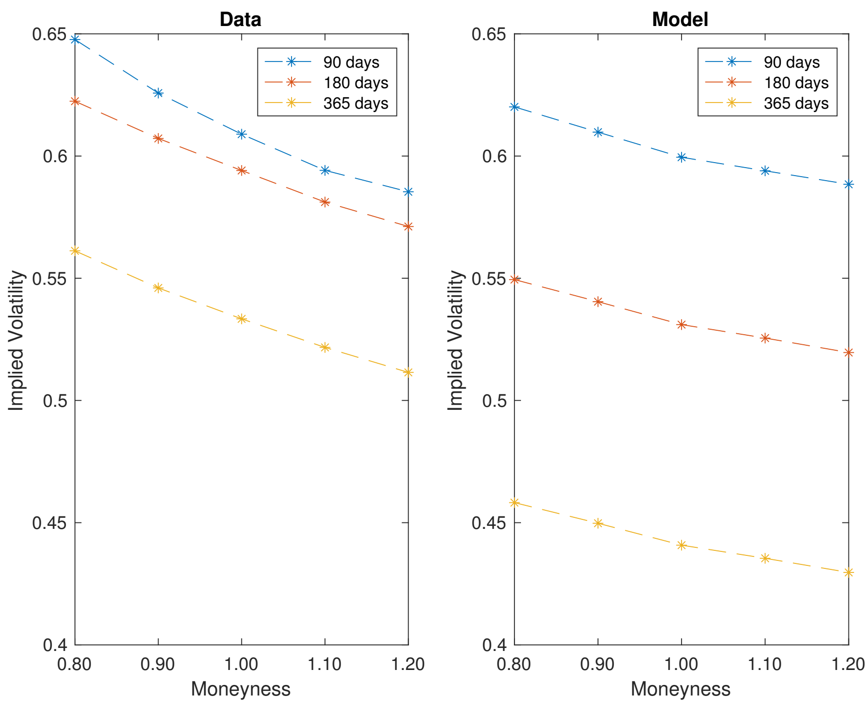
The dynamic model presented in the previous section is able to reproduce many features in the data. Notably as discussed in the previous section, the average 90 days ATM implied volatility is close the the corresponding number in the data. The model can create enough heterogeneity in the shapes on the implied distribution that it matches very closed what observed in the data at 90 days frequency. While it is remarkable that the model can do that, it is also true that we used those quantities as part of the calibration exercise. In this section, we present comparisons of the simulated economy with the real one along many other dimensions.

*7.1. Implied volatilities*

We start by comparing the average IV surfaces across all maturities considered (90, 180, and 360 days). Please remember that the model is only calibrated to fit the 90 days curve. Figure 9 juxtaposes the curves extracted from the data (left panel) to those extracted from the simulation. To obtain each curve, we first average across time, then across firms, and eventually across simulations.

**Figure 9: Implied volatility surface comparison**

The figure plots the average implied volatility surface extracted from the data (left panel) and from the simulation (right panel). The sample contains all industrial firms with options trading on their equity between 1996 and 2019. Data is sampled at quarterly frequency, and observations are removed if companies pay dividends in the next 12 months. A total of 3,536 firms are included.



The model is able to replicate the slope through moneyness and through maturities for all IV curves. Similarly to Geske (1979) and Toft and Prucyk (1997) who both incorporate leverage, the model can generate average downward sloping curves across moneyness levels. We show below that, differently from those other model, our set up can also create other IV surfaces. Interestingly, the model however can also generate a downward sloping surface across maturities without exogenously imposing a term-structure of volatility. Productivity shocks that affect the firm's value at short horizon tend to revert towards long run values, and as that the relationship between asset and equity volatility flattens. The total effect is to decrease prices for options at longer maturities, and henceforth producing a decreasing volatility surface.

**Table 2: Frequency of volatility surface types**

The table shows frequencies of implied volatility surface types extracted from the data (left side) and from the simulated economy (right side). Left smirk refers to an implied volatility curve that is downward sloping with moneyness, smile refers to a u-shape curve, right smirk to an upward sloping curve, and frown to an inverted u-shape. The sample contains all industrial firms with options trading on their equity between 1996 and 2019. Data is sampled at quarterly frequency, and observations are removed if companies pay dividends in the next 12 months. A total of 3,536 firms are included.

| | Data | | | Model | | |
|---|---|---|---|---|---|---|
| | 90 days | 180 days | 360 days | 90 days | 180 days | 360 days |
| Right smirk | 0.73 | 0.83 | 0.85 | 0.76 | 0.87 | 0.98 |
| Smile | 0.18 | 0.08 | 0.08 | 0.17 | 0.10 | 0.02 |
| Left smirk | 0.04 | 0.03 | 0.02 | 0.03 | 0.02 | 0.00 |
| Frown | 0.05 | 0.06 | 0.04 | 0.05 | 0.00 | 0.00 |

The model can also replicate much of the cross-sectional dispersion in IV surface types. In Table 2, we show average frequencies of IV surface types across maturities. Remember that the parameters are only calibrated to match the 90 days frequencies. Similarly to the data, in the simulated economy the most frequent IV curve type is downward sloping with maturity (i.e., Left smirk), with the frequency increasing with contract maturity.

*7.2. Risk-neutral moments*

As Panel B of Table 1 shows, the moments of the physical distribution of stock returns match quite well with the corresponding quantities in the data. Table 3 confirms that the higher moments of the risk-neutral distribution match as well. Risk-neutral skewness is negative at all horizons, and generate a substantial risk-premium as in Pederzoli (2020).

**Table 3: Model free risk-neutral moments**

The table compares summary statistics for model free risk-neutral skewness and kurtosis extracted from the data (left side) and from the simulated economy (right side). The sample contains all industrial firms with options trading on their equity between 1996 and 2019. Data is sampled at quarterly frequency, and observations are removed if companies pay dividends in the next 12 months. A total of 3,536 firms are included.

| | Data | | | | Model | | | |
|---|---|---|---|---|---|---|---|---|
| | 90 days maturity | | | | | | | |
| | Average | S.Dev | $5^{th}$ perc | $95^{th}$ perc | Average | St.Dev | $5^{th}$ perc | $95^{th}$ perc |
| Skewness | -0.51 | 0.47 | -1.22 | 0.10 | -0.27 | 0.10 | -0.45 | -0.10 |
| Kurtosis | 4.52 | 2.70 | 2.59 | 7.89 | 3.50 | 0.09 | 3.35 | 3.64 |
| | 180 days maturity | | | | | | | |
| | Average | St.Dev | $5^{th}$ perc | $95^{th}$ perc | Average | St.Dev | $5^{th}$ perc | $95^{th}$ perc |
| Skewness | -0.30 | 0.36 | -0.84 | 0.25 | -0.28 | 0.08 | -0.41 | -0.14 |
| Kurtosis | 3.35 | 1.58 | 2.28 | 4.84 | 3.22 | 0.10 | 3.03 | 3.33 |
| | 360 days maturity | | | | | | | |
| | Average | St.Dev | $5^{th}$ perc | $95^{th}$ perc | Average | St.Dev | $5^{th}$ perc | $95^{th}$ perc |
| Skewness | -0.22 | 0.36 | -0.72 | 0.40 | -0.30 | 0.04 | -0.36 | -0.21 |
| Kurtosis | 2.96 | 0.74 | 2.09 | 4.09 | 2.93 | 0.17 | 2.62 | 3.14 |

# 8. Model mechanisms

As we show in Section 5, in a simplified version of the main model there exists a relationship that links the equity volatility to the asset volatility through the equity elasticity. This relationship defines how the equity volatility changes with the firm's financial leverage and the moneyness of one growth options.

In our full model described in the previous section, the equity of the firm can be seen as an option on a bundle of assets: the production capacity that is already in place, the growth options related to possible future investments, and the contraction options related to possible future disinvestments. A financial option is priced on the firm's equity. The main model weighs all the effects of all those assets on the equity to produce a complex "elasticity" function. In states of the world where both investment and disinvestment options are worthless, the equity elasticity is driven entirely by the leverage effect. In that situation, Rathgeber, Stadler, and Stockl (2020) prove analytically that the resulting implied volatility curve of the financial option will be downward sloping relative to moneyness.

The existence of investment and disinvestment options, which value is state-dependent in our model, affect the equity elasticity and therefore also the implied volatility curve. OTM investment options (low moneynes) will tend to lower the implied volatility of financial options with low moneyness, while ITM investment options (high moneyness) tend to increase the implied volatility of ITM financial options. Conversely, ITM disinvestment options (low moneyness) tend to increase the implied volatility of financial options with low moneyness. What of the three effect (asset in place, option to grow, or option to shrink) prevails is entirely a numerical question and can only be observed through the lens of the model.

The existence of capital adjustment costs and operating leverage exacerbates the relationships making it possible for the quantitative model to match quantities observed in the data. Small firms that carry high leverage are particularly risky because they are more likely to default. Even when hit by a sequence of positive productivity shocks, which create large opportunity to invest, these firms resist investing to minimize capital adjustment costs. Nonetheless the prospect of adding additional capital in the future improves ability to keep production above operating costs, and thus safe. On the other hand, large firms that carry low leverage are safer (they have low levels of volatility): they are far from default as they can better absorb negative shocks. While being hit by a sequence of positive productivity shocks, they accumulate growth opportunities. However, because they are already large and cannot grow more than certain amount (as the production function is convex), these firms are particularly sensitive to downsizing adjustment costs (as in Kogan, 2004; Zhang, 2005; Cooper, 2006). These firms will have IV curves that have a less negative, or even positive, right tails.

There are many mechanisms in the model that affect how risk varies across states of the world: realization of profitability shocks, parameter choices, and functional relationships in the model. We start by giving a description of the exogenous conditions, and the related optimal choices, that lead to different volatility surfaces in the context of the simulated economy described in the previous section. We then conduct a comparative static experiments where we vary the key parameters of the model, one at a time and keep the remaining fix.

*8.1. Benchmark simulated economy*

As the firm parameters are determined by the calibration exercised, variability in the simulated economy in terms of implied volatility shapes is dictated by the optimal choices made by the firm relative to the realizations of the exogenous variables and the current state of capital and debt. Ultimately those choices determine the equity value relative to the capital in place and optimal amount of leverage.

A shock impacts optimal decisions in two ways: first, it determines the current period cash flow, which ultimately determines the financing needs of the firm. Second, because of persistence, it also determines the future prospects of the firm, and thus affects investment and financing decisions. Negative shocks create pressure for the firm to raise external financing and/or disinvest some of the capital in place. Financial and capital adjustment costs make these states of the world even riskier for the firm, thus depressing equity values. On the other hand, positive shocks free the firm for immediate cash flow needs and create growth opportunities that might be realized immediately or, because of financial and capital adjustment costs, in the near future.

When it comes to option pricing, the leverage effect is thus contrasted by the future prospects of the firm. The impact of states of the world where leverage is high because negative current (and future) productivity shocks are more likely push the implied volatility surface upward (high ATM IVs), to be downward sloping across maturities, and exhibit a mild term slope. In states of the world where shocks are positive, the firm is less risky and has large equity values because of good future opportunities. Those states of the world push the implied volatility downward (low ATM IVs), flatten the curve across moneyness, and create a large term spread.

Thus the model predicts that overall, the left slope of the IV surface (OTM minus ATM strikes) will be positively related to the level of the ATM IV and to book leverage and will be negatively related to growth opportunities (which we proxy by the market-to-book ratio). The implications for the right slope of the IV surface are diametrically opposite. The term spread is mostly affected by the temporal evolution of the productivity shocks, and less affected by leverage: thus it should be larger when firms have large equity values or when they experience unexpected negative shocks.

**Table 4: IV surface and firm decisions**

The table presents regression results in the actual (Data) and in the simulated economy (Model) of the slopes of the implied volatility surface against variables that summarize the state of the firm: size, book leverage, market to book ratio, and ATM IV. We consider the left slope of the 90 days IV curve (i.e., the difference between the IV measured at 0.8 moneyness level and the IV measured ATM), the right slope (i.e., the difference between the IV measured at 1.2 moneyness and the IV measured ATM), and the term slope (i.e., the difference between the 90 days ATM IV and the 360 days ATM IV. We report parameter estimates and standard errors. Regressions in the data include industry fixed effects, and rely on standard errors clustered at firm level. The sample contains all industrial firms with options trading on their equity between 1996 and 2019. Data is sampled at quarterly frequency, and observations are removed if companies pay dividends in the next 12 months. A total of 3,536 firms are included.

| | Left slope | | Right slope | | Term slope | |
|---|---|---|---|---|---|---|
| | Data | Model | Data | Model | Data | Model |
| Size | 0.25 | 0.16 | 0.03 | 0.24 | 0.90 | 5.76 |
| | (0.04) | (0.05) | (0.05) | (0.02) | (0.10) | (0.84) |
| Book Leverage | 1.12 | 6.93 | -0.49 | -0.99 | -1.16 | -13.71 |
| | (0.12) | (0.63) | (0.15) | (0.33) | (0.30) | (5.93) |
| Market-to-Book Ratio | -0.15 | -0.46 | 0.13 | 0.12 | 0.20 | 2.11 |
| | (0.02) | (0.02) | (0.03) | (0.01) | (0.04) | (0.13) |
| ATM IV | -2.51 | 8.61 | -5.84 | -13.44 | 14.35 | 99.62 |
| | (0.24) | (0.12) | (0.34) | (0.33) | (0.55) | (0.58) |

The model predictions are largely confirmed by the results reported in Table 4. In the simulated economy, we observe the correct signs for the variables explaining the three surface variables. With one notable exception, the same signs can be found also in the real data, thus confirming that the model is able to pick up fundamental relationships even if those were not imposed in the calibration process. The notable exception is represented by the fact that in the real data, the size of the left smile is negatively related to the level of ATM IV, which is in complete contrast with what the model predicts.

*8.2. Comparative static analysis*

Next we construct comparative statics relative to the baseline simulated economy by varying some of the model parameters that affect the investment and financing decisions of the firms. As our set up differs from other models that incorporate the leverage effect into option pricing because of the firm's ability to grow, we focus on the investment decisions. In this model growth options arise because of the persistency of the productivity shocks. When a firm is hit by a positive productivity shock, because of persistence, it is facing a very high probability of seeing another positive shock in the near future. Options to grow are therefore directly related to the degree of persistence of the profitability shocks. The first parameter that we change is therefore the auto-correlation of the idiosyncratic shock.[3]

Whether growth options are converted into productive capital depends on the marginal value (i.e., the slope of the production function) and the marginal cost (i.e., the model does have asymmetric capital adjustment costs) of the additional units of capital. Hence there are potentially three parameters that could aid a comparative static analysis: $\alpha$, $\lambda_1$, and $\lambda_2$. We focus on $\alpha$, because it has the easier interpretation. As the curvature of the production function increases, the firm average per-period cash flow increases as well as the steady state size of the firm and its debt capacity (hence leverage). Because the production function is convex, however, additional units of invested capital have decreasing marginal value. Thus, an increase in $\alpha$ should produce a decrease in investment, and in increase in growth options (market-to-book values) and leverage.

In Figure 10 we plot the frequency that the right tail of the implied volatility surface (i.e., IV at moneyness of 1.2 minus IV at monenyess of 1) is positive relative different values of $\rho_y$, $\alpha$, and $\theta$ centered around the calibrated coefficients reported in Table 1.

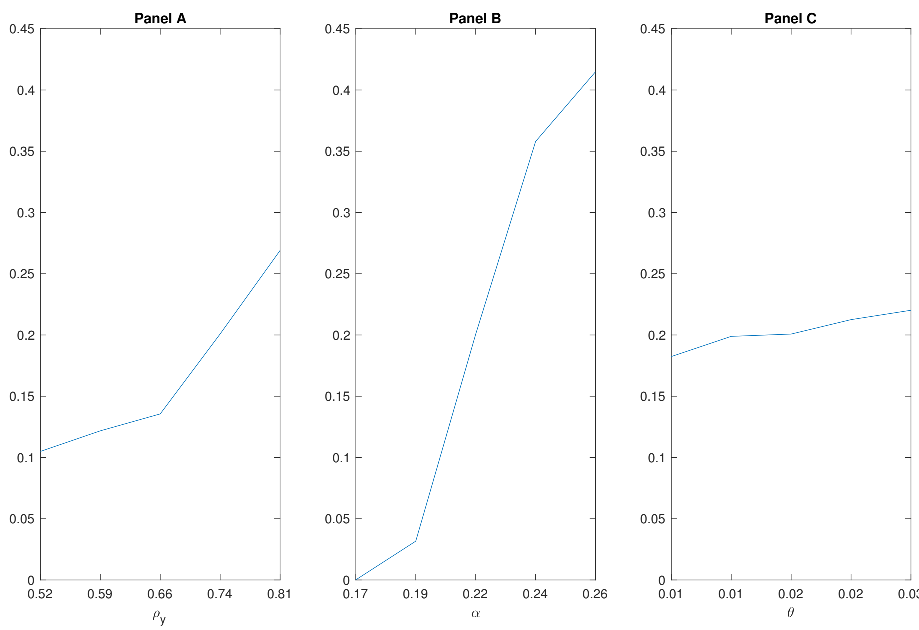
In Panel A and B, we consider variation in the autocorrelation of the firm-specific productivity shock and in the curvature of the production function. In both cases, an increase in the parameter leads to an increase in the growth opportunities available to firms. As the firm value increase so does the debt capacity, and hence the leverage ratio. The impact on the IV surface is nonetheless to decrease the left slope and increase the right one, thus leading to a less skewed risk-neutral distribution.

An even cleaner casual relationship can be obtained by varying the debt adjustment cost, $\theta$ (Panel C). In fact variation in $\theta$ modify the incentives to issue debt, but do not affect the firm's ability to accumulate growth options. Because the impact of leverage on the IV surface

[3]We could also vary the autocorrelation of the systematic productivity shock, but that also affect the pricing of securities, inducing a discount factor effect which is more complicated to interpret.

**Figure 10: Comparative Static**

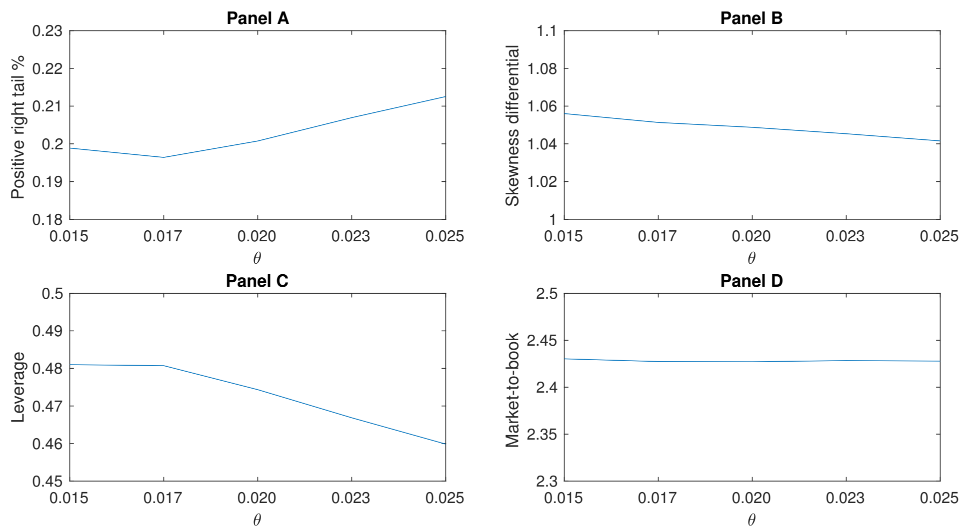
The figure plots the average frequency that the right tail of the IV surface (i.e., IV at moneyness of 1.2 minus IV at monenyess of 1) is positive for different values of the autocorrelation coefficient of the idiosyncratic productivity shock $\rho_y$ (Panel A), the curvature of the production function $\alpha$ (Panel B), and debt adjustment cost $\theta$ (Panel C) centered around the calibrated coefficients reported in Table 1.



diminishes as the debt adjustment cost increases, but leaves unaffected the impact of growth option, the net effect is to push the right side of the IV surface. Figure 11 gives a detailed account of the impact of variation of the debt adjustment costs on the IV surface, as well as leverage a market-to-book ratios.

**Figure 11: Comparative Static: Debt Adjustment Cost**

The figure plots the average frequency that the right tail of the IV surface (i.e., IV at moneyness of 1.2 minus IV at monenyess of 1) is positive (Panel A), risk-neutral skewness (Panel B), leverage (Panel C), and market-to-book ratio (Panel D) for different values of the autocorrelation coefficient of the debt adjustment cost $\theta$ centered around the calibrated coefficients reported in Table 1.




# 9. Conclusions

Traditional option pricing models often requires very strong assumptions about investor preferences and the dynamic of equity prices. We show that equity options can be priced in a production economy where we do not make strong exogenous assumptions about equity and volatility. In our set up the relation between risk and value arises endogenously through a dynamic sequence of optimal decisions that maximize the value of the firm. Thus we derive option prices that match many properties of those observed in the cross-section of US equities starting from a different set of assumptions that specify the functional forms of corporate trade-offs following the now long and established literature on dynamic corporate finance.

Our approach is not a better option pricing model, but rather an attempt to provide a link between fundamentals and derivative prices. We think that such link is important as it relates the primitives of the most successful finance models (i.e., those that price financial derivatives) to a large body of well understood economic mechanisms that describe the decision-making process within a typical firm.

Ultimately, we hope to provide an explanation for why option prices contain forward looking information about stock prices and corporate policies, despite being classically derived in models where such links should be uninformative unless one assumes some form of market segmentation.

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