Introduction
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• Motivation: hydrology generally driven by the need for accurate predictions or answers, i.e. quantitative analysis

• this requires mathematical description of fluid and contaminant movement in the subsurface

• for now we’ll examine water movement, beginning with an empirical (qualitative) relation: Darcy’s Law
Darcy’s Experiment
Darcy’s Observations

• Henry Darcy, studying public water supply development in Dijon, France, 1856. Trying to improve sand filters for water purification

• apparatus measured water level at both ends and discharge (rate of flow $\frac{L^3}{T}$) through a vertical column filled with sand (Fig. 1)
Figure 1: Simplified view of Darcy’s experimental apparatus. The central shaded area is a square sand-filled tube, in contact with water reservoirs on either side with water levels $h_1$ and $h_2$. 
Empirical Darcy Law

- Darcy found the following relationship

\[ Q \propto \frac{A(h_1 - h_2)}{L} \]

or

\[ Q = KA\frac{h_1 - h_2}{L} \]  \hspace{1cm} (1)

where \( K \) is a proportionality constant termed hydraulic conductivity \( \frac{L}{T} \)
Head
Head as Energy

- Head as a measure of energy \((1)[1]\) was used without much further consideration until the 1940’s when M. King Hubbert explored the quantitative meaning of head \(h\), and ultimately derived what he termed \textit{force potential} \(\Phi = g \cdot h\). He demonstrated that head is a measure of the \textit{mechanical energy} of a packet of fluid

- mechanical energy of a unit mass of fluid taken to be the sum of kinetic energy, gravitational potential energy and fluid pressure energy (work).

- Energy content found by computing work to get to current state from a standard state (e.g. \(z = v = p = 0\))

- energy units are \(\frac{\text{kg} \text{ m}^2}{\text{sec}^2} = \text{Nt} \cdot \text{m} = \text{joule}\)
Head Energy Components

• Kinetic Energy or “velocity work”. Energy required to accelerate fluid packet from velocity \( v_1 \) from velocity \( v_2 \).

\[
E_k = m \int_{v_1}^{v_2} v \, dv = \frac{1}{2}mv^2 \quad (2a)
\]

Remember that the integral sign is really just a fancy “sum”, saying “add up all the tiny increments of change between points 1 and 2”

• Gravitational work. Energy required to raise fluid packet
from elevation $z_1$ to elevation $z_2$.

$$ W = \int_{z_1}^{z_2} dz = mgz \quad (2b) $$

- Pressure work. Energy required to raise fluid packet pressure (i.e. squeeze fluid) from $P_1$ to $P_2$.

$$ P = \int_{P_1}^{P_2} V dP = m \int_{P_1}^{P_2} \frac{V}{m} = m \int_{P_1}^{P_2} \frac{1}{\rho} $$

$$ = \frac{1}{\rho} \int_{P_1}^{P_2} dP = \frac{P}{\rho} \quad (2c) $$

where (2c) assumes a unit mass of incompressible fluid.
Final Expression for Head

- the sum of (2a)–(2c) is the total mechanical energy for the unit mass (i.e. \( m = 1 \))

\[
E_{tm} = \frac{v^2}{2} + gz + \frac{P}{\rho} \tag{3}
\]

- in a real setting, energy is lost when flow occurs [Fetter [Sec. 4.2, 2001]], so \( E_{tm} = \text{constant} \) implies no flow. No flow (\( Q = 0 \)) in (1) implies constant head (\( h_1 = h_2 \)).

- Following the analysis of [Hubbert [1940]] define head \( h \) such that \( h \cdot g \) is the total energy of a fluid packet. Then

\[
hg = \frac{v^2}{2} + gz + \frac{P}{\rho} \tag{4a}
\]
\[ h = \frac{v^2}{2g} + z + \frac{P}{\rho g} \]  
\[ = z + \frac{P}{\rho g} \]  

(4b)  

(4c)

where (4c) assumes \( v \) is small (true for flow in porous media).

- head is a combination of \textit{elevation} (gravitational potential) head and \textit{pressure} head (where \( P = \rho g h_p \), and \( h_p \) is the pressure head, or pressure from the column of water above the point being considered)

- for a given head gradient (\( \frac{\Delta h}{\Delta x} \)) discharge flux \( Q \) is independent of path [Fig. 4.5, Fetter, 2001].
Head for Variable-Density Fluids

- (4c) indicates that head also depends on fluid density. This is only an issue for saline or hot fluids. For constant density (4c) can be written as

\[ h = z + h_p \]

where \( h_p \) is the height of water above the point of interest [eq. 4.11, Fetter, 2001].

- This implies that head is constant vs. \( z \) (for every meter change in \( z \) there is an equal but opposite change in \( h_p \)).

- When \( \rho \) isn’t constant, “fresh-water” head (the head for an equivalent-mass fresh water column, \( h_f = \frac{\rho_p}{\rho_f} h_p \)) should
be used for computing gradients, or in flow models [Pottorff et al., 1987]
Rock Properties
Applicability of Darcy’s Law

- Darcy’s Law makes some assumptions, which can limit its applicability
  - **Assumption:** kinetic energy can be ignored. **Limitation:** laminar flow is required (i.e. Reynold’s number low, \( R \leq 10 \), viscous forces dominate)
  - **Assumption:** average properties control discharge. **Limitation:** (1) applicable only on macroscopic scale (for areas \( \geq 5 \) to 10 times the average pore cross section, Fig. 2)
  - **Assumption:** fluid properties are constant. **Limitation:** (1) applicable only for constant temperature or salinity settings, or if head \( h \) is converted to fresh-water equivalent \( h_f \)
Figure 2: Variation of rock properties with scale. Note heterogeneous media have average properties that vary positively (shown) or negatively (not shown) with distance. See also Bear [Fig. 1.3.2, 1972].
True Fluid Velocity

- (1) gives total discharge through the cross-sectional area $A$

- specific discharge ($q$ traditionally, $v$ in the text) is the flux per unit area $q = \frac{Q}{A}$

- pore or seepage or average linear velocity ($v$ traditionally, $V_x$ in the text) is the velocity at which water actually moves within the pores $v = \frac{K \, dh}{\phi \, dl}$
Hydraulic conductivity ($K$)

- The proportionality constant in (1) depends on the properties of the fluid, a fact that wasn’t recognized until Hubbert [1956].

- Hubbert rederived Darcy’s Law from physical principles, obtaining a form similar to the Navier-Stokes equation. From this he found the following form for $K$:

\[
K = \frac{k \rho g}{\mu}
\]  

(5)

where $\rho$ is the density of the fluid and $\mu$ is its dynamic viscosity ($\frac{M}{LT}$). $k$ is the intrinsic permeability of the porous medium [Fetter [written as $K_i$, eqn. 3-19, 2001]].

\[\text{http://en.wikipedia.org/wiki/Navier-stokes}\]
• (5) demonstrates that hydraulic conductivity depends on the properties of both the fluid and the rock
Permeability \((k)\)

- \(k\) (units \(L^2\)) depends on the size and connectivity of the pore space, and can be expressed as \(k = Cd^2\), where \(C\) is the tortuosity of the medium (depends on grain size distribution, packing, etc.; “unmeasurable”), and \(d\) is the mean grain diameter (a proxy for the mean pore diameter). See also Fetter [sec. 3.4.3, 2001].

- Standard units are the darcy, \((1\ \text{darcy} = 9.87 \times 10^{-9}\text{cm}^2)\), which is defined as the permeability that produces unit flux given unit viscosity, head gradient, and cross-sectional area [Sec. 3.4.2, Fetter, 2001]
Measuring Rock Hydraulic Properties

- Porosity is measured in the lab with a *porosimeter*
  - sample is dried and weighed
  - then resaturate sample with water (or mercury), very difficult to do completely
  - measure volume or mass of liquid absorbed by sample
  - yields *effective porosity*

- Permeability/Hydraulic Conductivity are measured in the lab using *permeameters* (Fig. 3), which apply Darcy’s Law to determine $K$
  - *Constant Head* permeameter best used for high conductivity, indurated samples (rocks)
– *Falling Head* permeameter best used for low-permeability samples or soils
Permeameters

(a) Constant Head Permeameter: \[ K = \frac{V_L}{A t h} \]

(b) Falling Head Permeameter:
\[ K = \frac{d_i^2}{d_c^2} \frac{L}{t} \ln \left( \frac{h_o}{h} \right) \]

Figure 3: Constant and Falling-Head Permeameters. a) best for consolidated, high-permeability samples; b) best for unconsolidated and low-permeability samples. \( t \) is the duration of the experiment, \( V = Q t \) is the total volume discharged, and \( A \) is the cross-sectional area of the apparatus. After Fetter [Fig. 3.16-3.17, 2001], see also Freeze and Cherry [1979], Todd [1959].
Applications of Darcy’s Law

• find magnitude and direction of discharge through aquifer

• solve for head at heterogeneity boundary in composite (two-material) aquifer

• find total flux through aquitard in a multi-layer system
Derivation of the Flow Equation
The Flow Equation (Continuity Eqn.)

- **Approach:** use “control” volume, itemize flow in and out, and content of volume [sec. 4.7, Fetter, 2001], see Fig. 4

- **Simplifications:** assume steady-state, constant fluid properties

- **Mass flux**
  - must describe movement of mass across the sides of the control volume, i.e. determine \( \frac{\text{mass}}{\text{area} \cdot \text{time}} \) or \( \frac{m}{t \cdot l^3} \)
  - known fluid parameters: density \( \rho \ (m/l^3) \), velocity \( \vec{q} \ (l/t) \) (really specific discharge, or velocity averaged over a cross-sectional area; see references such as Bear [1972],
Schlichting [1979], Slattery [1972] for discussion of volume-averaging and REV’s)

- Then mass flux $= \rho \cdot \vec{q}$
Representative Control Volume

Figure 4: Mass fluxes for control volume in uniform flow field.
Control Volume Mass Balance

- Content of control volume (storage)
  - Steady-state ⇒ no change in content with time
  - then sum of in and outflows must be 0 (otherwise content would change)

- Sum of flows for control volume in uniform flow
  - from above know that form will be:

\[
\left\{ \begin{array}{c}
\text{rate of mass in} \\
\text{rate of mass out}
\end{array} \right\} - \frac{30}{30} = 0 \quad (6)
\]
– mass flux in

\[
\{ \text{rate of mass in} \} = \rho \vec{q}_x \cdot \Delta y \Delta z
\]

mass flux per unit area \cdot area of cube face

(7)

– mass flux out

* depends on \( \vec{q}_{out} \), let \( \vec{q}_{out} = \vec{q}_{in} + \text{change} \)
* express change as distance \cdot (\text{rate of change}) \text{ and rate of change as } \frac{\Delta q_x}{\Delta x}
* to be most accurate, determine rate of change over a very small distance, i.e. take limit as \( \Delta x \to 0 \)
* that’s a derivative

\[
\lim_{\Delta x \to 0} \frac{\Delta q_x}{\Delta x} = \frac{dq}{dx}
\]
* since $\vec{q}$ can vary as a function of $y$ and $z$ as well, we write it as a partial derivative, holding other variables constant

$$\frac{d\vec{q}}{dx}\bigg|_{y,z \text{ constant}} = \frac{\partial \vec{q}}{\partial x}$$

* then in the form required for (6)

$$\begin{cases}
\text{rate of mass out} \\
\text{mass out}
\end{cases} = \rho \left(q_x + \Delta x \frac{\partial q_x}{\partial x}\right) \Delta y \Delta z$$

– net flux in $x$-direction:

$$-\rho \frac{\partial q_x}{\partial x} \Delta x \Delta y \Delta z = -\rho \frac{\partial q_x}{\partial x} \Delta V$$
Continuity Equation

• sum of flows for general case \((y\text{ and } z\text{ sums are same form as for } x)\)

\[
-\rho \Delta V \left( \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} \right) = 0
\]

\[
\left( \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} \right) = 0
\]

\[
\nabla \cdot \vec{q} = 0
\]

(8)

• this is the divergence of the specific discharge, a measure of the fluids to diverge from or converge to the control volume

• assume this equation applies everywhere in problem domain,
i.e. that fluid is everywhere, and fluid/rock properties and variables are continuous

- then rock and fluid are overlapping continua
Flow Equation

• Approach: governing equations (8) and (1) contain similar variables, can we simplify?

• Combined equation
  – substituting Darcy’s Law (1) for $\vec{q}$ in (8):

$$\nabla \cdot \vec{q} = \nabla \cdot (-K \nabla h)$$

$$= \nabla^2 \cdot h$$

$$= \left( \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} \right)$$

$$= 0$$

  – This is our governing equation for groundwater flow
• Assumptions: many were made above
  – steady-state: no time variation
  – continuum: fluid and rock properties and variables continuous everywhere in problem domain
  – constant density: incompressible fluid, no compositional change, no temperature change
  – no viscous/inertial effects: low flow velocities
Bibliography


