SOLUTIONS of MATH 106 FINAL (FALL 2007)

1. a.) $f(x) = x\cdot e^{-x} \Rightarrow f'(x) = (1-x)e^{-x} \Rightarrow$ $\begin{array}{c|c} + & - \\ \hline & 1 \end{array}$ $\quad (-\infty,1)\ f(x)\nearrow,\ (1,\infty)\ f(x)\searrow \quad x=1$ max.

b.) $f''(x) = (x-2)\cdot e^{-x}$ $\begin{array}{c|c} - & + \\ \hline & 2 \end{array}$ $\quad (-\infty,2)$ concave down, $(2,\infty)$ concave up $\quad | \quad x=2$ inflection pt.

c.) $\lim\limits_{x\to\infty} x\cdot e^{-x} = \lim\limits_{x\to\infty} \frac{x}{e^x} \underset{\text{L'hopital}}{=} \lim\limits_{x\to\infty} \frac{1}{e^x} = 0 \quad | \quad \lim\limits_{x\to-\infty} x\cdot e^{-x} = \lim\limits_{x\to-\infty} \underset{-\infty}{x}\cdot \underset{\infty}{e^{-x}} = -\infty \quad | \quad x=0$ horizontal asymptote, no vertical asymptote

d.

2. $y = (\sin x)^{\tan x} \Rightarrow \ln y = \ln(\sin x)^{\tan x} = \tan x\cdot \ln(\sin x)$.

$\lim\limits_{x\to0^+} \ln y = \lim\limits_{x\to0^+} \tan x\cdot\ln(\sin x) = \lim\limits_{x\to0^+} \frac{\ln\sin x}{\cot x} \underset{\text{L'hopital}}{=} \lim\limits_{x\to0^+} \frac{\frac{1}{\sin x}\cos x}{\frac{-1}{\sin^2 x}} = \lim\limits_{x\to0^+} -\sin x\cdot\cos x = 0 \Rightarrow \ln y\to 0 \quad y\to e^0 = 1$ //

3. $\int (\ln x)^2 dx = x\cdot(\ln x)^2 - \int 2\ln x\, dx = x(\ln x)^2 - 2\left[x\cdot\ln x - \int dx\right] = x\cdot(\ln x)^2 - 2x\ln x + 2x$

$u=(\ln x)^2 \quad dv = dx$
$du = 2\ln x\cdot\frac{1}{x}dx \quad v = x$

$u=\ln x \quad dv = dx$
$du = \frac{1}{x}dx \quad v = x$

$\Rightarrow \int_1^e (\ln x)^2 dx = x\cdot(\ln x)^2 - 2x\cdot\ln x + 2x \Big|_1^e$

$= [(e\cdot1 - 2e + 2e) - (0 - 0 + 2)]$

$= e - 2$ //

4. a.) $\ln(1+x) = \int \frac{dx}{1+x} = \int \sum\limits_{n=0}^{\infty} (-x)^n dx = \int \sum\limits_{n=0}^{\infty} (-1)^n\cdot x^n dx = \sum\limits_{n=0}^{\infty} \frac{(-1)^n\cdot x^{n+1}}{n+1} + C \quad \left[\begin{array}{l} x=0 \Rightarrow \ln(1+x)=0 \\ C=0 \end{array}\right]$

$\ln(1+x) = \sum\limits_{n=0}^{\infty} (-1)^n\cdot\frac{x^{n+1}}{n+1} \underset{n-1\leftrightarrow n}{=} \sum\limits_{n=1}^{\infty} (-1)^{n-1}\cdot\frac{x^n}{n} \quad |x|<1$

b.) $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\cdot 2^n} = \sum\limits_{n=1}^{\infty} (-1)^{n-1}\cdot\frac{x^n}{n}$ at $x=\frac{1}{2} \Rightarrow \sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\cdot 2^n} = \ln\left(1+\frac{1}{2}\right) = \ln\frac{3}{2}$

5. $\int x^2\cdot e^{-x} dx = -x^2\cdot e^{-x} + \int 2x\cdot e^{-x} dx = -x^2\cdot e^{-x} + 2\left[-x\cdot e^{-x} + \int e^{-x} dx\right] = -x^2\cdot e^{-x} - 2xe^{-x} - 2e^{-x}$

$= -(x^2+2x+2)e^{-x}$

$u=x^2 \quad dv = e^{-x}dx$
$du = 2x dx \quad v = -e^{-x}$

$u=x \quad dv = e^{-x}$
$du = dx \quad v = -e^{-x}$

$\int_0^\infty x^2\cdot e^{-x} dx = \lim\limits_{t\to\infty} \int_0^t x^2\cdot e^{-x} dx = \lim\limits_{t\to\infty} -(x^2+2x+2)\cdot e^{-x}\Big|_0^t = \lim\limits_{t\to\infty} -(t^2+2t+2)\cdot e^{-t} - (-2) = 2$ //

$\to 0$ (L'hopital

6. a.)

$$V = \int_0^3 \pi \left(\frac{3}{\sqrt{x^2+9}}\right)^2 dx = \int_0^3 \frac{9\pi}{x^2+9}\,dx = \pi \int_0^3 \frac{dx}{1+\frac{x^2}{9}} = \pi \int_0^1 \frac{3\,du}{1+u^2}$$

$u = \frac{x}{3}$

$$= 3\pi \arctan u \Big|_0^1 = 3\pi(\arctan 1 - \arctan 0) = 3\pi\left(\frac{\pi}{4} - 0\right) = \frac{3\pi^2}{4} \quad //$$

b.)

$$V = \int_0^3 2\pi x \cdot \left(\frac{3}{\sqrt{x^2+9}}\right) dx = 6\pi \int_0^3 \frac{x}{\sqrt{x^2+9}}\,dx = 3\pi \int_9^{18} \frac{du}{\sqrt{u}} = 6\pi\sqrt{u}\Big|_9^{18} = 6\pi(3\sqrt{2}-3) \quad //$$

$u = x^2+9$

$du = 2x$

7. $V = \pi r^2 h = 8\pi \Rightarrow h = \frac{8}{r^2}$

$A = 2\pi r^2 + 2\pi r \cdot h$

$= 2\pi r(r+h)$

$A(r) = 2\pi r^2 + 2\pi r \cdot \frac{8}{r^2} = 2\pi r^2 + \frac{16\pi}{r}$

$A'(r) = 4\pi r - \frac{16\pi}{r^2} \Rightarrow A'(r) = 0 \Rightarrow r^3 = 4 \Rightarrow r = \sqrt[3]{4}$ (max)

$h = \frac{8}{\sqrt[3]{16}}$ //

8. a. $\sum_{n=3}^{\infty} \frac{1}{n \cdot (\ln n)^2} =$ INTEGRAL TEST $\sum_{n=3}^{\infty} \frac{1}{n \cdot (\ln n)^2} \sim \int_3^{\infty} \frac{1}{x \cdot (\ln x)^2}\,dx$ convergent!

$f(x) = \frac{1}{x \cdot (\ln x)^2}$ cts, +, ↘ ($f' < 0$) ✓

$\int_3^{\infty} \frac{dx}{x \cdot (\ln x)^2} = \int_{\ln 3}^{\infty} \frac{du}{u^2}$ → convergent by p-test

$u = \ln x$

b. $\sum_{n=1}^{\infty} \cos\frac{1}{n}$ $\lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = 1 \neq 0$ divergent by Divergence Test

9. a. $\sum_{n=1}^{\infty} \frac{n^2}{2^n} =$ RATIO TEST $\lim_{n\to\infty} \frac{\frac{(n+1)^2}{2^{n+1}}}{\frac{n^2}{2^n}} = \lim_{n\to\infty} \frac{(n+1)^2 \cdot 2^n}{n^2 \cdot 2^{n+1}} = \frac{1}{2} < 1$ convergent.

b. $\sum_{n=1}^{\infty} \frac{n+\sin n}{n^3+1} =$ LCT with $\sum \frac{1}{n^2}$ $\lim_{n\to\infty} \frac{\frac{n+\sin n}{n^3+1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^3+n^2\sin n}{n^3+1}$

$= \lim_{n\to\infty} \frac{n^3\left(1+\cancel{\frac{\sin n}{n}}^{\,0}\right)}{n^3\left(1+\cancel{\frac{1}{n^3}}\right)} = 1$ //

by LCT $\sum \frac{n+\sin n}{n^3+1} \sim \sum \frac{1}{n^2}$ convergent

10. a.) $\sum_{n=1}^{\infty} \frac{(x+3)^n}{4^n \cdot \sqrt{n}}$

RATIO TEST: $\lim_{n\to\infty} \left| \frac{\frac{(x+3)^{n+1}}{4^{n+1}\sqrt{n+1}}}{\frac{(x+3)^n}{4^n \cdot \sqrt{n}}} \right| = \lim_{n\to\infty} \left| \frac{\cancel{(x+3)^{n+1}} \cdot \cancel{4^n} \cdot \cancel{\sqrt{n}}}{\cancel{(x+3)^n} \cdot \cancel{4^{n+1}} \cdot \cancel{\sqrt{n+1}}} \right|$

$= \frac{|x+3|}{4} < 1 \Rightarrow |x+3| < 4$

$R = 4$ //

b.) $R = 4 \Rightarrow (-7, 1)$?

$x = -7 \Rightarrow \sum_{n=1}^{\infty} \frac{(-7+3)^n}{4^n \cdot \sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-4)^n}{4^n \cdot \sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} =$ $\frac{1}{\sqrt{n}} \to 0$, $\frac{1}{\sqrt{n}} \searrow$ by Alternating Series Test

$\sum \frac{(-1)^n}{\sqrt{n}}$ convergent

$x = 1 \Rightarrow \sum_{n=1}^{\infty} \frac{(1+3)^n}{4^n \cdot \sqrt{n}} = \sum_{n=1}^{\infty} \frac{4^n}{4^n \cdot \sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} =$ divergent by p-test

Interval of convergence: $[-7, 1)$ //