Problem 1) Consider the following lines:

$$l_1 : x = y = z = t \quad \text{and} \quad l_2 : x = t+1,\ y = 2t,\ z = 3t$$

a.) (8 pts) Find the distance between l_1 and l_2

$v_1 = \langle 1,1,1 \rangle \quad v_2 = \langle 1,2,3 \rangle \Rightarrow \vec{N} = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = \langle 1,-2,1 \rangle$ 4

$q_1 = (0,0,0) \quad q_2 = (1,0,0)$

$P_1: 1\cdot(x-0) - 2(y-0) + 1\cdot(z-0) = 0 \Rightarrow x - 2y + z = 0 \quad l_1 \subseteq P_1$ 2

$P_2: 1\cdot(x-1) - 2(y-0) + 1(z-0) = 0 \Rightarrow x - 2y + z = 1 \quad l_2 \subseteq P_2$

distance $\dfrac{|d_1 - d_2|}{\sqrt{a^2+b^2+c^2}} = \dfrac{1}{\sqrt{1^2+(-2)^2+1^2}} = \dfrac{1}{\sqrt{6}}$ ✓ 2

b.) (10 pts) Which points on l_1 and l_2 are closest to each other?

distance between (t,t,t) and $(s+1, 2s, 3s)$

$d(t,s) = \sqrt{(t-(s+1))^2 + (t-2s)^2 + (t-3s)^2}$ minimize $\hat{d} = d^2$

$\hat{d}(t,s) = t^2 - 2t(s+1) + (s+1)^2 + t^2 - 4st + 4s^2 + t^2 - 6ts + 9s^2$
$\quad -2st-2t + \quad s^2+2s+1$

$= 3t^2 + 14s^2 - 12st - 2t + 2s + 1$ 5

$d_t = 0 = 6t - 12s - 2 \Rightarrow \times 2 \quad 3t - 6s = 1$

$d_s = 0 = 28s - 12t + 2 \Rightarrow 6t - 14s = 1$

$2s = 1 \Rightarrow s = 1/2$
$t = 4/3$ 3

$\Rightarrow (4/3, 4/3, 4/3) \in l_1$ & $(3/2, 1, 3/2) \in l_2$ 2

$P_1 \qquad P_2$

$d(P_1, P_2) = \dfrac{1}{\sqrt{6}}$ ✓

Problem 2. (12 pts) Consider the points $A = (1, 0, 2)$, $B = (3, 5, 4)$ and $C = (2, 3, 0)$.

2.a) What is the area of the triangle ABC?

$\text{Area} = |\vec{v}\times\vec{w}| \Rightarrow \quad \vec{AB} = \langle 2, 5, 2\rangle$

$\vec{AC} = \langle 1, 3, -2\rangle$

$\text{Area} = \frac{1}{2}|\vec{AB}\times\vec{AC}| \Rightarrow \quad \vec{AB}\times\vec{AC} = \begin{vmatrix} i & j & k \\ 2 & 5 & 2 \\ 1 & 3 & -2 \end{vmatrix} = \langle -16, 6, 1\rangle$ 3

$|\vec{AB}\times\vec{AC}| = \sqrt{(-16)^2+6^2+1^2} = \sqrt{256+36+1} = \sqrt{293}$ 2

$\boxed{\text{Area} = \frac{\sqrt{293}}{2}}$ 1

2.b) Find an equation of the plane passing through the points A, B and C.

$\vec{N} = \vec{AB}\times\vec{AC} = \langle -16, 6, 1\rangle \qquad A \in \text{Plane}$

Plane equation: $-16(x-1)+6(y-0)+1\cdot(z-2) = 0$

$-16x+16+6y+z-2 = 0$

$\boxed{-16x+6y+z = -14}$

Problem 3.a) (7 pts) Find the following limit if it exists.

$$\lim_{(x,y)\to(0,0)} \frac{e^y x^2 y}{x^4+y^2}$$

① along the line $x=0$, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{e^y x^2 y}{x^4+y^2} = \lim_{y\to 0} \frac{0}{y^2} = \boxed{0.}$

② along the curve $y=x^2$, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{e^y x^2 y}{x^4+y^2} = \lim_{x\to 0} \frac{e^{x^2} x^4}{x^4+x^4} = \lim_{x\to 0} \frac{e^{x^2}}{2} = \boxed{\frac{1}{2}}$

Since on two different paths limit takes different values, the limit does not exist.

3.b) (7 pts) Determine all the points where the following function is continuous.

$$f(x) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & , \text{if } (x,y) \neq (0,0) \\ 0 & , \text{if } (x,y) = (0,0) \end{cases}$$

The function is continuous on $\mathbb{R}^2 - \{(0,0)\}$, the only point we need to check is $(0,0)$.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{r\to 0^+} \frac{r^2 \cos\theta \sin\theta}{r} = \lim_{r\to 0^+} r\cos\theta\sin\theta = 0$$

$x = r\cos\theta$
$y = r\sin\theta$

(since $\cos\theta\sin\theta$ is a bounded function, limit is 0 independent of θ)

$\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0 \Rightarrow f$ is continuous at $(0,0)$.

$\Rightarrow f$ is continuous on all $\mathbb{R}^2$.

Problem 4) (12 pts) Find an equation of the tangent plane to the given surface at the given point in each part.

a) $z = \sqrt{xy}$, $(1,1,1)$

$$\frac{\partial z}{\partial x} = \frac{1}{2}\frac{1}{\sqrt{xy}}y \qquad \left.\frac{\partial z}{\partial x}\right|_{\substack{x=1\\y=1}} = \frac{1}{2}$$

$$\frac{\partial z}{\partial y} = \frac{1}{2}\frac{1}{\sqrt{xy}}x \qquad \left.\frac{\partial z}{\partial y}\right|_{\substack{x=1\\y=1}} = \frac{1}{2}$$

$$\longrightarrow \quad z - z_0 = \frac{\partial z}{\partial x}(x_0,y_0)\cdot(x-x_0) + \frac{\partial z}{\partial y}(x_0,y_0)(y-y_0)$$

$$z - 1 = \frac{1}{2}(x-1) + \frac{1}{2}(y-1)$$

$$z = \frac{x}{2} + \frac{y}{2}$$

b) $x^2 - 2y^2 + z^2 = 2 - yz$, $(2,1,-1)$

Let $f(x,y,z) = x^2 - 2y^2 + z^2 + yz - 2$

$f_x = 2x$, $\quad f_y = -4y + z$, $\quad f_z = y + 2z$

$f_x(2,1,-1) = 4$, $\quad f_y(2,1,-1) = -5$, $\quad f_z(2,1,-1) = -1$

$$\longrightarrow \quad f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0) = 0$$

$$4(x-2) - 5(y-1) - 1\cdot(z+1) = 0$$

$$4x - 5y - z = 4$$

Solution **Problem 5.a)** (7 points) Given $x - z = \arctan yz$ evaluate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(2,0,2)$.

By Implicit Differentiation:

$$1-\frac{\partial z}{\partial x} = \frac{1}{1+(yz)^2}\cdot y\cdot\frac{\partial z}{\partial x} \text{ ; then}$$

$$\frac{\partial z}{\partial x}\left(1+\frac{y}{1+(yz)^2}\right) = 1 \text{ ; so that}$$

$$\frac{\partial z}{\partial x} = \frac{1+y^2z^2}{1+y+y^2z^2}$$

$$0-\frac{\partial z}{\partial y} = \left(\frac{1}{1+(yz)^2}\right)\left(z+y\cdot\frac{\partial z}{\partial y}\right)$$

then,

$$-\frac{\partial z}{\partial y} = \frac{z}{1+y^2z^2} + \frac{y}{1+y^2z^2}\frac{\partial z}{\partial y}$$

So that

$$\frac{\partial z}{\partial y} + \frac{y}{1+y^2z^2}\frac{\partial z}{\partial y} = \frac{-z}{1+y^2z^2}$$

and

$$\frac{\partial z}{\partial y}\left(1+\frac{y}{1+y^2z^2}\right) = \frac{-z}{1+y^2z^2}$$

hence

$$\frac{\partial z}{\partial y} = \frac{-z/1+y^2z^2}{1+y+y^2z^2/1+y^2z^2}$$

and

$$\frac{\partial z}{\partial y} = \frac{-z}{1+y^2z^2+y}$$

Then

$$\frac{\partial z}{\partial x}(2,0,2) = \frac{1+0^2.2^2}{1+0+0^2.2^2} = 1 \quad \&$$

$$\frac{\partial z}{\partial y}(2,0,2) = \frac{-2}{1+0+0^2.2^2} = -2 \quad \square$$

Second Method: Using the formulas

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad \& \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

where

$$F(x,y) = x - z - \arctan yz$$

then

$$\frac{\partial z}{\partial x} = -\frac{1}{-1-\frac{1}{1+y^2z^2}\cdot y}$$

and

$$\frac{\partial z}{\partial x}(2,0,2) = -\frac{1}{-1-\frac{1}{1+0^2.0^2}.0} = 1$$

&

$$\frac{\partial z}{\partial y} = -\frac{\frac{-1}{1+y^2z^2}\cdot z}{-1-\frac{1}{1+y^2z^2}\cdot y}$$

and

$$\frac{\partial z}{\partial y}(2,0,2) = -\frac{\frac{-1}{1+0^2.2^2}.2}{-1-\frac{1}{1+0^22^2}.0}$$

$$= -2 \quad \square$$

5.b) (8 points) Given $f(x,y) = 3xy^4 + x^3y^2$, find $f_{xyx}(x,y), \nabla f(x,y)$. Find $D_u f$ at $(1,2)$ for the vector $u = \langle 3/\sqrt{13}, -2/\sqrt{13} \rangle$.

Solution

- $f_x(x,y) = 3y^4 + 3x^2y^2$, $f_y(x,y) = 12xy^3 + 2x^3y$

then $f_{xy}(x,y) = 12y^3 + 6x^2y$

then $f_{xyx}(x,y) = 12xy$.

- $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle$
$= \langle 3y^4+3x^2y^2, 12xy^3+2x^3y\rangle$

- $D_u f(1,2) = \nabla f(1,2) \bullet u$ → where u is unit vector.

dot product.

$\nabla f(1,2) = \langle 3.16+3.1^2.2^2 , 12.1.2^3+2.1^3.2\rangle$
$= \langle 60, 100\rangle$

Note that the vector $u = \langle \frac{3}{\sqrt{13}}, \frac{-2}{\sqrt{13}}\rangle$ is already a unit vector. Then;

dot products

$$D_u f(1,2) = \nabla f(1,2) \bullet u$$
$$= \langle 60,100\rangle \bullet \langle \frac{3}{\sqrt{13}}, \frac{-2}{\sqrt{13}}\rangle$$
$$= \frac{180}{\sqrt{13}} - \frac{200}{\sqrt{13}} = \frac{-20}{\sqrt{13}} \quad \square$$

Problem 6) Consider the function $f(x,y) = x^3 + 8y^3 - 12xy$.

(a) (6 pts) Find the critical points of this function.

$$\left.\begin{array}{l} f_x = 3x^2 - 12y = 0 \\ f_y = 24y^2 - 12x = 0 \end{array}\right\}$$

$y = \frac{x^2}{4}$

$x = 2y^2 = 2\frac{x^4}{16}$

$x^4 - 8x = 0$

$x = 0$ or $x = 2$

$y = 0$ $\quad$ $y = 1$

$(0,0)$ and $(2,1)$ are the crit. pts.

(b) (8 pts) Determine the nature of each critical point (local maximum/minimum or saddle).

$$D(x,y) = f_{xx} f_{yy} - f_{xy} f_{yx} = 6x \cdot 48y - (-12)^2$$
$$= 288xy - 144$$

$D(0,0) = -144 < 0$ So $(0,0)$ is a saddle point

$$\left.\begin{array}{l} D(2,1) = 288 \cdot 2 - 144 > 0 \\ f_{xx}(2,1) = 12 > 0 \end{array}\right\}$$ So $(2,1)$ gives a local minimum.

Problem 7) (20 points) Find the absolute maximum and absolute minimum values of $f(x,y) = y^3 - x^2y - y$ on the region $\{(x,y) : x^2 + 4y^2 \leq 4\}$.

Critical pts: $f_x = -2xy = 0$, $f_y = 3y^2 - x^2 - 1 = 0$ $\;\Rightarrow\;$ $x = 0$ or $y = 0$

$x = 0 \Rightarrow y = \mp \frac{1}{\sqrt{3}}$ $\qquad$ $y = 0 \Rightarrow \cancel{x^2 = -1}$

$\boxed{\left(0, \frac{1}{\sqrt{3}}\right) \text{ and } \left(0, -\frac{1}{\sqrt{3}}\right)}$ are the crit. pts.

Boundary: $x^2 + 4y^2 = 4$

Lagrange multipliers method: $g(x,y) = x^2 + 4y^2$

$\nabla f(x,y) = \langle -2xy, 3y^2 - x^2 - 1 \rangle = \lambda \langle 2x, 8y \rangle = \lambda \cdot \nabla g(x,y)$

$$\begin{cases} -2xy = 2x\lambda \\ 3y^2 - x^2 - 1 = 8y\lambda \\ x^2 + 4y^2 = 4 \end{cases}$$

$\rightsquigarrow$ $x = 0$ or $y = -\lambda$

$x = 0 \Rightarrow y = \mp 1$ $\qquad$ $\boxed{(0, \mp 1)}$

$y = -\lambda$:

$\lambda = \mp \frac{1}{\sqrt{3}}$

$y = \mp \frac{1}{\sqrt{3}}$ $\quad$ $x = \mp \frac{2\sqrt{2}}{\sqrt{3}}$ $\qquad$ $\boxed{\left(\mp \frac{2\sqrt{2}}{\sqrt{3}}, \mp \frac{1}{\sqrt{3}}\right)}$

$f\left(0, \frac{1}{\sqrt{3}}\right) = -\frac{2}{3\sqrt{3}}$ $\qquad$ $f\left(0, -\frac{1}{\sqrt{3}}\right) = \frac{2}{3\sqrt{3}}$

$f(0, \mp 1) = 0$ $\qquad$ $f\left(\mp \frac{2\sqrt{2}}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \boxed{-\frac{10}{3\sqrt{3}}}$ $\leftarrow$ abs. min.

$f\left(\mp \frac{2\sqrt{2}}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) = \boxed{\frac{10}{3\sqrt{3}}}$ $\leftarrow$ abs. max.