2) (5 pts each) For each of (a)-(d) below: If the proposition is true, write TRUE. If the proposition is false, write FALSE. No explanations are required for this problem.

2a) Let $f : X \to Y$ be a continuous map. If $A \subset Y$ is compact, then $f^{-1}(A)$ is compact.

FALSE. $\left(\begin{array}{l} f: \mathbb{R} \to \mathbb{R} \\ \quad\; x \mapsto 1 \end{array}, \; A = \{1\} \;\; f^{-1}(A) = \mathbb{R} \right)$

2b) Any bounded sequence in a complete metric space has a convergent subsequence.

FALSE, ($X = \mathbb{R}$ with discrete metric
$a_n = n$ bdd but not convergent)

2c) $A = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$ is a compact subset of $\mathbb{R}$.

TRUE. A closed and bounded.

2d) Let A be compact subset of X, and $f : A \to Y$ be continuous map. Then, there is a continuous extension of f such that $\widehat{f} : X \to Y$.

TRUE. f uniformly cts (QS)
+ uniform extension theorem.

3) Prove or give a counterexample for the following statements.

a) (10 pts) $x \in \bar{A}$ if and only if $d(x, A) = \inf\{d(x, a) : a \in A\} = 0$.

TRUE.

$\Rightarrow$ $x \in \bar{A} \Rightarrow \exists (a_n) \subseteq A \quad a_n \to x$

$\Rightarrow d(a_n, x) \to 0 \Rightarrow d(x, A) = 0$

$\Leftarrow$ $d(x, A) = 0 \Rightarrow \forall n \ \exists a_n \in A$ with $d(x, a_n) < \frac{1}{n}$

$\Rightarrow a_n \to x \Rightarrow x \in \bar{A}$

b) (10 pts) Let A and B be disjoint subsets of X. If A is closed, and B is compact, then $d(A, B) = \inf\{d(a, b) : a \in A,\ b \in B\} > 0$

TRUE.

Let $\varphi : B \to \mathbb{R}$

$b \quad d(b, A)$

Claim: φ is continuous.

$\varepsilon_0 > 0$ given. let $\delta_0 = \frac{\varepsilon_0}{2}$ and $d(b, b_0) < \delta_0 = \frac{\varepsilon_0}{2}$

$\forall a \quad d(b, a) \leq d(b, b_0) + d(b_0, a)$

$\Rightarrow d(b, A) \leq d(b, b_0) + d(b_0, A)$ (similarly other side)

$\Rightarrow |d(b, A) - d(b_0, A)| = |\varphi(b) - \varphi(b_0)| \leq d(b, b_0) < \varepsilon_0 \Rightarrow \varphi$ cts.

Since φ cts & B compact. $\exists b_0 \in A$ with $d(b_0, A) = \inf(d(b, A)) = d(A, B)$

if $d(A, B) = 0 \Rightarrow d(b_0, A) = 0 \Rightarrow b_0 \in \bar{A} = A$ (by part a)

$b_0 \in A \cap B$. $\times$.

4) (20 pts) Prove or give a counterexample for the following statement.

Let $f : X \to Y$ be a one-to-one and continuous map, and Y be compact. If $(f(x_n))$ is convergent, then (x_n) is convergent.

FALSE.

$$f : (0,1) \to [0,1] \qquad x_n = \frac{1}{n} \qquad f(x_n) = \frac{1}{n}$$

$$x \qquad\qquad x \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$? \qquad\qquad 0 \text{ in } [0,1]$$

or

$$f : [1,\infty) \to [0,1] \qquad x_n = n \qquad f(x_n) = \frac{1}{n} \to 0$$

$$x \qquad\qquad \frac{1}{x}$$

5) (20 pts) Prove or give a counterexample for the following statement.

Let $f : X \to Y$ be continuous. If X is compact, then f is uniformly continuous.

TRUE.

Let $\varepsilon_0 > 0$ given. f cts $\Rightarrow$ $\forall x \in X \ \exists \delta_x \quad f(B_{\delta_x}(x)) \subseteq B_{\frac{\varepsilon_0}{2}}(f(x))$. (*)

$\bigcup B_{\frac{\delta_x}{2}}(x)$ is an open cover for X

$\Rightarrow \exists$ finite subcover $\quad B_{\frac{\delta_{x_1}}{2}}(x_1) \cup \dots \cup B_{\frac{\delta_{x_k}}{2}}(x_k)$

Let $\delta = \min\left\{ \frac{\delta_{x_1}}{2}, \dots \frac{\delta_{x_k}}{2} \right\}$

Claim: $d(x,y) < \delta \Rightarrow \rho(f(x), f(y)) < \varepsilon_0$. (Hence f uniformly cts.

Let $d(x,y) < \delta$. $\exists x_i \quad d(x, x_i) < \frac{\delta_{x_i}}{2}$

$\Rightarrow \quad d(y, x_i) \leq \underbrace{d(y,x)}_{<\delta} + \underbrace{d(x, x_i)}_{<\frac{\delta_{x_i}}{2}} < \delta_{x_i}$

$\overset{(*)}{\Rightarrow} \quad \rho(f(x), f(x_i)) < \frac{\varepsilon_0}{2}$ and $\rho(f(y), f(x_i)) < \frac{\varepsilon_0}{2}$

$\rho(f(x), f(y)) \leq \underbrace{\rho(f(x), f(x_i))}_{<\frac{\varepsilon_0}{2}} + \underbrace{\rho(f(x_i), f(y))}_{<\frac{\varepsilon_0}{2}} < \varepsilon_0$

$\square$.

Bonus) (20 pts) Let X be a compact metric space, and $\{U_\alpha\}$ be an open cover of X. Show that there is a $\delta > 0$ such that for any $x \in X$, there is an α_x with $B_\delta(x) \subset U_{\alpha_x}$.

Assume that there is no such $\delta > 0$.

$\Rightarrow \forall \delta > 0 \quad \exists x \in X$ s.t. $B_\delta(x) \not\subseteq U_\alpha$ for any α.

$\Rightarrow \forall \delta_n = \frac{1}{n} \quad \exists x_n \in X$ s.t. $B_{1/n}(x_n) \not\subseteq U_\alpha$ for any α. (*)

$\Rightarrow (x_n) \subseteq X$ seq. $\overset{X \text{ compact}}{\Rightarrow}$ $\exists$ convergent subsequence $x_{n_k} \to p \in X$.

$p \in U_{\alpha_0}$ for some α_0. $\Rightarrow \exists \varepsilon_0 > 0 \quad B_{\varepsilon_0}(p) \subseteq U_{\alpha_0}$.

Since $x_{n_k} \to p \quad \exists N_1$ s.t. $\forall n_k > N_1 \quad d(x_{n_k}, p) < \frac{\varepsilon_0}{2}$

Let N_2 s.t. $\frac{1}{N_2} < \frac{\varepsilon_0}{2}$ (Archimedian Property).

$\Rightarrow$ let $n_{k_0} > \max\{N_1, N_2\}$. Then $B_{\frac{1}{n_{k_0}}}(x_{n_{k_0}}) \subseteq B_{\varepsilon_0}(p) \subseteq U_{\alpha_0}$

$\times$

contradicts with (*)