Math 301

HW6 Solutions

(4) Suppose $\lim x_n = x$, $\lim y_n = y$ so $\lim d(x,x_n)=0$ and $\lim d(y,y_n)=0$.
$d(x_n,y_n) \leq d(x_n,x)+d(x,y)+d(y,y_n)$ by triangle inequality.
Similarly $d(x,y) \leq d(x,x_n)+d(x_n,y_n)+d(y_n,y)$. Hence
$-d(x_n,x)-d(y_n,y) \leq d(x_n,y_n)-d(x,y) \leq d(x_n,x)+d(y_n,y)$ so
$|d(x_n,y_n)-d(x,y)| \leq d(x_n,x)+d(y_n,y)$. Given $\varepsilon>0$, $\exists N_1, N_2 \in \mathbb{N}$ s.t. $d(x_n,x)<\varepsilon/2$ $\forall n \geq N_1$ and $d(y_n,y)<\varepsilon/2$ $\forall n \geq N_2$. Let $N=\max(N_1,N_2)$.
Then $|d(x_n,y_n)-d(x,y)|<\varepsilon$ $\forall n \geq N$. Thus $\lim d(x_n,y_n)=d(x,y)$.

---

(6) Suppose c is a cluster point of (x_n). We construct a subsequence of (x_n) converging to c. Let $\varepsilon_1 = 1/2$ Then $\exists x_{n_1}$ s.t. $d(x_{n_1},c)<\varepsilon = 1/2$.
Let $\varepsilon_2 = d(x_{n_1},c)/2^2$, then $\exists x_{n_2}$ s.t. $d(x_{n_2},c)<\varepsilon_2$. Having choosen $x_{n_1},\dots,x_{n_{k-1}}$, choose x_{n_k} as: Let $\varepsilon_k = d(x_{n_{k-1}},c)/2^k$, then $\exists x_{n_k}$ s.t. $d(x_{n_k},c)<\varepsilon_k$. As $d(x_{n_k},c)<1$ $\forall k$, $\varepsilon_k < \frac{1}{2^k}$ so $\varepsilon_k \to 0$ as $k\to\infty$. Hence $d(x_{n_k},c)\to 0$ as $k\to\infty$. Thus $\lim_{k\to\infty} x_{n_k}=c$.
Suppose $\exists (x_{n_k})$ s.t. $\lim_{k\to\infty} x_{n_k}=c$. i.e. $\lim d(x_{n_k},c)=0$.
Given $\varepsilon>0$, $\exists N\in\mathbb{N}$ s.t. $d(x_{n_k},c)<\varepsilon$ $\forall k\geq N$ i.e. $x_{n_k}\in B_\varepsilon(c)$ $\forall k \geq N$.
Thus c is a cluster point of (x_n).

---

(11) $\mathbb{Q}$ and $\mathbb{R}\setminus\mathbb{Q}$ are not complete as they are dense in $\mathbb{R}$.
$\mathbb{Z}$ is complete since any cauchy sequence in $\mathbb{Z}$ is eventually constant hence convergent. $(a,b]$ is not complete as $a+\frac{1}{n}\to a$ but $a\notin(a,b]$ so $a+\frac{1}{n}$ is not convergent in $(a,b]$. Similarly (a,b) is not complete.

(17) Let $f: X \longrightarrow \mathbb{R}$ be a continuous function. Suppose $f(x)=0$ $\forall x \in A$. Let $x \in \bar{A}$, then $\exists (x_n)_n$ in A s.t. $\lim x_n = x$. $f(x_n)=0$ $\forall n \geq 0$ since $x_n \in A$ $\forall n \geq 0$. Since f is continuous $\lim f(x_n) = f(x)$ and $\lim f(x_n) = 0$ as $f(x_n)=0$ $\forall n \geq 0$. Thus $f(x)=0$.

---

(21) Let $x \in [0,1]$. Since f is continuous $\exists \delta > 0 : |f(x)-f(y)| < 1$ whenever $|x-y|<\delta$ and $x, y \in [0,1]$. Hence $f(y) < f(x)+1$ for $|x-y|<\delta$.

$$|\Psi_f(x) - \Psi_f(y)| = \left| \int_0^x f(t)dt - \int_0^y f(t)dt \right| = \left| \int_0^x f(t)dt - \int_0^x f(t)dt - \int_x^y f(t)dt \right|$$

$$= \left| \int_x^y f(t)dt \right| \leq \int_x^y |f(t)|dt \ (*)$$. Given $\varepsilon > 0$, choose $\delta_1 > 0$ s.t.

We assume $x<y$, otherwise take the integral y to x.

$\delta_1 < \min\left(\delta, \varepsilon / (f(x)+1)\right)$. Then for $|x-y|<\delta_1$ we have $f(y)<f(x)+1$

so if $|x-y|<\delta_1$ then by $(*)$ $\int_x^y |f(t)|dt < (f(x)+1)(y-x)$

$< (f(x)+1)\,\delta_1 < (f(x)+1)\dfrac{\varepsilon}{(f(x)+1)} = \varepsilon$. Thus $|\Psi_f(x) - \Psi_f(y)| < \varepsilon$

whenever $|x-y|<\delta_1$, i.e. Ψ_f is continuous at x.

Define $\Psi: C[0,1] \longrightarrow C[0,1]$ by $f \longmapsto \Psi_f$. Given $\varepsilon > 0$.

$\|\Psi_f - \Psi_g\| = \sup_{x\in[0,1]} |\Psi_f(x) - \Psi_g(x)|$. $|\Psi_f(x) - \Psi_g(x)| = \left| \int_0^x f(t)dt - \int_0^x g(t)dt \right|$

$$= \left| \int_0^x (f(t)-g(t))dt \right| \leq \int_0^x |f(t)-g(t)|\,dt \leq \int_0^x \sup_{a\in[0,1]} |f(a)-g(a)|\,dt$$

$$= \sup_{a\in[0,1]} |f(a)-g(a)| \int_0^x dt = \sup_{a\in[0,1]} |f(a)-g(a)|\, \underset{0<x\leq 1}{\underbrace{x}} \leq \sup_{a\in[0,1]} |f(a)-g(a)| = \|f-g\|$$

Choose $\delta = \varepsilon$. So $|\Psi_f(x) - \Psi_g(x)| \leq \|f-g\| < \delta = \varepsilon \quad \forall x \in [0,1]$ whenever $\|f-g\| < \delta$. Hence $\sup_{x\in[0,1]} |\Psi_f(x) - \Psi_g(x)| < \varepsilon$. Thus

$\|\Psi_f - \Psi_g\| < \varepsilon$ whenever $\|f-g\| < \delta$. and δ depends only on ε.

i.e. Ψ is uniformly continuous.

---

(22) Let $f: \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = x^2$. Then $f(x)$ is continuous on $\mathbb{R}$ but not uniformly continuous. To see this, assume that f is uniformly continuous. Let $0 < \varepsilon < 2$, then $\exists \delta = \delta(\varepsilon) > 0$ s.t. $|f(x) - f(y)| < \varepsilon$ for all $x, y \in \mathbb{R}$ s.t. $|x-y| < \delta$. Choose $n \in \mathbb{N}$ s.t. $\frac{1}{n} < \delta$, and put $x = n$, $y = n + \frac{1}{n}$, then $|x-y| = |\frac{1}{n}| < \delta$, but $|f(x) - f(y)| = |n^2 - n^2 - 2 - \frac{1}{n^2}| = 2 + \frac{1}{n^2} > \varepsilon$, contradiction.

---

(23) Let $f: \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & x \neq 0 \\ a & x = 0 \end{cases}$

- f is continuous at $x=0$ iff for every sequence $(x_n)_{n\in\mathbb{N}}$ converging to 0, $f(x_n) \longrightarrow f(0)$. Take $x_n = \dfrac{1}{n\pi + \frac{\pi}{2}}$.

Then $\lim x_n = 0$ and $f(x_n) = \sin\left(n\pi + \frac{\pi}{2}\right) = (-1)^n \quad \forall n \geq 0$.

Hence $\lim f(x_n) = \lim (-1)^n$ does not exists so f can not be continuous at $x=0$ for any value of a.