**Math 301 - Problem Set # 10 - Fall 2010**

**Homework Problems:** 1, 3, 10, 17, 19, 26.

In the following problems $(X, d)$, $(Y, \rho)$ are assumed to be metric spaces, $A, B$ are subsets of $X$, $f$ is a mapping from $X$ to $Y$.

1. Suppose that $X$ is compact and $(A_n)$ is a sequence of non-empty closed subsets of $X$ with $A_{n+1} \subseteq A_n$. Prove that $\cap_{n=0}^{\infty} A_n$ is non-empty.

2. The *diameter* $\delta(A)$ of a non-empty bounded set $A$ is defined to be $\sup\{d(a, a') : a, a' \in A\}$. Prove Cantor Nested Sets Theorem: *Suppose that* $X$ *is complete and* $(A_n)$ *is a sequence of non-empty closed subsets of* $X$ *with* $A_{n+1} \subseteq A_n$ *and* $\lim \delta(A_n) = 0$. *Then* $\cap_{n=0}^{\infty} A_n$ *consists of a single point.*

3. Let $f_n : [0, 1] \to \mathbb{R}$ be defined by $f_n(x) = x^n$ for each $n \in \mathbb{Z}_+$ and $f_0 : [0, 1] \to \mathbb{R}$ be defined by $f_0(x) = 1$. Prove that $(f_n(x))$ is convergent for each $x \in [0, 1]$, and find this limit for every $x$. Is the sequence $(f_n)$ uniformly convergent, i.e. is it convergent in $B[0, 1]$ with the uniform convergence metric (sup-metric)?

4. For two subsets $E$ and $F$ of $\mathbb{R}$, $E + F$ is defined to be $\{e + f : e \in E, f \in F\}$. Prove that $E + F$ is compact if $E$ and $F$ are compact. Prove that $E + F$ is closed if $E$ is compact and $F$ is closed.

5. Let $\phi : \mathbb{R} \to \mathbb{R}$ be defined by $\phi(x) = 0$ if $x \notin \mathbb{Q}$ and $\phi(m/n) = 1/n$ whenever $m$ and $n$ are relatively prime integers and $n > 0$. Prove that $\phi$ is continuous at irrational numbers and discontinuous at rational numbers.

6. Prove that $X$ is connected iff it has no nonempty, proper subset that is both open and closed.

7. Prove that $X$ is connected iff every non-empty proper subset of $X$ has non-empty boundary.

8. Prove that a discrete metric space is connected iff it consists of a single point.

9. Suppose that $A$ and $B$ are nonempty, closed and disjoint. Prove that $A \cup B$ is disconnected.

10. Suppose that $A$ and $B$ are connected.

    (a) Prove that if $\bar{A} \cap B \neq \emptyset$, then $A \cup B$ is connected.

    (b) Prove or disprove: if $\bar{A} \cap \bar{B} \neq \emptyset$, then $A \cup B$ is connected.

11. Prove that either the interior or the exterior of $A$ is empty if $X \setminus \partial(A)$ is connected.

12. Prove that any convex set in $\mathbb{R}^k$, in particular any ball in $\mathbb{R}^k$ and $\mathbb{R}^k$ itself are all connected.

13. Prove that $\{(x, y) : 1 \leq x^2 + y^2 \leq 4\}$ is a compact connected subset of $\mathbb{R}^2$.

14. Prove that if $\phi : X \to \mathbb{R}$ is continuous, $0, 2 \in \phi(A)$ and $1 \notin \phi(A)$, then $A$ is disconnected.

15. Suppose that $X$ is disconnected. Prove that there is a continuous function $\phi : X \to \mathbb{R}$ such that $\phi(X)$ consists of two real numbers.

16. Let $\sim$ be a relation on $X$ defined by $x \sim y$ if the connected component of $x$ is the same as that of $y$. Prove that this is an equivalence relation. What is the equivalence class of an element $x$?

17. Suppose that $X$ is connected and not bounded, and $x_0 \in X$. Prove that for every $r > 0$ there exists $x \in X$ with $d(x_0, x) = r$.

18. Let $X = [0, 1]$, $Y = \{1, 2, 3\}$, $d$ be the absolute value metric and $\rho$ be the discrete metric. Considering $\rho_\infty$ metric on the cartesian product $\mathcal{X} = X \times Y$, give examples of compact, non-compact, connected and disconnected subsets of $\mathcal{X}$.

19. Check whether the following sets are compact or connected. Justify your claim.

    (a) $\{\sin(1/n) : n \in \mathbb{Z}_+\} \subseteq \mathbb{R}$.
    (b) $\{(x, e^x) : 0 < x < 1\} \subseteq \mathbb{R}^2$.
    (c) $\{(x, y, z) : x^2 + y^2 \leq 1\} \cap \{(x, y, z) : z^2 + y^2 \leq 4\} \subseteq \mathbb{R}^3$.
    (d) $\{z : z = x^2 \sin y, x^2 + y^2 \leq 1\} \subseteq \mathbb{R}$.

20. Prove that every open subset of $\mathbb{R}$ is the union of countably many disjoint open intervals.

21. Suppose that $X$ is path-connected and $f$ is continuous. Prove that $f(X)$ is path-connected.

22. Let $\sim$ be the relation defined on $X$ by $x \sim y$ if there is a path from $x$ to $y$. Prove that $\sim$ is an equivalence relation.

23. Suppose that $A_i$ is connected for $i = 1, 2, 3$. Prove that $A_1 \cup A_2 \cup A_3$ is connected if $A_1 \cap A_2$ and $A_2 \cap A_3$ are non-empty.

24. Let $\phi : X \to \mathbb{R}$ be continuous, $X$ be compact, $x \in X$ and $E_x$ be the connected component of $x$ in $X$. Prove that $f(E_x)$ is a closed and bounded interval.

25. Let $K$ be a convex subset of $R^k$ and $\phi : K \to X$ be continuous. Prove that the connected components of any two points in $\phi(K)$ are the same, i.e. $\phi(K)$ is contained in a single connected component.

26. Let $A$ be the compact subset of $\mathbb{R}^2$ consisting of the vertical line segment

$$E = \{(0, y) : -1 \leq y \leq 1\}$$

together with the portion of the graph of $\sin(1/x)$ given by

$$F = \{(x, \sin(1/x)) : 0 < x \leq 1\}\ .$$

Prove that $A$ is connected. Is it path-connected?