Math 301 - Problem Set # 3 - Fall 2010

Homework Problems: 2, 8, 10, 12, 17, 21

In the following exercises all sequences are assumed to be in $\mathbb{R}$ and (a_n) denote a sequence of real numbers. As usual, you should justify your answers.

1. Prove that any subsequence of a convergent sequence is convergent and has the same limit as the sequence.

2. Find all the cluster points of the following sequences if they exist.

   (a) $a_n = 5$ for every natural number n.

   (b) $a_n = (-1)^n$ for every natural number n.

   (c) For every natural number n, a_n is the remainder when n is divided by 3.

   (d) $a_n = n$ for every natural number n.

   (e) $a_n = f(n)$ for every natural number n, where f is any bijection from $\mathbb{N}$ to $\mathbb{Q} \cap [0, 1]$. (*How do we know that there is such a bijection?*)

3. Prove that the limit of a convergent sequence is a cluster point and in fact the only cluster point of that sequence. If possible, find a sequence which diverges even though it has a unique cluster point.

4. If possible, give an example of a set which has a maximum,i.e. a greatest element, and an example of a bounded set which does not have a maximum.

5. If possible, find a convergent sequence with more than one cluster point.

6. Prove that any bounded sequence has at least one cluster point.

7. Prove that (a_n) converges iff both (a_{2n}) and (a_{2n+1}) converges to the same limit.

8. If possible, give an example of a Cauchy sequence which isn't convergent and an example of a sequence which isn't Cauchy. What would change if we had considered sequences in $\mathbb{Z}$ or $\mathbb{Q}$?

9. Prove that if (a_n) is monotone and unbounded, then $\lim \frac{1}{a_n} = 0$.

   Definition. Let S be a subset of $\mathbb{R}$. A real number a is called an *accumulation point of S* if for every $\epsilon > 0$ the interval $(a - \epsilon, a + \epsilon)$ contains infinitely many points of S.

10. Given a sequence (a_n), what can you say about the accumulation points of the set $\{a_n : n \in \mathbb{N}\}$?

11. (a) Prove that $\mathbb{N}$ has no accumulation points.
    (b) Find the accumulation points of $S = (0,1) \cup \{2\}$.
    (c) Prove that any real number is an accumulation point of $\mathbb{Q}$.

12. Prove that a is an accumulation point of S iff there is a sequence (a_n) in S such that $\lim a_n = a$ and $a_n \neq a_m$ whenever $n \neq m$.

13. Prove that if S is a bounded subset of $\mathbb{R}$ with infinitely many elements, then S has at least one accumulation point.

14. Find a sequence (A_n) of non-empty subsets of $\mathbb{R}$ such that A_0 is bounded, $A_{n+1} \subseteq A_n$ for every $n \in \mathbb{N}$ and the intersection $\cap_{n\in\mathbb{N}} A_n$ is empty.

15. Let $[a_0, b_0] \supseteq [a_1, b_1] \supseteq \cdots \supseteq [a_n, b_n] \supseteq \cdots$ be a sequence of closed and bounded intervals in $\mathbb{R}$. Prove that the intersection $\cap_{n\in\mathbb{N}}[a_n, b_n]$ contains only one element iff for every $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $b_n - a_n < \epsilon$.

16. If possible, find an unbounded sequence whose limit inferior and limit superior both exist.

17. Find limit superior and limit inferior of each of the following sequences, if they exist.
    (a) $a_n = (-1)^n$ for every natural number n.
    (b) For every natural number n, a_n is the remainder when n is divided by 4.
    (c) $a_n = \frac{1}{n+1}$ for every natural number n.
    (d) $a_n = (-1)^n \frac{2n}{n+1}$ for every natural number n.
    (e) $a_n = \frac{\sin((n+1)\pi)}{n+1}$ for every natural number n.
    (f) $a_n = f(n)$ for every natural number n, where f is any bijection from $\mathbb{N}$ to $\mathbb{Q} \cap [0,1]$.
    (g) $a_n = n$ for every natural number n.
    (h) $a_0 = 0$, $a_{2n+1} = \frac{1}{3} + a_{2n}$ and $a_{2n+2} = \frac{1}{3} a_{2n+1}$ for every natural number n.

18. Prove that if $\liminf a_n = \limsup a_n$, then (a_n) is convergent.

19. Prove that if (a_n) is convergent, then $\limsup a_n = \liminf a_n = \lim a_n$.

20. In each of the following, either give an example of a sequence (a_n) in $\mathbb{R}$ satisfying the given condition or prove that there is no such sequence.
    (a) $\limsup a_n \leq a_m$ for every $m \in \mathbb{N}$
    (b) $(\limsup a_n)^2 < (\liminf a_n)^2$
    (c) $\liminf a_n < \limsup a_n$ and $\lim a_n = 2$.

21. Let (a_n) and (b_n) be two bounded sequences. Prove that
    (a) $\liminf a_n + \liminf b_n \leq \liminf(a_n + b_n)$
    (b) $\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n$.