Math 301 - Problem Set # 4 - Fall 2010

Homework Problems: 1, 6, 14, 15, 23, 24, 27, 30, 34, 35.

In the following exercises all sequences are assumed to be in $\mathbb{R}$, (a_n) denote a sequence of real numbers, A denotes a subset of $\mathbb{R}$. As usual, you should justify your answers.

1. Prove that neither $(x, y]$ nor $[x, y)$ is closed in $\mathbb{R}$ for $x, y \in \mathbb{R}$ with $x < y$. Are they open?
2. Prove that the infimum of a nonempty, closed and bounded subset A of $\mathbb{R}$ belongs to A.
3. Give an example of a subset A of $\mathbb{R}$ which is not closed even though both $\sup(A)$ and $\inf(A)$ belong to A.
4. Let $x, y \in \mathbb{R}$ with $x < y$. Find the closure, interior, accumulation and isolated points of (x, y), $[x, y]$, $(x, y]$, $[x, y)$, $(-\infty, x)$, (x, ∞), $\varnothing$, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R} \setminus \mathbb{Q}$ and $\{\frac{1}{n+1} : n \in \mathbb{N}\}$. Which one of these sets are open? Which of them are closed? Which ones are discrete? Which ones are perfect?
5. Suppose that A is open and bounded. Prove that neither the supremum nor the infimum of A belongs to A.
6. Are there any subsets of $\mathbb{R}$ which are both open and closed? If yes, find all of them.
7. Prove that the union of finite number of closed subsets of $\mathbb{R}$ is closed in $\mathbb{R}$.
8. Prove that the intersection of any family of closed subsets of $\mathbb{R}$ is closed in $\mathbb{R}$.
9. Prove that the union of any family of open subsets of $\mathbb{R}$ is open in $\mathbb{R}$.
10. Prove that the intersection of finite number of open sets in $\mathbb{R}$ is open in $\mathbb{R}$.
11. Prove that $A = \bar{A}$ iff A is closed.
12. Prove that $\bar{A} = A \cup A'$.
13. Prove that A is open iff it is the union of a family of open intervals.
14. Find the closure, interior, derived set and the isolated points of $A = \{0\} \cup (1, 2] \cup \{4\}$. Is this set open? Is it closed? Is it perfect? Is it discrete?
15. Prove that A' is closed.
16. Prove that $A \subseteq \bar{A}$.
17. Prove that the following two definitions of an accumulation point are equivalent.

(a) $x \in \mathbb{R}$ is an *accumulation point of* A if for every $\epsilon > 0$, $(x-\epsilon, x+\epsilon) \cap (A \setminus \{x\}) \neq \varnothing$.

(b) $x \in \mathbb{R}$ is an *accumulation point of* A if for every $\epsilon > 0$, $(x-\epsilon, x+\epsilon) \cap A$ contains infinitely many elements.

18. Prove that if x is an isolated point of A, then x is an accumulation point of $\mathbb{R} \setminus A$.

19. Suppose that $x \in \bar{A}$. Prove that $x \in A'$ iff there is a sequence (a_n) in A with $a_n \neq a_m$ for $n \neq m$ that converges to x.

20. Let $A = \{a_n : n \in \mathbb{N}\}$ and C be the set of cluster points of (a_n). Prove that $\bar{A} = A \cup C$. Prove that $A' \subseteq C$.

21. Give an example of a sequence (a_n) which has a cluster point that is not an accumulation point of $\{a_n : n \in \mathbb{N}\}$.

22. Let $a_n \neq a_m$ for $n \neq m$, $A = \{a_n : n \in \mathbb{N}\}$ and (a_n) converges to L. Prove that $A' = \{L\}$ and each element of A is an isolated point.

23. Let (a_n) be a strictly increasing sequence and $A = \{a_n : n \in \mathbb{N}\}$. Prove that A is discrete. Also prove that A is closed iff (a_n) is unbounded.

24. Prove the following.

    (a) A^o is open.

    (b) $A^o \subseteq A$.

    (c) A^o is the largest open set contained in A.

    (d) $A^o = A$ iff A is open.

25. Let d be a real-valued function on $X \times X$ for a nonempty set X. Prove that d is a metric on X iff d satisfies the following conditions

    (a) $d(x, y) = 0$ iff $x = y$

    (b) $d(x, z) \leq d(y, x) + d(y, z)$ for every x, y and $z \in X$

26. Let X be a nonempty set. Prove that the function d on $X \times X$ defined by

$$d(x,y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{cases}$$

is a metric on X.

27. Let (X, d) be a metric space and Y be a nonempty subset of X. Prove that the restriction of d to $Y \times Y$ is a metric on Y.

28. Let (X, d) be a metric space and ρ be defined by

$$\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)} \text{ for every } x, y \in X \ .$$

Prove that ρ is a metric on X.

29. Let (X, d) be a metric space and ρ be defined by

$$\rho(x, y) = \min\{1, d(x, y)\} \text{ for every } x, y \in X \ .$$

Prove that ρ is a metric on X.

30. Let $f : X \to \mathbb{R}$ be a one-to-one function on a nonempty set X. Prove that d defined by $d(x, y) = |f(x) - f(y)|$ for every $x, y \in X$ is a metric on X.

31. Prove that the function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$d(x, y) = |\arctan x - \arctan y|$$

is a metric on $\mathbb{R}$.

32. Let (X, d) be a metric space and $x, y, z, w \in X$. Prove the following.

    (a) $|d(x, z) - d(y, w)| \leq d(x, y) + d(z, w)$

    (b) $|d(x, z) - d(y, z)| \leq d(x, y)$

33. Let X be a nonempty set and $\sim$ be the relation on the set of all metrics on X defined by $d \sim \rho$ iff d and ρ are equivalent metrics on X. Prove that $\sim$ is an equivalence relation.

34. Let m be a positive integer and the functions d_1, d_2 and d_∞ on $\mathbb{R}^m \times \mathbb{R}^m$ be defined by

$$d_1(x, y) = \sum_{i=1}^{m} |x_i - y_i|$$

$$d_2(x, y) = \sqrt{\sum_{i=1}^{m} |x_i - y_i|^2}$$

$$d_\infty(x, y) = \max\{|x_i - y_i| : 1 \leq i \leq m\}$$

for all vectors $x = (x_1, x_2, \ldots, x_m)$ and $y = (y_1, y_2, \ldots, y_m)$ in $\mathbb{R}^m$. Prove that d_1, d_2 and d_∞ are metrics on $\mathbb{R}^m$.

35. Let S be any nonempty set. A function $f : S \to \mathbb{R}$ is called *bounded* if $f(S)$ is a bounded subset of $\mathbb{R}$. Let $B(S)$ be the set of all bounded real-valued functions on S. Prove that the function $\rho : B(S) \times B(S) \to \mathbb{R}$ defined by

$$\rho(f, g) = \sup\{|f(s) - g(s)| : s \in S\}$$

is a metric on $B(S)$ (ρ is called the *uniform convergence metric* or the *sup metric*).

36. Let $C^1[0, 1]$ be the set of all differentiable functions $f : [0, 1] \to \mathbb{R}$ which have continuous derivatives. Prove that the function d defined by

$$d(f, g) = |f(0) - g(0)| + \sup\{|f'(t) - g'(t)| : 0 \leq t \leq 1\}$$

is a metric on $C^1[0, 1]$. Also prove that $\rho(f, g) \leq d(f, g)$ for every $f, g \in C^1[0, 1]$, where ρ is the uniform convergence metric on the set of all bounded functions on $[0, 1]$.

37. Let $C[0, 1]$ be the set of all continuous functions $f : [0, 1] \to \mathbb{R}$. Prove that the function d defined by

$$d(f, g) = \int_0^1 |f(t) - g(t)| \, dt$$

is a metric on $C[0, 1]$.