Math 301 - Problem Set # 5 - Fall 2010

Homework Problems: 5, 6, 13, 15, 20, 22.

In the following problems (X, d) is assumed to be a metric space, $A \subseteq X$ and $B \subseteq X$.

1. Prove that A is bounded iff the subset $\{d(x, y) : x, y \in A\}$ of $\mathbb{R}$ is bounded from above.
2. Prove that the subset of a bounded set is bounded.
3. Prove that the closure of a bounded set is bounded.
4. Prove that the union of a finite number of bounded sets is bounded. Find an infinite family of bounded sets whose union is not bounded.
5. Give a counterexample to this statement: for every $x \in X$ and $r > 0$, the closure of the open ball $B_r(x)$ is the closed ball $\bar{B}_r(x)$.
6. Prove that if A is open and $A \cap B = \emptyset$, then $A \cap \bar{B} = \emptyset$.
7. Prove that A is closed iff for every $x \in X \setminus A$ there exists $\epsilon > 0$ such that $B_\epsilon(x) \cap A = \emptyset$.
8. Let τ_d, also called the topology on X induced by the metric d, denote the collection of all the open sets in X. Prove the following.
    (a) X and $\emptyset$ are both in τ_d.
    (b) If U and V are in τ_d, then so is $U \cap V$.
    (c) If $\{U_\alpha\}_{\alpha \in I}$ is a family of sets in τ_d, then $\cup_{\alpha \in I} U_\alpha \in \tau_d$.
9. If A is open, then it is the union of some open balls.
10. Give an example of a metric space with a family of open sets such that the intersection of these open sets is not open. For the same metric space find a family of closed sets such that the union of these closed sets is not closed.
11. Give an example of a metric space which contains a subset that is neither open nor closed.
12. Give an example of a metric space which contains a proper nonempty subset that is both open and closed.
13. Prove that $x \in \bar{A}$ iff $\inf\{d(x, a) : a \in A\} = 0$.
14. Consider a subset S of $\mathbb{R}$. We have defined the interior, closure, openness and closedness of S in Section 1.7. Verify that these definitions are consistent with those for S when it is considered as a subset of the metric space $(\mathbb{R}, d)$, where d is the absolute value metric.

15. Consider $\mathbb{R}$ as a metric space with the absolute value metric. What are the interior, closure, exterior and boundary of $\mathbb{Q}$? What are the interior, closure, exterior and boundary of $\mathbb{N}$? What are the interior, closure, exterior and boundary of a finite interval ? Does it matter what kind of an interval we consider?

16. Prove the following.

    (a) A^o is open.

    (b) $A^o \subseteq A$.

    (c) A^o is the largest open set contained in A.

    (d) $A^o = A$ iff A is open.

    (e) $(A^0)^0 = A^0$

17. Prove the following.

    (a) $\bar{A}$ is closed.

    (b) $A \subseteq \bar{A}$.

    (c) $\bar{A}$ is the smallest closed set containing A.

    (d) $\bar{A} = A$ iff A is closed.

    (e) $\bar{\bar{A}} = \bar{A}$.

18. Prove the following.

    (a) $A^0 \cup B^0 \subseteq (A \cup B)^0$

    (b) $A^0 \cap B^0 = (A \cap B)^0$

    (c) $\overline{A \cup B} = \bar{A} \cup \bar{B}$

    (d) $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$

19. Prove that $\partial A = \{x \in X : \text{for every } r > 0,\ B_r(x) \cap A \neq \emptyset \text{ and } B_r(x) \cap (X \setminus A) \neq \emptyset\}$.

20. Prove that $\partial A \subseteq \bar{A}$, in fact $\bar{A} = A \cup \partial A$.

21. Prove that $\partial A = \bar{A} \cap \overline{X \setminus A} = \partial(X \setminus A)$.

22. Prove that ∂A is closed.

23. Prove that A is open iff $A \cap \partial A = \emptyset$.

24. Prove that A is closed iff $\partial A \subseteq A$.

25. Prove that A^0 is the same as the exterior of $X \setminus A$.

26. Prove or disprove: for every $x \in X$ and $r > 0$ we have $\partial B_r(x) = \{y \in X : d(x, y) = r\}$.