Math 301 - Homework # 6 - Fall 2010

Homework Problems: 4, 6, 11, 17, 21, 22, 23.

In the following problems (X, d), (Y, ρ) and (Z, u) are assumed to be metric spaces, $A, B \subseteq X$ and x_n, y_n are sequences in X.

1. Prove that a sequence in a metric space cannot converge to two distinct elements.
2. Prove that every convergent sequence in a metric space is bounded.
3. Prove that A is closed iff $\lim a_n$ belongs to A whenever (a_n) is a convergent sequence in X with $a_n \in A$ for every $n \in \mathbb{N}$.
4. Prove that if $\lim x_n = x$ and $\lim y_n = y$, then $\lim d(x_n, y_n) = d(x, y)$.
5. Prove that every subsequence of a convergent sequence (x_n) converges to $\lim x_n$.
6. Generalizing the notion of a cluster point of a sequence of real numbers, $c \in X$ is called *a cluster point of* (x_n) if for every $\epsilon > 0$ there exists infinitely many $n \in \mathbb{N}$ with $x_n \in B_\epsilon(c)$. Prove that c is a cluster point of (x_n) iff (x_n) has a subsequence that converges to c.
7. Prove that if (x_n) is convergent, then its limit is its only cluster point.
8. Prove that a Cauchy sequence in X either has no cluster point and divergent or has a single cluster point and convergent.
9. Prove that $C[a, b]$, the set of all continuous functions on the interval $[a, b]$, is complete when considered with the sup-metric.
10. Choose a subset S of $\mathbb{R}$ which is not closed and then give an example of a Cauchy sequence in S which doesn't converge to any number in S.
11. Is $\mathbb{Q}$ with the absolute value metric complete? How about $\mathbb{R} \setminus \mathbb{Q}$, $\mathbb{Z}$, $[a, b]$, $(a, b]$ and (a, b), where a and b are real numbers ?
12. Prove that if d is the discrete metric on X, then (X, d) is complete.
13. Prove that $\mathbb{R}^m$ is complete with each of the metrics d_1, d_2 and d_∞.
14. Consider $\mathbb{R}$ with the metric d' defined by $d'(a, b) = |\arctan a - \arctan b|$ for all $a, b \in \mathbb{R}$. Prove that $(\mathbb{R}, d')$ is incomplete, i.e. give an example of a Cauchy sequence in $(\mathbb{R}, d')$ which is not convergent.
15. Let S be the set of all sequences of real numbers which converge to 0 and let $\underline{d} : S \times S \to \mathbb{R}$ be defined by $\underline{d}((a_n), (b_n)) = \sup\{|a_n - b_n| : n \in \mathbb{N}\}$ for all (a_n) and $(b_n) \in S$. Prove that $\underline{d}$ is a metric and in fact a complete one.

16. Let d be the discrete metric on X. Prove that any mapping from X to Y is continuous. Also prove that if $g : Y \to X$ is continuous at an element $y \in Y$, then there exists $\delta > 0$ such that $g(y) = g(z)$ whenever $\rho(y, z) < \delta$.

17. Let $f : X \to \mathbb{R}$ be a continuous function. Prove that if $f(x) = 0$ for every $x \in A$, then $f(x) = 0$ for every $x \in \bar{A}$.

18. Let $f : X \to Y$ and $g : Y \to Z$ be continuous mappings. Prove that $g \circ f : X \to Z$ is continuous. Also prove that $g \circ f$ is uniformly continuous if both f and g are uniformly continuous.

19. Let d_1 and d_2 be two equivalent metrics on a nonempty set S. Show that if a sequence in S is convergent with respect to the metric d_1, then it is convergent with respect to the metric d_2. Conclude that a continuous function $f : (X; d) \to (Y; \rho)$ remains continuous if d and ρ are replaced by equivalent metrics.

20. Let f and g be continuous (real-valued) functions on X. Prove that $f + g$ and $f \cdot g$ are also continuous.

21. For each continuous function $f : [0, 1] \to \mathbb{R}$ define $\psi_f : [0, 1] \to \mathbb{R}$ by

$$\psi_f(x) = \int_0^x f(t)dt \quad \text{for every} \quad x \in [0, 1] \, .$$

Prove that ψ_f is continuous. Define $\psi : C[0, 1] \to C[0, 1]$ by $\psi(f) = \psi_f$. Prove that ψ is uniformly continuous with respect to the sup-metric on $C[0, 1]$.

22. Give an example of a mapping f between two metric spaces which is continuous but not uniformly continuous on its domain.

23. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \sin(\frac{1}{x}) & , \quad x \neq 0 \\ a & , \quad x = 0 \end{cases}$$

where a is a constant. Is there any value a which makes f continuous at 0?
Hint: Use the fact that $\lim \frac{1}{n\pi + \frac{\pi}{2}} = 0$.