Math 301 - Problem Set # 9 - Fall 2010

Homework Problems: 2, 3, 5, 6, 8.

In the following problems (X, d), (Y, ρ) are assumed to be metric spaces, A, B are non-empty subsets of X, f is a mapping from X to Y, and (x_n) is a sequence in X. Remember that, for each $x \in X$ *the distance between* x *and* A, $d(x, A)$, is defined as $\inf\{d(x, a) : a \in A\}$, and *the distance between* A *and* B, $d(A, B)$, is defined as $\inf\{d(a, B) : a \in A\}$.

1. Prove that any completion of $(\mathbb{Q}, d)$ is isometric to $(\mathbb{R}, d)$, where d is the absolute value metric.
2. Find a completion of $(0, 1)$
   (a) with the absolute value metric.
   (b) with the discrete metric.
3. Suppose that (X, d) is complete and totally bounded. Prove that if f is continuous, then it is uniformly continuous.
4. Suppose that X is compact , f is one-to-one and continuous, and $(f(x_n))$ is convergent. Prove that (x_n) is convergent.
5. Suppose that A is compact. Prove that for each $x \in X$ there is $y \in A$ such that $d(x, y) = d(x, A)$.
6. (a) Give an example of two disjoint closed sets of a metric space with zero distance between them.
   (b) Suppose that A and B are disjoint, A is closed, and B is compact. Prove that the distance between A and B is nonzero.
7. Prove that $C(X)$, the set of all continuous functions on a compact metric space X with the sup-metric, is a complete metric space.
8. Suppose that X is compact and $\phi : X \to X$ is a mapping that satisfies

   $$d(\phi(x), \phi(y)) < d(x, y)$$

   whenever $x \neq y$. Prove that ϕ has a unique fixed point.
   Hint: Prove and use that the function $\theta : X \to \mathbb{R}$ *defined by* $\theta(x) = d(x, \phi(x))$ *is continuous.*
9. Suppose that $\phi : [0, 1] \to [0, 1]$ is continuous. Prove that ϕ has at least one fixed point.