Math 301

HW7 - Solutions

① $f: \mathbb{R} \longrightarrow \mathbb{R}$ $\quad f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$. Then f is not continuous at any point. Let $x \in \mathbb{R}$. Let $\varepsilon = 1/2$ and $\delta > 0$. If $x \in \mathbb{Q}$, then $\exists y \in \mathbb{R} \setminus \mathbb{Q}$ s.t. $|x-y| < \delta$ and $|f(x) - f(y)| = 1 > 1/2$. If $x \notin \mathbb{Q}$, then $\exists y \in \mathbb{Q}$ s.t. $|x-y| < \delta$ and $|f(x) - f(y)| = 1 > 1/2$. Hence for $\varepsilon = 1/2$, there is no $\delta > 0$ s.t. $|x-y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$.

---

② Let $x \in \mathbb{R}$. For any $n \in \mathbb{Z}$, we have $f(nx) = f(x + \dots + x) = nf(x)$. Moreover $f(x) = f(\frac{x}{n} \cdot n) = nf(\frac{x}{n})$, hence $f(\frac{x}{n}) = \frac{f(x)}{n}$.
Let $q \in \mathbb{Q}$, then $f(qx) = f(\frac{m}{n} x) = \frac{m}{n} f(x) = qf(x)$. Let $x \in \mathbb{R}$ and choose $(q_n)_{n \in \mathbb{N}}$, $q_n \in \mathbb{Q}$ s.t. $q_n \longrightarrow x$. As f is continuous, $f(q_n) \to f(x)$. $f(q_n) = f(q_n \cdot 1) = q_n f(1)$, $q_n f(1)$ converges to $x f(1)$ as $q_n \to x$ so $f(x) = x f(1)$. Thus $c = f(1)$.

---

⑧ (a, b): Consider $f: (\mathbb{R}, d) \longrightarrow (\mathbb{R}, |\ |)$ where d is the discrete metric, and $f(x) = x$. Then f is continuous, but not open and not closed, as every subset $(\mathbb{R}, d)$ is both open and closed.
(c, d): Consider $f: (\mathbb{R}, |\ |) \longrightarrow (\mathbb{R}, d)$, d is the discrete metric and $f(x) = x$. Then f is closed and open but not continuous.
(e): $f(x) = x^2$ is closed but not open.
(f): Consider $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$, $(x, y) \longmapsto x$. Then $\{(x, 1/x) : x \in \mathbb{R}^+\}$ is closed but $f(\{(x, 1/x) : x \in \mathbb{R}^+\}) = (0, \infty)$ is open. Hence f is not closed, but open.

(g) Consider the example in (f). f is continuous and open but not homeomorphism as it is not one-to-one.

(h) Consider $f: \mathbb{R} \longrightarrow \mathbb{R}$, $f(x)=x^2$. Then f is continuous and closed but not homeomorphism.

(i, j) Every homeomorphism is both open and closed as its inverse is continuous and it is a bijection.

(k, l, m): Consider the function in (a). It is continuous and bijection but not open, not closed and so not homeomorphism.

---

(12) $\mathbb{R}$ is homeomorphic to $(0,1)$ open interval. $\mathbb{R}$ is complete but $(0,1)$ is not complete.

---

(15) Consider the function $f_A: X \longrightarrow \mathbb{R}$ defined by $f_A(x) = \inf_{a\in A} d(x,a)$.

(a) Given $\varepsilon > 0$, choose $\delta = \varepsilon$. Then $|f_A(x) - f_A(y)| = |\inf_{a\in A} d(x,a) - \inf_{a\in A} d(y,a)|$ $\leq \inf_{a\in A} d(x,a) + \inf_{a\in A} d(y,a) \leq \inf_{a\in A} (d(x,a) + d(y,a)) \leq d(x,y)$. Hence $|f_A(x) - f_A(y)| \leq d(x,y)$. Therefore $|f_A(x) - f_A(y)| < \varepsilon$, whenever $d(x,y) < \delta$. i.e. f_A is uniformly continuous.

(b) Let $x \in \bar{A}$, then given $\varepsilon > 0$ $\exists a \in A : d(x,a) < \varepsilon$. Hence $\inf_{a\in A} d(x,a) < \varepsilon$. Since ε is arbitrary $\inf_{a\in A} d(x,a) = 0$, i.e. $f_A(x) = 0$, i.e. $x \in f_A^{-1}(0)$.
Let $x \in f_A^{-1}(0)$. Then $f_A(x) = \inf_{a\in A} d(x,a) = 0$. Given $\varepsilon > 0$, as $\inf_{a\in A} d(x,a) = 0$ $\exists a \in A$; $d(x,a) < \varepsilon$. Hence $x \in \bar{A}$. Thus $\bar{A} = f_A^{-1}(0)$.

(c) Suppose $A \cap B \neq \emptyset$. $\exists x_0 \in A \cap B$. Then the distance between A and B $= \inf_{b\in B} f_A(x) = \inf_{b\in B} (\inf_{a\in A} d(b,a))$. Since $x_0 \in A\cap B$, and $d(x_0, x_0) = 0$, $\inf_{b\in B} \inf_{a\in A} d(b,a) \leq 0$. We also have $\inf_{b\in B} \inf_{a\in A} d(a,b) \geq 0$ so the distance between A and B is 0.

(d) Let $A \subseteq X$ be an open set and let $B = \bar{A} \setminus A$. Then $A \cap B = \emptyset$. the distance between A and B is $\inf_{b \in B} \inf_{a \in A} d(a,b)$. Let $b \in B$, $\varepsilon > 0$. then $\exists x_0 \in A : d(x_0, b) < \varepsilon$ as $b \in \bar{A} \setminus A$. Hence $\inf_{a \in A} d(a,b) < \varepsilon$. As ε is arbitrary $\inf_{a \in A} d(a,b) = 0$ so $\inf_{b \in B} \inf_{a \in A} d(a,b) = 0$, i.e. the distance between A and B is 0.

(e) Let $A \cap B = \emptyset$ and $A, B \subseteq X$ are closed. Put $g(x) = f_A(x) - f_B(x)$. Then g is uniformly continuous as f_A and f_B are uniformly continuous by (a). $g^{-1}(0, \infty) = \{x \in X : g(x) > 0\} = \{x \in X : f_A(x) > f_B(x)\}$

Let $a \in A$, then $g(a) = f_A(a) - f_B(a) = -f_B(a) < 0$ as B is closed.

$\Rightarrow a \in g^{-1}(-\infty, 0)$ so $A \subseteq g^{-1}(-\infty, 0)$. Let $b \in B$, $g(b) = f_A(b) - f_B(b)$ $= f_A(b) > 0$ as A is closed and $A \cap B = \emptyset$. Hence $b \in g^{-1}(0, \infty)$ so $B \subseteq g^{-1}(0, \infty)$. It is clear that $g^{-1}(0, \infty) \cap g^{-1}(-\infty, 0) = \emptyset$ and they are open as g is continuous.