2) (5 pts each) For each of (a)-(d) below: If the proposition is true, write TRUE. If the proposition is false, write FALSE. No explanations are required for this problem.

2a) Product of two convergent series is convergent.

FALSE $\sum \frac{-1}{\sqrt{n}}, \sum \frac{-1}{\sqrt{n}} \to \sum \frac{1}{n}$

2b) If a series is not unconditionally convergent, then it has a rearrangement which is divergent.

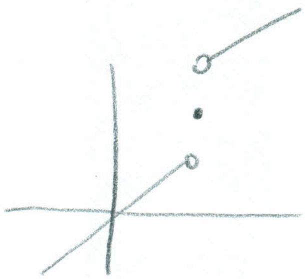
TRUE.

2c) Let $f : \mathbb{R} \to \mathbb{Q}$. If $lim_{x \to a} f(x)$ exists, then f satisfies the Cauchy condition at a.

TRUE.

2d) If $f : [a, b] \to \mathbb{R}$ is monotone, then it is either left continuous or right continuous at any point $p \in [a, b]$.

FALSE



3a) (10 pts) Determine the values of $x \in \mathbb{R}$ for which $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ is convergent.

$a_n > 0$ $\quad a_n \searrow 0$ & $\sum b_n < C \Rightarrow \sum a_n b_n$ convergent (Dirichlet Test)

let $a_n = \frac{1}{n}$ and $b_n = \sin nx$

$\sum \sin nx = \sin x + \sin 2x + \sin 3x + \cdots + \sin nx \Rightarrow \cos\frac{x}{2}(\sin x + \sin 2x + \cdots + \sin nx + \cdots) =$

$\cos\frac{x}{2} \cdot \sin x + \cos\frac{x}{2} \cdot \sin 2x + \cdots + \cos\frac{x}{2} \cdot \sin nx + \cdots = \left(\frac{\sin\frac{3x}{2} - \sin\frac{x}{2}}{2}\right) + \left(\frac{\sin\frac{5x}{2} - \sin\frac{3x}{2}}{2}\right) + \cdots$

$\cos a \sin b = \frac{\sin(a+b) - \sin(a-b)}{2}$ $\quad = \frac{1}{2}\sum \sin(n+\frac{1}{2})x - \sin(n-\frac{1}{2})x = \lim_{n\to\infty} \sin(n+\frac{1}{2})x - \sin\frac{x}{2}$

$\Rightarrow \sum b_n$ bdd for any x

$\Rightarrow \sum a_n \cdot b_n$ convergent for any x.

3b) (10 pts) Prove or give a counterexample for the following statement.

Let $a_n \geq 0$. If $\sum a_n$ converges, then $\sum a_n^2$ converges.

$\sum a_n$ convergent $\Rightarrow a_n \to 0 \Rightarrow \exists N_0$ s.t. $1 \geq a_n \quad \forall n \geq N_0$

$\Rightarrow \forall n \geq N_0 \quad a_n^2 \leq a_n \Rightarrow \sum_{N_0}^{\infty} a_n^2 \leq \sum_{N_0}^{\infty} a_n$ $\quad (a_n \geq 0)$ comparison test

$\Rightarrow \sum a_n^2$ convergent.

4) (15 pts) Prove or give a counterexample for the following statement.

Let $\sum a_n$ be a convergent series and (b_n) be a bounded sequence. Then, $\sum a_n b_n$ converges.

Counterexample:

$a_n = \frac{(-1)^n}{n}$. $b_n = (-1)^n$

$\Rightarrow \quad \sum a_n \cdot b_n = \sum \frac{1}{n}$ divegent !

5a) (15 pts) Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Show that if f has bounded derivative on (a,b), then f is a function of bounded variation.

Let $|f'(x)| \leq M$ for any $x \in [a,b]$.

Let P be a partition of $[a,b]$. $P = \{a = x_0, x_1 \ldots x_n = b\}$.

$$V(f,P) = \sum_{i=0}^{n-1} |f(x_{i+1}) - f(x_i)|$$

by MVT $\dfrac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} = f'(a_i)$ $a_i \in (x_i, x_{i+1}) \Rightarrow |f(x_{i+1}) - f(x_i)| \leq M.(x_{i+1} - x_i)$

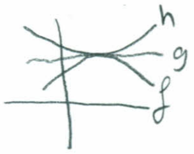
$\Rightarrow V(f,P) \leq M.(b-a)$

5a) (10 pts) Give a counterexample to the converse of the statement above.

$$f : [-1,1] \to \mathbb{R}, \quad x \mapsto \sqrt[3]{x}$$

f BV but $f'(0) = \infty$

Bonus) (15 pts) Let $f, g, h : [a, b] \to \mathbb{R}$ be three functions such that $f(x) \leq g(x) \leq h(x)$ for any x in $[a, b]$. If f and h are continuous on $[a, b]$, and $f(c) = h(c)$ for some $c \in (a, b)$, show that g is continuous at c.



WANT: $\forall \varepsilon > 0 \ \exists \delta > 0 \quad |x - c| < \delta \Rightarrow |g(x) - p| < \varepsilon$ where $p = f(c) = h(c)$

Let $\varepsilon_0 > 0$ given.

f cts $\Rightarrow \exists \delta_1 > 0 \quad |x-c| < \delta_1 \Rightarrow |f(x) - p| < \varepsilon_0$ (*)

h cts $\Rightarrow \exists \delta_2 > 0 \quad |x-c| < \delta_2 \Rightarrow |g(x) - p| < \varepsilon_0$ (**)

Let $\delta_0 = \min\{\delta_1, \delta_2\}$

$|x - c| < \delta_0 \Rightarrow$ by (*) $\quad p - \varepsilon_0 < f(x) < p + \varepsilon_0$

by (**) $\quad p - \varepsilon_0 < h(x) < p + \varepsilon_0$

$\Rightarrow p - \varepsilon_0 < f(x) \leq g(x) \leq h(x) < p + \varepsilon_0$

$\Downarrow$

$p - \varepsilon_0 < g(x) < p + \varepsilon_0$

$\Downarrow$

$|g(x) - p| < \varepsilon_0$

$\Rightarrow \left[\, |x - c| < \delta_0 \Rightarrow |g(x) - p| < \varepsilon_0 \,\right]$.

Since $\varepsilon_0 > 0$ arbitrary, $\square$.