Math 302 - Problem Set # 1 - Spring 2011

Homework Problems: 1, 2, 5, 8, 12.

1. Prove that the terms of the partial sum sequence (s_n) of the geometric series $\sum_{n=0}^{\infty} ar^n$ are $s_n = a\frac{1-r^{n+1}}{1-r}$.

2. Prove that the series $\sum_{n=0}^{\infty} \frac{1}{n!}$ is convergent and its sum is an irrational number between 2 and 3.

3. Prove that the sum of the series in the previous problem is the same as $\lim(1+\frac{1}{n})^n$.

4. Find the sum of the series $\sum_{n=3}^{\infty} \frac{1}{n(n+1)}$.

5. Prove that the harmonic series $\sum \frac{1}{n}$ is divergent.

6. Prove that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is convergent and its sum is less than 1.

7. Prove that for each $p \geq 2$, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent and its sum is less than 2.

8. Prove the following.

   (a) If $p > 0$, then $\lim \frac{1}{n^p} = 0$.

   (b) If $p > 0$, then $\lim \sqrt[n]{p} = 1$.

   (c) $\lim \sqrt[n]{n} = 1$.

   (d) If $p > 0$ and α is real, then $\lim \frac{n^\alpha}{(1+p)^n} = 0$.

   (e) If $|x| < 1$, then $\lim x^n = 0$.

9. Let $\sum a_n$ and $\sum b_n$ be two positive series such that $\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}$ for all but finitely many $n \in \mathbb{N}$. Prove that the series $\sum a_n$ converges provided that $\sum b_n$ converges.

10. Let (a_n) be a positive bounded sequence and x be a real number with $|x| < 1$. Prove that $\sum a_n x^n$ converges.

11. Let (a_n) be a positive sequence. Prove that $\sum a_n$ converges iff $\sum \frac{a_n}{1+a_n}$ converges.

12. Prove that a sequence (a_n) converges if the series $\sum |a_{n+1} - a_n|$ converges. Give a counterexample to the converse of this statement.

13. Let (a_n) be a positive sequence. Prove that if $\lim \frac{a_{n+1}}{a_n} = L$, then $\lim \sqrt[n]{a_n} = L$.