Math 302 - Problem Set # 2 - Spring 2011

Homework Problems: 1, 3, 8, 10, 11.

1. Let $\sum a_n$ be a convergent nonnegative series. Prove that $\sum \frac{\sqrt{a_n}}{n}$ is also convergent.
2. Let $\sum a_n$ be a convergent series and (b_n) be a monotone and bounded sequence. Prove that $\sum a_n b_n$ converges.
3. Let $\sum_{n=1}^{\infty} a_n$ be a positive convergent series. Prove that $\lim \frac{\sum_{k=1}^{n} k a_k}{n} = 0$.
4. Prove that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$ is convergent provided that $p > 0$.
5. Determine the values of $x \in \mathbb{R}$ for which $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ is convergent.
6. Determine the values of $x \in \mathbb{R}$ for which $\sum_{n=1}^{\infty} \frac{\cos nx}{n}$ is convergent.
7. Prove that $\sum_{n=1}^{\infty} \frac{\sin nx}{n^p}$ is convergent for every $x \in \mathbb{R}$ and every $p > 0$.
8. (a) Prove that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is absolutely convergent for every $x \in \mathbb{R}$.
   (b) Let $S(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for every $x \in \mathbb{R}$. Prove that $S(x) \cdot S(y) = S(x+y)$ for every $x, y \in \mathbb{R}$.
9. (a) Prove that $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ and $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ are absolutely convergent for every $x \in \mathbb{R}$.
   (b) Let $S(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ and $C(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ for every $x \in \mathbb{R}$. Prove that $S(x) \cdot C(x) = S(2x)/2$ for every $x \in \mathbb{R}$.
10. Let $\sum a_n$ and $\sum b_n$ be two series satisfying the following properties.
    (a) $\sum |a_{n+1} - a_n|$ is convergent.
    (b) The partial sum sequence of $\sum b_n$ is bounded.
    (c) $\lim a_n = 0$.

    Prove that $\sum a_n b_n$ is convergent.
11. Suppose that $\sum a_n^2$ and $\sum b_n^2$ are convergent. Prove that $\sum |a_n b_n|$ is also convergent and moreover $(\sum_{n=0}^{\infty} |a_n b_n|)^2 \leq \sum_{n=0}^{\infty} a_n^2 \cdot \sum_{n=0}^{\infty} b_n^2$.
12. Let a be a real number with $|a| < 1$. Prove that $\sum a^n \cos nx$ and $\sum n a^n \sin nx$ are convergent for every $x \in \mathbb{R}$.
13. Let (a_n) be a sequence of real numbers and x be a real number. Prove that $\sum a_n x^n$ is convergent if $\limsup \sqrt[n]{|a_n|} < 1/|x|$ and divergent if $\liminf \sqrt[n]{|a_n|} > 1/|x|$.