Math 302 - Problem Set # 3 - Spring 2011

Homework Problems: 1, 2, 9, 10, 12.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying $\lim_{h\to 0}[f(x+h) - f(x-h)] = 0$ for every $x \in \mathbb{R}$. Does this imply that f is continuous?
2. Let $A = \mathbb{R} \setminus \{0\}$ and $f : A \to \mathbb{R}$ be defined by $f(x) = \sin(1/x)$ for every $x \in A$. Prove that $\lim_{x\to 0} f(x)$ does not exist.
3. Let $A = [-1, 1] \setminus \{0\}$ and $f : A \to \mathbb{R}$ be defined by $f(x) = -1$ for every $x \in [-1, 0)$ and $f(x) = 1$ for every $x \in (0, 1]$. Prove that $\lim_{x\to 0} f(x)$ does not exist.
4. Let $A = \mathbb{R}^2 \setminus \{(0, 0)\}$ and $f : A \to \mathbb{R}$ be defined by $f(x, y) = \frac{xy}{x^2+y^2}$ for every $(x, y) \in A$. Prove that $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist.
5. Let A be a bounded subset of $\mathbb{R}$ and $f : A \to \mathbb{R}$. Prove that if $\lim_{x\to\bar{x}} f(x)$ exists for every $\bar{x} \in \bar{A}$, then f is a bounded function on A, i.e. $f(A)$ is a bounded subset of $\mathbb{R}$.
6. Let A be a subset of $\mathbb{R}$ and $f : A \to \mathbb{R}$. Prove that if $\lim_{x\to\bar{x}} f(x)$ exists for every $\bar{x} \in \bar{A}$, then f is a locally bounded function, i.e. every point in A has a neighborhood on which f is bounded.
7. Let f be a function on R defined by $f(x) = 1/x$ if $x \neq 0$ and $f(0) = 0$. Does 0 have a neighborhood on which f is bounded? Does $\lim_{x\to 0} f(x)$ exist?
8. For each real number x let $[x]$ denote the greatest integer less than x and define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x - [x]$. Determine whether $f(x+)$ and $f(x-)$ exist for each real number x and calculate whenever they exist.
9. Let a, b be real numbers, $f : [a, b] \to \mathbb{R}$, and $y \in (a, b)$. Prove the following.
   (a) f is continuous at y if and only if for every monotone sequence (y_n) in $[a, b]$ converging to y, the sequence $(f(y_n))$ converges to $f(y)$.
   (b) f is continuous at y from the right if and only if for every decreasing sequence (y_n) in $[a, b]$ converging to y, the sequence $(f(y_n))$ converges to $f(y)$.
   (c) f is continuous at y from the left if and only if for every increasing sequence (y_n) in $[a, b]$ converging to y, the sequence $(f(y_n))$ converges to $f(y)$.
10. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that for any two rational numbers a and b, $f(a+b) = f(a)+f(b)$. Prove that if $\lim_{x\to 0} f(x)$ exists, then $\lim_{x\to 0} f(x) = 0$.
11. For a given real number x, prove that the sequence $(\sin^2 nx)$ is convergent if and only if $(\cos^2 nx)$ is convergent. Prove or disprove the following statement: For any real number x, the sequence $(\sin nx)$ is convergent if and only if $(\cos nx)$ is convergent. Determine the set of all $x \in \mathbb{R}$ for which the sequence $(\sin nx)$ is convergent.
12. Let $a, b \in \mathbb{R}$ and $f : (a, b) \to \mathbb{R}$ be a uniformly continuous function. Prove that both $\lim_{x\to a} f(x)$ and $\lim_{x\to b} f(x)$ exist.