Math 302 - Problem Set # 4 - Spring 2011

Homework Problems: 1, 3, 9, 15, 17.

1. Let (Y, ρ) be a metric space and $f : [a, \infty) \to Y$. Prove that $\lim_{x\to\infty} f(x) = L$ if and only if $(f(x_n))$ converges to L for every increasing unbounded sequence (x_n) in $[a, \infty)$.
2. Let (X, d) be a metric space, $A \subseteq X$, $p \in A$, and $f : X \to \mathbb{R}$. Prove that $\lim_{x\to p} f(x) = \infty$ if and only if $(f(x_n))$ diverges to ∞ for every sequence (x_n) in A converging to p.
3. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Prove that the following are equivalent.
   (a) $\lim_{|x|\to\infty} |f(x)| = 0$.
   (b) For each $\epsilon > 0$ there is a compact subset K of $\mathbb{R}$ such that $|f(x)| < \epsilon$ whenever $x \in K^c$.
   (c) For each $\epsilon > 0$ there is a continuous function $\varphi : \mathbb{R} \to \mathbb{R}$ with compact support (support of φ is the closure of the set $\varphi^{-1}(\mathbb{R} \setminus \{0\})$) such that $\sup\{|f(x) - \varphi(x)| : x \in \mathbb{R}\} < \epsilon$.
4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \frac{x}{1+x^2}$ for every $x \in \mathbb{R}$. Prove that $\lim_{|x|\to\infty} |f(x)| = 0$. Let $\epsilon > 0$ be given and find a continuous function $\varphi : \mathbb{R} \to \mathbb{R}$ with compact support such that $\sup\{|f(x) - \varphi(x)| : x \in \mathbb{R}\} < \epsilon$.
5. Let a be a positive real number, $f : [a, \infty) \to \mathbb{R}$ and $g : (0, 1/a] \to \mathbb{R}$ be defined by $g(x) = f(1/x)$ for every $x \in (0, 1/a]$. Prove that $\lim_{x\to\infty} f(x) = L$ if and only if $g(0+) = L$.
6. Let $f : [0, \infty) \to \mathbb{R}$. Prove that if $\lim_{x\to\infty} f(x) = L$, then $\lim_{x\to\infty}[f(x + 1) - f(x)] = 0$.
7. Let $f : (a, b) \to \mathbb{R}$ be a continuous function. Prove that f is one-to-one if and only if it is strictly monotone.
8. Let f be a continuous strictly increasing function on a nonempty open interval I. Prove that f is an open mapping. Also prove that $J = f(I)$ is an open interval and f^{-1} is defined and continuous on J.
9. Let $f : [a, b] \to \mathbb{R}$ be a continuous function and define $g : [a, b] \to \mathbb{R}$ by $g(a) = f(a)$ and $g(x) = \sup\{f(y) : y \in [a, x]\}$ for every $x \in (a, b]$. Prove that g is increasing and continuous.
10. Let $f : [a, b] \to \mathbb{R}$ be a decreasing function. Prove the following.
    (a) For every $p \in (a, b)$, $f(p+) = \sup\{f(x) : x \in (p, b]\}$, $f(p-) = \inf\{f(x) : x \in [a, p)\}$, and $f(p-) \geq f(p) \geq f(p+)$.
    (b) $f(a+) = \sup\{f(x) : x \in (a, b]\}$ and $f(b-) = \inf\{f(x) : x \in [a, b)\}$.
    (c) For any $p, q \in [a, b]$ with $p < q$ we have $f(p+) \geq f(q-)$.
11. Prove that the set of all points of discontinuity of a decreasing function on a closed interval is a countable set.
12. Prove that the function f defined by $f(0) = 0$ and $f(x) = x^2 \sin(1/x)$ if $x \neq 0$ is of bounded variation on $[0, 1]$.
13. Let f be a function of bounded variation on $[a, b]$. Prove that f is constant on $[a, b]$ if and only if $V(f, [a, b]) = 0$.
14. Let f be a function of bounded variation on $[a, b]$. Prove that $f = 0$ on $[a, b]$ if and only if $V(f, [a, b]) + |f(a)| = 0$.
15. Prove that every function of bounded variation on a closed interval is bounded on that interval.
16. Prove that the product of two functions of bounded variation on a closed interval is also of bounded variation on that interval.
17. Let f be defined by $f(x) = x^4 + x^3 - 3x^2 - x + 2$. Prove that f is of bounded variation on $[-3, 3]$ and express f as the difference of two increasing functions on $[-3, 3]$.