Math 302 - Problem Set # 6 - Spring 2011

Homework Problems: 2, 4, 7, 11, 12.

1. Let A and B be two nonempty subsets of $\mathbb{R}$. Suppose that for every $a \in A$ and $b \in B$, $a \leq b$. Prove that $\sup(A) = \inf(B)$ if and only if for every $\epsilon > 0$ there exist $a_\epsilon \in A$ and $b_\epsilon \in B$ such that $b_\epsilon - a_\epsilon < \epsilon$.

2. Let A be a nonempty bounded subset of $\mathbb{R}$. Prove that

$$\sup(A) - \inf(A) = \sup\{a - b : a, b \in A\} = \sup\{|a - b| : a, b \in A\} = -\inf\{a - b : a, b \in A\} .$$

3. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Prove that f is R-integrable if and only if for every $\epsilon > 0$ there exist a partition $\mathfrak{P}_\epsilon$ of $[a, b]$ such that $U(f, \mathfrak{P}) - L(f, \mathfrak{P}) < \epsilon$ whenever $\mathfrak{P}$ is a partition of $[a, b]$ with $\mathfrak{P}_\epsilon \subseteq \mathfrak{P}$.

4. Let $f : [a, b] \to \mathbb{R}$ be a bounded function.
   (a) Prove that if f is R-integrable, then for every $\epsilon > 0$ there exist a partition $\mathfrak{P}_\epsilon$ of $[a, b]$ for which $|R(f, \mathfrak{P}) - \int_a^b f(x)\, dx| < \epsilon$ whenever $\mathfrak{P}$ is a partition of $[a, b]$ with$\mathfrak{P}_\epsilon \subseteq \mathfrak{P}$ for any choice of ξ_i's in the definition of a Riemann sum $R(f, \mathfrak{P})$.
   (b) Prove that if there is a real number I such that for every $\epsilon > 0$ there exist a partition $\mathfrak{P}_\epsilon$ of $[a, b]$ for which $|R(f, \mathfrak{P}) - I| < \epsilon$ whenever $\mathfrak{P}$ is a partition of $[a, b]$ with $\mathfrak{P}_\epsilon \subseteq \mathfrak{P}$ for any choice of ξ_i's in the definition of a Riemann sum $R(f, \mathfrak{P})$, then f is R-integrable and $I = \int_a^b f(x)\, dx$.

5. Let $f \in R([a, b])$ and $c \in \mathbb{R}$. Prove that $cf \in R([a, b])$ and $\int_a^b c\, f(x)\, dx = c \int_a^b f(x)\, dx$.

6. Let $f, g \in R([a, b])$. By using only the definition of the Riemann integral, prove that if $f(x) \geq g(x)$ for every $x \in [a, b]$, then $\int_a^b f(x)\, dx \geq \int_a^b g(x)\, dx$.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(0) = 0$ and $f(x) = \sin(1/x)$ if $x \neq 0$. Prove that f is R-integrable on $[0, 1]$.

8. Give an example of a bounded function f which is not R-integrable even though f^2 is.

9. Let $f \in R([a, b])$. Suppose that $g : [a, b] \to \mathbb{R}$ is different from f only at finitely many points in $[a, b]$. Prove that $g \in R([a, b])$ and $\int_a^b g(x)\, dx = \int_a^b f(x)\, dx$.

10. Let $[c, d] \subseteq [a, b]$. Prove that $\chi_{[c,d]} \in R([a, b])$ and moreover $\int_a^b \chi_{[c,d]}(x)\, dx = d - c$.

11. Let $f \in R([a, b])$ and $c \in (a, b)$. Prove that $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$.

12. Consider Riemann's zeta function $\zeta : (1, \infty) \to \mathbb{R}$ defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} .$$

(a) Prove that ζ is well-defined, i.e. prove that the series $\sum_{n=1}^{\infty} \frac{1}{n^s}$ converges for every $s \in (1, \infty)$.

(b) Prove that $\zeta(s) = s \int_1^{\infty} \frac{[x]}{x^{s+1}}\, dx$, where $[x]$ denotes the greatest integer less than or equal to x. (*Hint: Look at the difference between the integral over* $[1, N]$ *and the* N*th partial sum of the series in the definition of* $\zeta(s)$.)

(c) Prove that $\zeta(s) = \frac{s}{s-1} - s \int_1^{\infty} \frac{x-[x]}{x^{s+1}}\, dx$.