Math 302 - Problem Set # 8 - Spring 2011

Homework Problems: 4, 6, 8.

1. Prove that a function series is uniformly convergent if and only if it is uniformly Cauchy.
2. If $\sum_{n=0}^{\infty} f_n$ is uniformly convergent, then $(f_n) \to 0$ uniformly.
3. Give an example of a pointwise convergent series which is not uniformly convergent.
4. Is the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ uniformly convergent on a compact interval $[a, b]$? Is it uniformly convergent on $\mathbb{R}$?
5. Study the convergence (pointwise or uniform) of the series $\sum_{n=0}^{\infty} x^n$ on $\mathbb{R}$, $(-1, 1)$ and $[-a, a]$, where a is a real number in $(0, 1)$.
6. Study the convergence (pointwise or uniform) of the series $\sum_{n=1}^{\infty} \frac{x^n}{n}$ on $\mathbb{R}$, $[-1, 1)$ and $[-a, a]$, where a is a real number in $(0, 1)$.
7. Prove that the series $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$ is uniformly convergent on any interval $[a, b] \subseteq (0, 2\pi)$.
8. Let (f_n) be a sequence of functions defined on $[0, 1]$ by
$$f_n(x) = \begin{cases} \frac{1}{n}, & \text{if } \frac{1}{2^{n+1}} < x \leq \frac{1}{2^n}; \\ 0, & \text{otherwise.} \end{cases}$$
Prove that the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly even though it fails the Weierstrass' M-test.
9. For a given real number a in $(-1, 1)$ define $f, g : \mathbb{R} \to \mathbb{R}$ by $f(x) = \sum_{n=0}^{\infty} a^n \cos(nx)$ and $g(x) = -\sum_{n=0}^{\infty} na^n \sin(nx)$. Prove that $g = f'$.
10. Prove that $\sum_{n=1}^{\infty} \frac{1}{n^x}$ converges uniformly on $[p, \infty]$ for any $p > 1$.
11. Consider $f(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2 x}$. For what values of x does the series converge absolutely? On what intervals does it converge uniformly? On what intervals does it fail to converge uniformly? Is f continuous wherever the series converges ? Is f bounded?
12. Prove that the series $\sum_{n=1}^{\infty} (-1)^n \frac{x^2+n}{n^2}$ converges uniformly in every bounded interval, but does not converge absolutely for any value of x.
13. Let
$$\sigma(x) = \begin{cases} 0, & \text{if } x \leq 0; \\ 1, & \text{if } x > 0. \end{cases}$$
Let (x_n) be a sequence of real numbers and $\sum_{n=0}^{\infty} c_n$ be an absolutely convergent series. Prove that the series $\sum_{n=0}^{\infty} c_n \sigma(x - x_n)$ converges uniformly and the sum function is continuous at every $x \notin \{x_n : n \in \mathbb{N}\}$.