2) (5 pts each) For each of (a)-(d) below: If the proposition is true, write TRUE. If the proposition is false, write FALSE. No explanations are required for this problem.

2a) If $\sum a_n$ and $\sum b_n$ are divergent, then $\sum a_n \cdot b_n$ is also divergent.

FALSE. $\sum a_n = \sum (-1)^n$ but $\sum a_n \cdot b_n = \sum \frac{(-1)^n}{n}$ ✓

$\sum b_n = \sum \frac{1}{n}$

2b) Any $f \in C([0,1])$ can be written as uniform limit of a sequence $\{f_n\} \subset C([0,1])$ where $f_n(x) \neq f_m(x)$ for any $n \neq m$.

TRUE. $f \in C[0,1] \Rightarrow f_n = f(x) + \frac{1}{n}$

2c) There is no function $f : \mathbb{R} \to \mathbb{R}$ with $C_f = \mathbb{Q}$.

TRUE. Class Notes.

2d) Intersection of two second category set is second category.

No. $X = \mathbb{R}$

$A = (0,1)$ $\quad A \cap B = \emptyset$

$B = (2,3)$

3a) (10 pts) Let $a, b \in \mathbb{R}$ and $f : (a, b) \to \mathbb{R}$ be a uniformly continuous function. Prove that both $\lim_{x \to a} f(x)$ and $\lim_{x \to b} f(x)$ exist.

HW 3-12

3b) (10 pts) Prove or give a counterexample for the following statement.

Let $\{f_n\} \subset C([0, 1])$, and $f_n \to f$ uniformly. Then, f is uniformly continuous.

Class Notes $\Rightarrow$ f cts

since $[0,1]$ compact $\Rightarrow$ f uniformly cts.

4) (15 pts) Evaluate $\int_0^1 x\, dx$ first by the definition of the Riemann integral.

Text book.

5) (15 pts) Let $f_n : [0,1] \to [0,1]$ be a decreasing function for each $n \in \mathbb{N}$. Prove that if (f_n) converges pointwise to a continuous function, then it converges uniformly.

WANT: given $\varepsilon_0 > 0$. $\exists \underline{\underline{N_0}}$ s.t. $\forall n \geq N_0$ $|f_n(x) - f(x)| < \varepsilon_0$.

$\forall x$ $f_n(x) \to f(x)$.

Claim: f is decreasing, too.

otherwise, $\exists a, b$ with $a < b$ and $f(a) < f(b)$. Let $\varepsilon_1 = \frac{f(b) - f(a)}{3}$

$\Rightarrow \exists N_1$ s.t. $\forall n \geq N_1$ $|f_n(a) - f(a)| < \varepsilon_1$
$\exists N_2$ s.t. $\forall n \geq N_2$ $|f_n(b) - f(b)| < \varepsilon_1$ $\Rightarrow \exists m \geq \max\{N_1, N_2\}$ with $f_m(a) < f_m(b)$ $\times$.

Hence, $f : [0,1] \to [0,1]$ cts & decreasing function.

Let $\frac{1}{K} < \frac{\varepsilon_0}{3}$. Then let $1 \geq f(0) = b_0 > b_1 > b_2 \cdots > f(1) = b_K \geq 0$.
and let $f(a_i) = b_i$.

Since $f_n(0) \to f(0) = b_0$, $\exists M_0$ s.t. $\forall n \geq M_0$ $|f_n(0) - f(0)| < \frac{\varepsilon_0}{3}$

since $f_n(a_1) \to f(a_1) = b_1$, $\exists M_1 > M_0$ s.t. $\forall n > M_1$ $|f_n(b_1) - f(b_1)| < \frac{\varepsilon_0}{3}$

Claim: $\forall x \in [0, a_1]$ $\forall n \geq M_1$ $|f_n(x) - f(x)| < \varepsilon_0$

$|f_n(x) - f(x)| \leq \max\{|f_n(0) - f(1)|, |f(0) - f_n(1)|\}$ since both f_n and f $\searrow$

$|f_n(0) - f(1)| \leq |f_n(0) - f(0) + f(0) - f(1)| \leq \underbrace{|f_n(0) - f(0)|}_{< \frac{\varepsilon_0}{3}} + \underbrace{|f(0) - f(1)|}_{= b_0 - b_1 < \frac{\varepsilon_0}{3}} < \frac{2\varepsilon_0}{3}$

$|f(0) - f_n(1)| \leq |f(0) - f(1) + f(1) - f_n(1)| \leq \underbrace{|f(0) - f(1)|}_{= b_0 - b_1 < \frac{\varepsilon_0}{3}} + \underbrace{|f(1) - f_n(1)|}_{< \frac{\varepsilon_0}{3}} < \frac{2\varepsilon_0}{3}$

$\Rightarrow f_n \to f$ uniformly on $[0, a_1]$. By iterating the process, $f_n \to f$ uniformly on $[0,1]$.

6) (15 pts) State and prove one version of the Baire Category Theorem.
(In the proof, do not use other versions of the theorem.)

Class Notes or Textbook.

Bonus) (15 pts) Let (X, d) be a compact metric space and $\mathfrak{L}$ be the collection of all real-valued Lipschitz functions on X, i.e.,

$\mathfrak{L} = \{f \in C(X) :$ there exists $K \geq 0$ such that $|f(x) - f(y)| \leq Kd(x, y)$ for every $x, y \in X\}$

Prove that $\mathfrak{L}$ is dense in $C(X)$.

WANT: $\mathcal{L}$ unital subalgebra & separating pts ($\overset{\text{STONE-WEIERSTRASS}}{\Longrightarrow}$ ✓)

① $\mathcal{L}$ separates points.

$\forall x \in X$ consider $f_x(y) = d(x,y)$. Claim: $f_x \in \mathcal{L}$

$$|f_x(y) - f_x(z)| = |d(x,y) - d(x,z)| \underbrace{\leq}_{\text{Triangle Ineq.}} d(y,z) \Rightarrow f_x \in \mathcal{L} \text{ with } K=1$$

Hence, $\forall x,y \in X$ $f_x(x) \neq f_x(y)$ ✓

② $\mathcal{L}$ is unital subalgebra. $f(x)=1$ is unit. ✓

$f \in \mathcal{L} \Rightarrow \alpha f \in \mathcal{L}$ with $K_{\alpha f} = \alpha \cdot K_f$ ✓

$f,g \in \mathcal{L} \Rightarrow f+g \in \mathcal{L}$ with $K_{f+g} = \max\{K_f, K_g\}$ ✓

$f,g \in \mathcal{L} \Rightarrow f \cdot g \in \mathcal{L}$

$$\begin{aligned}|f(x)\cdot g(x) - f(y)\cdot g(y)| &= |f(x)\cdot g(x) - f(x)\cdot g(y) + f(x)g(y) - f(y)\cdot g(y)| \\ &\leq |f(x)|\,|g(x)-g(y)| + |g(y)|\cdot|f(x)-f(y)| \\ &\leq |f(x)|\,K_g\, d(x,y) + |g(y)|\cdot K_f\, d(x,y)\end{aligned}$$

since X compact, $\exists M_f$ with $|f(x)| < M_f$
f,g cts $\exists M_g$ with $|g(y)| < M_g$ $\Rightarrow$ $\leq (M_f \cdot K_g + M_g \cdot K_f)\, d(x,y)$

$\Rightarrow f \cdot g \in \mathcal{L}$ ✓