Math 302

HW3 - Solutions.

1- Consider the function $f(x) = \begin{cases} x & x<1 \text{ and } x>1 \\ 0 & x=1 \end{cases}$. Then

$\lim_{h\to\infty}(f(x+h) - f(x-h)) = 0 \quad \forall x \in \mathbb{R}$ but at $x=1$, $\lim_{x\to 1} f(x) = 1 \neq f(1) = 0$

i.e. $f$ is not continuous at $x=1$.

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2- Let $A = \mathbb{R}\setminus\{0\}$, and $f: A \longrightarrow \mathbb{R}$, $f(x) = \sin(1/x)$.

Take $x_n = \frac{1}{\pi/2 + n\pi}$. If $\lim_{x\to 0} f(x)$ were exist, then $\lim_{n\to\infty} f(x_n)$ would exists as $x_n \longrightarrow 0$. Now $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \sin\left(\pi/2 + n\pi\right) = \lim_{n\to\infty} (-1)^n$

so $\lim_{n\to\infty} f(x_n)$ does not exist. Thus $\lim_{x\to 0} f(x)$ does not exist.

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3- Let $a, b \in \mathbb{R}$, $f: [a,b] \longrightarrow \mathbb{R}$, $y \in (a,b)$.

(a) ($\Rightarrow$) Suppose $f$ is continuous at $y$. Then $\lim_{x\to y} f(x) = f(y)$. Hence for every sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \to y$, we have $\lim_{n\to\infty} f(x_n) = f(y)$.

In particular, this is true for every monotone sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \to y$.

($\Leftarrow$) Suppose for every monotone sequence $(y_n)$ in $[a,b]$ converging to $y$ $f(y_n) \longrightarrow f(y)$. For a contradiction assume $f$ is not continuous at $y$.

i.e. $\exists \varepsilon > 0 : \forall \delta > 0 \ \exists x \in [a,b]$ with $|x-y| < \delta$ and $|f(x) - f(y)| \geq \varepsilon$

Hence one can construct a sequence $(x_n)_{n\in\mathbb{N}}$ s.t. $|x_n - y| < 1/n$ and $|f(x_n) - f(y)| \geq \varepsilon$. i.e. $x_n \longrightarrow y$ and $f(x_n) \not\longrightarrow f(y)$. Now as every sequence in $\mathbb{R}$ has a monotone subsequence, there exists a monotone subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$. Hence $x_{n_k} \longrightarrow y$ and $f(x_{n_k}) \not\longrightarrow f(y)$

contrary to hypothesis. Thus $f$ is continuous at $y$.

(b) ($\Rightarrow$) Suppose $f$ is continuous at $y$ from the right, i.e. $\lim_{\substack{x \to y \\ x > y}} f(x) = f(y)$.

Let $(y_n)_{n \in \mathbb{N}}$ be a decreasing sequence with $y_n \longrightarrow y$. Given $\varepsilon > 0$.

Since $\lim_{\substack{x \to y \\ x > y}} f(x) = f(y)$, $\exists \delta > 0 : \forall x \in ]y, y+\delta[$, $|f(x) - f(y)| < \varepsilon$.

Since $y_n \longrightarrow y$, $\exists N \in \mathbb{N} : \forall n \geq N$, $0 < y_n - y < \delta$. Hence $\forall n \geq N$ $|f(y_n) - f(y)| < \varepsilon$. i.e. $f(y_n) \longrightarrow f(y)$.

($\Leftarrow$). Suppose for every decreasing sequence $(y_n)$ in $[a, b]$ converging to $y$ $f(y_n) \longrightarrow f(y)$. Assume that $f$ is not continuous at $y$ from the right.

i.e. $\exists \varepsilon > 0 : \forall \delta > 0$ $\exists x \in ]y, y+\delta[ : |f(x) - f(y)| \geq \varepsilon$.

Choose $x_1 \in ]y, y+1[$ s.t. $|f(x_1) - f(y)| \geq \varepsilon$. Then choose $x_2 \in ]y, y + \frac{1}{2}[ \cap ]y, x_1[$, s.t. $|f(x_2) - f(y)| \geq \varepsilon$. In this way we obtain a decrasing sequence $(x_n)_{n \in \mathbb{N}}$ and as $|x_n - y| < \frac{1}{n}$, $x_n \longrightarrow y$. but $f(x_n) \not\longrightarrow f(y)$. This contradicts our hypothesis. Thus $f$ is continuous at $y$ from the right.

(c) Similar to part (b).

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10- Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function s.t. $\forall a, b \in \mathbb{Q}$ $f(a+b) = f(a) + f(b)$.

Suppose $\lim_{x \to 0} f(x)$ exists. Since $f(0) = f(0+0) = f(0) + f(0)$, $f(0) = 0$.

Let $x \in \mathbb{Q}$. Then $0 = f(0) = f(x - x) = f(x) + f(-x)$. i.e. $-f(x) = f(-x)$.

Let $\lim_{x \to 0} f(x) = L$, then $f(0+) = f(0-) = L$. Let $(r_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{Q}$ with $r_n > 0$ $\forall n \geq 0$ and $r_n \longrightarrow 0$. Then $\lim_{n \to \infty} f(r_n) = L$. We also have $-r_n \longrightarrow 0$ and $-r_n < 0$, so $\lim_{n \to \infty} f(-r_n) = L$. Hence $L = \lim_{n \to \infty} f(-r_n) = \lim_{n \to \infty} -f(r_n) = -\lim_{n \to \infty} f(r_n) = -L$, i.e. $L = 0$, 12- Let $a, b \in \mathbb{R}$ and $f : (a,b) \longrightarrow \mathbb{R}$ be a uniformly continuous function. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $(a,b)$ s.t. $x_n \longrightarrow a$. Then $(x_n)_{n\in\mathbb{N}}$ is Cauchy and as $f$ is uniformly continuous $(f(x_n))_{n\in\mathbb{N}}$ is Cauchy. As $\mathbb{R}$ is complete $(f(x_n))_{n\in\mathbb{N}}$ converges to some point. We need to show that the limit of $(f(x_n))_{n\in\mathbb{N}}$ is independent from the particular sequence $(x_n)_{n\in\mathbb{N}}$. Let $(y_n)_{n\in\mathbb{N}}$ be another such sequence. We have to show $\lim f(x_n) = \lim f(y_n)$. Let $\alpha = \lim_{n\to\infty} f(x_n)$ and $\beta = \lim_{n\to\infty} f(y_n)$. Construct a new sequence $(z_n)_{n\in\mathbb{N}}$ s.t. $z_0 = x_0$, $z_1 = y_0$, $z_2 = x_1$, $z_3 = y_1$, .... Then $z_n \longrightarrow a$, hence $\lim_{n\to\infty} f(z_n)$ exists say $\gamma = \lim_{n\to\infty} f(z_n)$. Now $(f(x_n))_{n\in\mathbb{N}}$ and $(f(y_n))_{n\in\mathbb{N}}$ are subsequences of $(f(z_n))_{n\in\mathbb{N}}$, hence $\alpha = \gamma = \beta$. Thus for every sequence $(x_n)_{n\in\mathbb{N}}$ converging to $a$, $\lim_{n\to\infty} f(x_n)$ exists and $\lim_{n\to\infty} f(x_n) = L$.

Thus $\lim_{x\to a} f(x) = L$, i.e. $\lim_{x\to a} f(x)$ exists.

$\lim_{x\to b} f(x)$ exists by similar reasonings.