Math 302

HW4 - Solutions

1- Let (Y, ρ) be a metric space and $f: [a, \infty) \longrightarrow Y$.

$(\Rightarrow)$ Suppose $\lim_{x \to \infty} f(x) = L$. Let $(x_n)_{n \in \mathbb{N}}$ be an increasing unbounded sequence in $[a, \infty)$. Given $\varepsilon > 0$. Since $\lim_{x \to \infty} f(x) = L$, $\exists M \in \mathbb{R}: x > M \Rightarrow \rho(f(x), L) < \varepsilon$. Since (x_n) is unbounded $\exists N \in \mathbb{N}: x_n > M \; \forall n \geq N$ as (x_n) is increasing. For $n \geq N$ $\rho(f(x_n), L) < \varepsilon$ i.e. $f(x_n) \longrightarrow L$ in Y.

$(\Leftarrow)$ Suppose $(f(x_n))_{n \in \mathbb{N}}$ converges to L for every increasing unbounded sequence $(x_n)_{n \in \mathbb{N}}$ in $[a, \infty)$. For a contradiction assume that $\lim_{x \to \infty} f(x) \neq L$, i.e. $\exists \varepsilon > 0: \forall M \in \mathbb{N} \; \exists x > M : \rho(f(x), L) \geq \varepsilon$. Choose $x_1 > 1$ s.t. $\rho(f(x_1), L) \geq \varepsilon$. choose $x_2 > \max(x_1, 2)$ s.t. $\rho(f(x_2), L) \geq \varepsilon$, and so on. Now $(x_n)_{n \in \mathbb{N}}$ is an increasing unbounded sequence, hence by hypothesis $f(x_n) \longrightarrow L$ but $\rho(f(x_n), L) \geq \varepsilon \; \forall n \geq 1$ so we get a contradiction. Thus $\lim_{x \to \infty} f(x) = L$.

3- Let $f: \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function.

<u>(a) $\Rightarrow$ (b)</u>: Suppose $\lim_{|x| \to \infty} |f(x)| = 0$. Let $\varepsilon > 0$. Then $\exists M > 0 : |f(x)| < \varepsilon$ whenever $|x| > M$. Put $K = [-M, M]$. Then K is compact and $|f(x)| < \varepsilon$ for $x \in K^c$.

<u>(b) $\Rightarrow$ (c)</u>: Suppose for each $\varepsilon > 0$ there is a compact subset K of $\mathbb{R}$ s.t. $|f(x)| < \varepsilon$ whenever $x \in K^c$. Let $\varepsilon > 0$. Then $\exists K \subseteq \mathbb{R}$ compact with $|f(x)| < \varepsilon$ whenever $x \in K^c$. Since K is compact, it is bounded. Hence $K \subseteq]-M, M[$ for some $M \in \mathbb{R}$. Put $O =]-M, M[$. By Urysohn Lemma, there exists a continuous function $\varphi: \mathbb{R} \longrightarrow \mathbb{R}$ satisfying.

i) $0 \leq \varphi(x) \leq 1 \quad \forall x \in \mathbb{R}$

ii) $\varphi(x) = 0 \quad \forall x \in O^c$

iii) $\varphi(x) = 1 \quad \forall x \in K$

Then support of φ is compact. Put $h(x) = \varphi(x) f(x)$. Then h is continuous with compact support and $|f(x) - h(x)| = |f(x) - \varphi(x) f(x)| = |f(x)||1 - \varphi(x)| < \varepsilon \quad \forall x \in \mathbb{R}$, as for $x \in K$ $|1 - \varphi(x)| = 0$ and for $x \in K^c$, $|f(x)||1 - \varphi(x)| \leq |f(x)| < \varepsilon$.

(c) $\Rightarrow$ (a) : Suppose for each $\varepsilon > 0$ there is a continuous function $\varphi : \mathbb{R} \to \mathbb{R}$ with compact support s.t. $\sup \{ |f(x) - \varphi(x)| : x \in \mathbb{R} \} < \varepsilon$. Let $\varepsilon > 0$ then $\exists \varphi : \mathbb{R} \to \mathbb{R}$ continuous with compact support and $\sup \{ |f(x) - \varphi(x)| : x \in \mathbb{R} \} < \varepsilon$. Put $K = supp(\varphi)$. Since K is compact $\exists M > 0 : K \subseteq [-M, M]$. Now for $|x| > M$, $x \in K^c$ so $\varphi(x) = 0$ and $|f(x) - \varphi(x)| = |f(x)| < \varepsilon$ i.e. $\lim_{|x| \to \infty} |f(x)| = 0$.

9- Let $f : [a, b] \to \mathbb{R}$ be a continous function and define $g : [a, b] \to \mathbb{R}$ by $g(a) = f(a)$ and $g(x) = \sup_{y \in [a, x]} f(y) \quad \forall x \in (a, b]$. If $a \leq x_1 \leq x_2 \leq b$ then $[a, x_1] \subset [a, x_2]$ so $\sup_{y \in [a, x_1]} f(y) \leq \sup_{y \in [a, x_2]} f(y)$ i.e. $g(x_1) \leq g(x_2)$

So g is increasing. Let $x_0 \in [a, b]$ and $\varepsilon > 0$. Since f is continuous $\exists \delta > 0 : \forall x \in [a, b]$, $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$. We show that $|x - x_0| < \delta \Rightarrow |g(x) - g(x_0)| < \varepsilon$. Let $x \in [a, b]$ with $x_0 < x < x_0 + \delta$. Then $|g(x) - g(x_0)| = \sup_{y \in [a, x]} f(y) - \sup_{y \in [a, x_0]} f(y)$. If $\sup_{y \in [a, x]} f(y) = \sup_{y \in [a, x_0]} f(y)$, then

$|g(x)-g(x_0)| = 0 < \varepsilon$. Suppose $\sup\limits_{y\in[a,x]} f(y) > \sup\limits_{y\in[a,x_0]} f(y)$. Then

$\sup\limits_{y\in[a,x]} f(y) - \sup\limits_{y\in[a,x_0]} f(y) \leq \sup\limits_{y\in[a,x]} f(y) - f(x_0) = f(x_1) - f(x_0)$ for some $x_1 \in (x_0, x]$

since $x_1 \in (x_0-\delta, x_0+\delta)$, $f(x_1)-f(x_0) < \varepsilon$. i.e. $|g(x)-g(x_0)| < \varepsilon$.

Similarly for $x_0 - \delta < x < x_0$, $|g(x)-g(x_0)| < \varepsilon$. Thus g is continuous.

15- Let f be a function of bounded variation on $[a,b]$. Then $V_a^b(f) < M$ for some $M>0$. Let $x \in [a,b]$. Take the partition $P: \{a, x, b\}$

Then $V(f,P) = |f(x)-f(a)| + |f(b)-f(x)| \leq V_a^b(f) < M$. Hence $|f(x)-f(a)| < M$

i.e. $f(a) - M < f(x) < f(a) + M$. Thus f is bounded.

17- Let $f(x) = x^4 + x^3 - 3x^2 - x + 2$. f is of bounded variation iff it can be written as the difference of two increasing function.

Let $g(x) = x^5 + x^3 + 19x + 2$ and $h(x) = x^5 - x^4 + 3x^2 + 20x$ Then $f = g - h$

and $g'(x) = 5x^4 + 3x^2 + 19 \geq 0$ on $[-3,3]$ and

$h'(x) = 5x^4 - 4x^3 + 6x + 20 \geq 0$ on $[-3,3]$.

Hence g and h are increasing. Thus f is of bounded variation.