Math 302

HW7 - Solutions

1- Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions defined on $[-\pi/2, \pi/2]$ by

$$f_n(x) = \begin{cases} \frac{\sin^2(nx)}{n\sin x} & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

a) It suffices to check continuity at $x=0$, as $\frac{\sin^2(nx)}{n\sin x}$ is continuous when $x \neq 0$. $\lim_{x\to 0} f_n(x) = \lim_{x\to 0} \frac{\sin^2(nx)}{n\sin x} = \lim_{x\to 0} \frac{2\sin(nx).n\cos(nx)}{n\cos x} = 0$. Hence $f_n(x)$ is continuous at $x=0$.

For $x=0$, $f_n(x)=0$ so $\lim_{n\to\infty} f_n(0) = 0$. For $x\neq 0$, as $\left|\frac{\sin^2(nx)}{n\sin x}\right| \leq \left|\frac{1}{n}\right| \frac{1}{|\sin x|}$ $\lim_{n\to\infty} f_n(x) = 0$. Thus $(f_n)_{n\in\mathbb{N}}$ converges pointwise to 0 function.

b) Let $a \in (0, \pi/2)$. Let $x\in[a, \pi/2]$. Then $\left|\frac{\sin^2(nx)}{n\sin x}\right| \leq \frac{1}{n|\sin x|} \leq \frac{1}{n\sin a}$ as $\sin x$ is increasing on $[a, \pi/2]$. Thus $\sup_{x\in[a,\pi/2]} |f_n(x)| \leq \frac{1}{n\sin a} \to 0$ as $n\to\infty$.
Thus $(f_n)_{n\in\mathbb{N}}$ converges uniformly on $[a, \pi/2]$.

c) $\lim_{n\to\infty} f_n(\pi/2n) = \lim_{n\to\infty} \frac{\sin^2 \pi/2}{n\sin(\pi/2n)} = \lim_{n\to\infty} \frac{1}{\frac{\pi}{2}\cdot\frac{\sin(\pi/2n)}{\pi/2n}} = \frac{2}{\pi}$.

Then as $|f_n(\pi/2n)| \leq \sup_{x\in[-\pi/2,\pi/2]} |f_n(x)|$, $\lim_{n\to\infty} \sup_{x\in[-\pi/2,\pi/2]} |f_n(x)| \geq \frac{2}{\pi}$.

Thus $(f_n)_{n\in\mathbb{N}}$ does not converge uniformly.

3- Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of differentiable functions with $|f_n'| \leq 1$ on an interval $[a,b]$. Suppose that $(f_n)_{n\in\mathbb{N}}$ converges pointwise. Hence $\forall x \in [a,b]$, $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence. Let $x\in[a,b]$. Given $\varepsilon > 0$ $\exists M_x \in \mathbb{N}$ s.t. $|f_n(x) - f_m(x)| < \varepsilon/3$ $\forall n,m \geq M_x$. For each $n \geq 1$ by MVT, $|f_n(x) - f_n(y)| \leq |x-y|$ as $|f_n'| \leq 1$. Let $y \in [a,b]$ s.t. $|x-y| < \varepsilon/3$. Then

$$|f_n(y) - f_m(y)| \leq \underbrace{|f_n(y) - f_n(x)|}_{<|x-y|<\varepsilon/3} + \underbrace{|f_n(x) - f_m(x)|}_{<\varepsilon/3} + \underbrace{|f_m(x) - f_m(y)|}_{<|x-y|<\varepsilon/3}$$

$< \varepsilon$ for $n,m \geq M_x$.

Now $[a,b] \subseteq \bigcup_{x\in[a,b]} (x-\varepsilon/3, x+\varepsilon/3)$. As $[a,b]$ is compact, $\exists \{x_1, \dots, x_n\}$ s.t. $[a,b] \subseteq \bigcup_{i=1}^{n} (x_i - \varepsilon/3, x_i + \varepsilon/3)$. Let $M = \max_{1\leq i\leq n} M_{x_i}$. Let $x\in[a,b]$. Then $x \in (x_i - \varepsilon/3, x_i + \varepsilon/3)$ for some $i \in \{1,\dots,n\}$. Hence we have

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f_n(x_i)| + |f_n(x_i) - f_m(x_i)| + |f_m(x_i) - f_m(x)|$$
$$\leq |x - x_i| + \varepsilon/3 + |x - x_i| < \varepsilon$$

for $n,m \geq M$. Thus $\sup_{x\in[a,b]} |f_n(x) - f_m(x)| < \varepsilon$, $\forall n,m \geq M$. Thus $(f_n)_{n\in\mathbb{N}}$ is uniformly cauchy, hence $(f_n)_{n\in\mathbb{N}}$ is uniformly convergent.

6- Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions defined on $[-1,1]$ by $f_n(x)=\frac{x^{2n}}{1+x^{2n}}$. If $x\in\{-1,1\}$, then $f_n(x)=\frac{1}{2}$ $\forall n\in\mathbb{N}$, hence $f_n(\pm 1)\longrightarrow \frac{1}{2}$ as $n\to\infty$. If $x\in(-1,1)$, then $f_n(x)=1-\frac{1}{1+x^{2n}}\longrightarrow 0$ as $n\to\infty$ since $x^{2n}\longrightarrow 0$.

Hence $f_n(x)\longrightarrow f(x)=\begin{cases}0 & x\in(-1,1)\\ 1/2 & x=\pm 1\end{cases}$ pointwise.

$$\int_0^1 \lim_{n\to\infty} f_n(x)\,dx=\int_0^1 f(x)\,dx=0.$$

Let $g(x)=\frac{x^2}{1+x^2}$. Then $g(x)$ is continuous on $[0,1]$.

Hence by question 8 of PS5, $\lim\limits_{n\to\infty}\int_0^1 g(x^n)\,dx=g(0)$. Hence

$$\lim_{n\to\infty}\int_0^1\frac{x^{2n}}{1+x^{2n}}dx=\lim_{n\to\infty}\int_0^1 g(x^n)\,dx=g(0)=0,$$

This exercise gives an example that although $f_n\to f$ not uniformly, it may happen that $\lim\limits_{n\to\infty}\int f_n(x)dx=\int f(x)dx$.

9- Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions defined on $[0,\infty)$ by

$f_n(x) = \dfrac{x e^{-x/n}}{n}$.

a) For $x=0$, $f_n(x)=0$ $\forall n\geq 1$ so $\lim\limits_{n\to\infty} f_n(x)=0$.

For $x\neq 0$, $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \dfrac{x}{n e^{x/n}} = 0$. Thus $f_n \longrightarrow 0$ pointwise.

b) Let $b>0$. $|f_n(x)| = \left|\dfrac{x e^{-x/n}}{n}\right| \leq \dfrac{b}{n}$ $\forall x\in[0,b]$. Hence

$\sup\limits_{x\in[0,b]} |f_n(x)| \leq \dfrac{b}{n} \longrightarrow 0$ as $n\to\infty$. Thus (f_n) converges uniformly

on $[0,b]$.

c) $$\int_0^b f_n(x)\,dx = \int_0^b \frac{x e^{-x/n}}{n}\,dx \underset{u=\frac{x}{n}}{=} n\int_0^{b/n} u e^{-u}\,du = n\left[-ue^{-u}\Big|_0^{b/n} + \int_0^{b/n} e^{-u}\,du\right]$$

$$= -n\frac{b}{n}e^{-b/n} - n\left[e^{-b/n}-1\right] = -be^{-b/n} - ne^{-b/n} + n.$$

$$\lim_{n\to\infty}\lim_{b\to\infty}\int_0^b f_n(x)\,dx = \lim_{n\to\infty} n = \infty.$$

$$\lim_{b\to\infty}\lim_{n\to\infty}\int_0^b f_n(x)\,dx \underset{\text{uniform convergence}}{=} \lim_{b\to\infty}\int_0^b \lim_{n\to\infty} f_n(x)\,dx = 0, \quad \text{as } \lim_{n\to\infty} f_n(x) = 0.$$

11- a) $f_n(x)=a_n x^2$, $x\in\mathbb{R}$, $a_n \longrightarrow 1$. $f_n(x)\longrightarrow x^2$ pointwise, but not uniformly. Let $\varepsilon=1$. $|a_n x^2 - x^2| = |x^2||a_n-1|$. For any $n\in\mathbb{N}$ choose x large enough that $|x^2||a_n-1|>1$. Hence $\sup\limits_{x\in\mathbb{R}} |a_n x^2 - x^2|>1$ $\forall n\in\mathbb{N}$, i.e. $\sup\limits_{x\in\mathbb{R}} |a_n x^2 - x^2| \not\longrightarrow 0$, i.e. $f_n(x)\longrightarrow x^2$ not uniformly.

b) $f_n(x)=\frac{nx}{1+n^2x^2}$, $x\in[0,\infty)$. $f_n(x)\longrightarrow 0$ pointwise but not uniformly.

Since $\frac{1}{n}\in[0,\infty)$ $\forall n\in\mathbb{N}$, and $f_n\left(\frac{1}{n}\right)=\frac{1}{2}$, hence $\sup\limits_{x\in[0,\infty)} |f_n(x)|\geq\frac{1}{2}$ $\forall n\in\mathbb{N}$. Thus convergence is not uniform.

c) $f_n(x)=\frac{nx}{1+n^2x^2}$, $x\in[a,\infty)$, $a>0$. As in (b) $f_n(x)\longrightarrow 0$ pointwise on $[a,\infty)$. Let $x\in[a,\infty)$. Then

$$|f_n(x)| = \left|\frac{nx}{1+n^2x^2}\right| \leq \left|\frac{nx}{n^2x^2}\right| = \frac{1}{|nx|} \leq \frac{1}{na}.$$

Hence $\sup\limits_{x\in[a,\infty)} |f_n(x)| \leq \frac{1}{na} \longrightarrow 0$ as $n\longrightarrow\infty$. Hence $\sup\limits_{x\in[a,\infty)} |f_n(x)|\longrightarrow 0$ i.e. $f_n\longrightarrow 0$ uniformly on $[a,\infty)$.

d) $f_n(x)=nx^r e^{-nx}$, $x\in[0,\infty)$ $r\in(0,1]$. $f_n(x)\longrightarrow 0$ pointwise. $f_n\left(\frac{1}{n}\right)=n^{1-r}e^{-1}>e^{-1}$. Thus $\sup\limits_{x\in[0,\infty)} |f_n(x)|>e^{-1}$ $\forall n\in\mathbb{N}$. Thus convergence is not uniform.

e) $f_n(x) = nx^r e^{-nx}$, $x \in [a,\infty)$, $r \in (0,1]$, $a>0$.

As in (d) $f_n(x) \longrightarrow 0$ pointwise. Let $x \in [a,\infty)$. $f_n'(x) = nx^{r-1}e^{-nx}(r-nx)$ $\leqslant nx^{r-1}e^{-nx}(r-na)$. $\exists N \in \mathbb{N}$ s.t. $r-na<0$ $\forall n \geq N$. Hence $f_n'(x)<0$ $\forall n \geq N$ i.e. f_n is decreasing $\forall n \geq N$. Then

$|f_n(x)| = |nx^r e^{-nx}| \leq |na^r e^{-na}| \longrightarrow 0$ as $n \longrightarrow \infty$. Hence $\exists N' \in \mathbb{N}$ s.t. $|na^r e^{-na}| < \varepsilon$ $\forall n \geq N'$. Put $M = \max\{N, N'\}$. Thus $|f_n(x)| < \varepsilon$ $\forall n \geq M$ and $\forall x \in [a,\infty)$. i.e. $\sup\limits_{x \in [a,\infty)} |f_n(x)| \longrightarrow 0$ i.e. $f_n \longrightarrow 0$ uniformly.

f) $f_n(x) = \dfrac{x^n}{1+x^n}$, $x \in [0,\infty)$

For $x \in [0,1)$, $f_n(x) = 1 - \dfrac{1}{1+x^n} \longrightarrow 0$ as $x^n \longrightarrow 0$.

For $x=1$, $f_n(x) = \frac{1}{2}$ so $f_n(x) \longrightarrow \frac{1}{2}$

For $x \in (1,\infty)$, $f_n(x) = 1 - \dfrac{1}{1+x^n} \longrightarrow 1$ as $\dfrac{1}{1+x^n} \longrightarrow 0$.

Hence $f_n(x) \longrightarrow f(x) = \begin{cases} 0 & x \in [0,1) \\ 1/2 & x=1 \\ 1 & x \in (0,\infty) \end{cases}$

f_n is continuous for each n but $f(x)$ is not continuous

Thus convergence is not uniform.