Math 302

HW8 - Solutions

4- $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges uniformly on a compact interval $[a,b]$. Indeed,

$$\sup_{x\in[a,b]} \left| \sum_{k=n}^{n+p} \frac{x^k}{k!} \right| \leq \sum_{k=n}^{n+p} \frac{\max\{|a|,|b|\}^k}{k!} \longrightarrow 0 \quad \text{as } n\to\infty. \text{ Hence}$$

the series converges uniformly.

$\sum_{n=0}^{\infty} \frac{x^n}{n!}$ does not converges uniformly on $\mathbb{R}$. Indeed, if the convergence were uniform then this would imply that $f_n(x) = \frac{x^n}{n!} \longrightarrow 0$ uniformly on $\mathbb{R}$.

But $\sup_{x\in\mathbb{R}} \left|\frac{x^n}{n!}\right| \geq \frac{n^n}{n!} \longrightarrow \infty$ as $n\to\infty$. Hence $f_n(x) \longrightarrow 0$ not uniformly, and so $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ does not converge uniformly.

6- $\sum_{n=1}^{\infty} \frac{x^n}{n}$. Let $a\in(0,1)$. For $|x|\leq a$ $|x^n/n| \leq \frac{a^n}{n}$. By root test, $\lim \sqrt[n]{a^n/n} = a < 1$, $\sum_{n=1}^{\infty} \frac{a^n}{n}$ converges. Hence by weierstrass M-test $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges uniformly on $[-a,a]$.

$\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges pointwise for every $x\in[-1,1)$ and diverges for $x\notin[-1,1)$, by root test.

The convergence is not uniform on $[-1,1)$: If $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges uniformly then $\forall \varepsilon>0\ \exists N\in\mathbb{N}\ \forall n,m\geq N,\ \forall x\in[-1,1)$ $\left|\sum_{k=n}^{m} \frac{x^k}{k}\right| < \varepsilon$.

Letting $x\to 1$, $x\in[-1,1)$ we get $\left|\sum_{k=n}^{m} \frac{1}{k}\right| < \varepsilon$ controdiction as $\sum_{n=1}^{\infty} \frac{1}{n}$ is not convergent.

8- Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions defined on $[0,1]$ by

$$f_n(x) = \begin{cases} 1/n & 1/2^{n+1} < x \le 1/2^n \\ 0 & \text{otherwise.} \end{cases}$$

$\sum_{n=1}^{\infty} f_n(x)$ converges uniformly if $\forall \varepsilon > 0,\ \exists N\in\mathbb{N}: \sup_{x\in[0,1]} \left|\sum_{k=n}^{n+p} f_k(x)\right| < \varepsilon$

$\forall n \geq N,\ \forall p \in \mathbb{N}$. Let $\varepsilon > 0$. Choose $N\in\mathbb{N}$ s.t. $\frac{1}{N} < \varepsilon$.

Let $x\in[0,1]$. Then for any $n \geq N$ and $p\in\mathbb{N}$ we have

$$\left|\sum_{k=n}^{n+p} f_k(x)\right| = \begin{cases} f_{n+m}(x) & \text{if } 1/2^{m+1} < x \le 1/2^m \text{ for some } m, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $x\in[0,1]$ can be lie in only one such interval.

Hence $\left|\sum_{k=n}^{n+p} f_k(x)\right| \le f_{n+m}(x) \le f_N(x) \le \frac{1}{N} < \varepsilon$. Thus $\sum_{k=0}^{\infty} f_k(x)$ converges uniformly.