Mehmet

COMPLEX 8

(9.1) (8) Suppose $f(z) \to \infty$ as $z \to z_0$, z_0 - isolated singularity. Show that f has a pole at z_0.

By Casorati-Weierstrass, as $f(z) \to \infty$ as $z \to z_0$, z_0 can't be an essential singularity. (4)

By Laurent expansion and $C_{-k} = 0$ for $k > 0$ when z_0 is a removable singularity, $f(z) \not\to \infty$ as $z \to z_0$, so z_0 can't be a remov. sing. (4)

(9.4) (8) Suppose f is analytic in punctured plane $z \neq 0$ and satisfies $|f(z)| \leq \sqrt{|z|} + \frac{1}{\sqrt{|z|}}$. Prove f is constant.

$\lim_{z \to 0} |z| \cdot \frac{|z|+1}{\sqrt{|z|}} = \lim_{z \to 0} \sqrt{|z|}\,(|z|+1) = 0$, by Riemann's principle, the singularity is removable (4) So consider the extension of f, call g, that is entire (2)

Now, $|g(z)| \leq A|z|$ for large z implies that g is a linear polinom

But $|g(z)| \leq \sqrt{|z|}$ for large z implies that g is constant. (2)

(8.1) (8) A set $S \subseteq \mathbb{C}$ is star-shaped if $\exists \underset{\text{center}}{\alpha} \in S$ s.t. the line segment connecting α and z is contained in S $\forall z \in S$. Show that star-shaped $\Rightarrow$ simply connec

Show that $\boxed{\gamma : \gamma(t) = tz + (1-t)\alpha \quad , t \geq 1}$ (2) is contained, in the complement for any z in the complement.

γ represents the portion of the ray from α thru z to ∞, starting at z. (2)

(4) So if $z \in S^c$, then so is all of γ. Otw, if any $z_1 \in \gamma$ is in S, so would the entire segment connecting α and z_1, including z. But $z \in S^c$, hence a contradiction.

(8.8) Show that $\pi i + \int_{-1}^{z} \frac{d\zeta}{\zeta}$ defines an analytic branch of $\log z$ in the plane slit along the non-neg. axis, call D with $0 < \operatorname{Im} \log z = \operatorname{Arg} z < 2\pi$.

D is simply connected, $0 \notin D$. Choose $-1 \in D$.

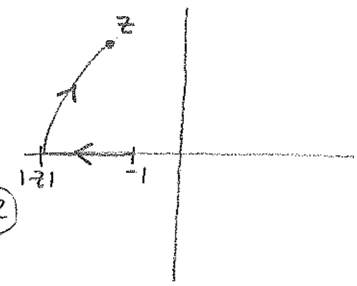
$\log -1 = \cancel{\log 1} + i\pi$. Fix this: $\boxed{\log -1 = i\pi}$

$\Rightarrow f(z) = \int_{-1}^{z} \frac{d\zeta}{\zeta} + \pi i$ is an analytic branch of $\log z$ in D. 2

Let $z \in D$. $f(z) = \int_{-1}^{z} \frac{d\zeta}{\zeta} + \pi i = \int_{-1}^{-|z|} \frac{d\zeta}{\zeta} + \int_{-|z|}^{z} \frac{d\zeta}{\zeta} + \pi i$ 2

$\int_{-1}^{-|z|} \frac{d\zeta}{\zeta} = \int_{1}^{R} \frac{\cancel{e^{i\pi}}}{r\cancel{e^{i\pi}}} dr = \ln R = \ln |z|$ 2

$\int_{-|z|}^{z} \frac{d\zeta}{\zeta} = i\phi$ for $\boxed{\zeta = |z| e^{i\phi}, \ -\pi < \phi < \pi}$ 2



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$\Rightarrow \operatorname{Im} \log z = i(\pi + \phi)$ but $-\pi < \phi < \pi \Rightarrow 0 < \operatorname{Im} \log z < 2\pi$.

(8.9) Define a fnc. f analytic in the plane minus the non-pos. real axis & s.t. $f(x) = x^x$ on the pos. axis. Find $f(i), f(-i)$. Show that $f(\bar{z}) = \overline{f(z)} \ \forall z$

2 $\boxed{f(z) = e^{z \log z}}$ Show $f(x) = x^x$ 1 for $x \in \mathbb{R}^+$.

$f(i) = e^{i \log i} = e^{i(i\frac{\pi}{2})} = e^{-\frac{\pi}{2}}$

$f(-i) = e^{-i \log -i} = e^{-i(i\frac{3\pi}{2})} = e^{\frac{3\pi}{2}} = e^{-\frac{\pi}{2}}$ 1

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See that $\log \bar{z} = \overline{\log z}$.
Also, $e^{\bar{z}} = e^{x-iy} = e^x(\cos -y + i\sin -y) = e^x(\cos y - i \sin y) = \overline{e^{x+iy}} = \overline{e^z}$
So, $f(\bar{z}) = e^{\bar{z}\log\bar{z}} = e^{\bar{z}\,\overline{\log z}} = e^{\overline{z\log z}} = \overline{e^{z\log z}} = \overline{f(z)}$