Math 402/571 Topology

Midterm 1

October 30, 2013

1a) (5 pts) Define topology.
1b) (5 pts) Define metric, and metric space.
1c) (5 pts) Define topological equivalence between two topological spaces.
1d) (5 pts) Define connectedness.

2) (5 pts each) For each of (a)-(d) below: If the proposition is true, write TRUE. If the proposition is false, write FALSE. No explanations are required for this problem.
2a) Let (X, τ) be a topological space. If A is compact in X, then it is closed.
2b) Let Y be a closed set in X. If Z is closed in Y, then Z is closed in X.
2c) Any indiscrete space is path connected.
2d) If $X \times Y$ compact, then both X and Y are compact.

3) Prove or give a counterexample for the following statements:
3a) (7 pts) Every metric space is Hausdorff.
3b) (13 pts) Let (X, d) be a metric space, and $A \subset X$. Then,

$$d(x, A) = 0 \text{ if and only if } x \in \overline{A}.$$

4) Prove or give a counterexample for the following statements:
4a) (10 pts) Every closed subset of a metric space is the intersection of countable number of open sets.
4b) (10 pts) If every function $f : X \to \mathbf{R}$ is continuous, then X has discrete topology.

5) (20 pts) Let S^1 be the circle. Let $\mathcal{C}S^1$ be the cone on circle. Let D^2 be the closed unit disk in $\mathbf{R}^2$. Show that $\mathcal{C}S^1 \simeq D^2$.

Bonus) (20 pts) Prove or give a counterexample for the following statement:
Let (X, τ) be a topological space, and $A \subset X$.
If A is compact in X, then $\overline{A}$ is compact in X.

KEY

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1a) (5 pts) Define topology.

check lecture notes
and ~~t~~textbook.

1b) (5 pts) Define metric, and metric space.

1c) (5 pts) Define topological equivalence between two topological spaces.

1d) (5 pts) Define connectedness.

2) (5 pts each) For each of (a)-(d) below: If the proposition is true, write TRUE. If the proposition is false, write FALSE. No explanations are required for this problem.

2a) Let (X, τ) be a topological space. If A is compact in X, then it is closed.

FALSE: Take (X,τ) indiscrete topology, A is proper subset.

2b) Let Y be a closed set in X. If Z is closed in Y, then Z is closed in X.

TRUE. lecture notes.

2c) Any indiscrete space is path connected.

True: $x, y \in X$ any map $f : [0,1] \to X$, $f(0) = x$, $f(1) = y$ cts.

2d) If $X \times Y$ compact, then both X and Y are compact.

True. lecture notes.

3) Prove or give a counterexample for the following statements:

3a) (7 pts) Every metric space is Hausdorff.

(X,d). $x,y \in X$ $x \neq y$ $\Rightarrow a = d(x,y) > 0$

$B_{\frac{a}{2}}(x) \cap B_{\frac{a}{2}}(y) = \emptyset$

3b) (13 pts) Let (X, d) be a metric space, and $A \subset X$. Then,

$d(x, A) = 0$ if and only if $x \in \overline{A}$.

see HW solutions.

4) Prove or give a counterexample for the following statements:

4a) (10 pts) Every closed subset of a metric space is the intersection of countable number of open sets.

A closed.

$U_n = B_{1/n}(A)$ open.

show. $A = \bigcap U_n$

4b) (10 pts) If every function $f : X \to \mathbb{R}$ is continuous, then X has discrete topology.

let $A \subseteq X$.

let $f_A : X \to \mathbb{R} : \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$

f_A cts $\Rightarrow$ $f^{-1}(1) = f^{-1}((\frac{1}{2}, \frac{3}{2})) = A$ open in X.

$\forall A$ A open in X $\Rightarrow$ $\cancel{X}$ discrete top.

5) (20 pts) Let S^1 be the circle. Let CS^1 be the cone on circle. Let D^2 be the closed unit disk in $\mathbb{R}^2$. Show that $CS^1 \simeq D^2$.

$$CS^1 = S^1 \times [0,1] \Big/ S^1 \times \{1\}$$

Let $f : S^1 \times [0,1] \longrightarrow D^2$
$(\theta, t) \qquad ((1-t)\cos t, (1-t)\sin t)$

f cts. $S^1 \times [0,1]$ compact, D^2 Hausdorff.

f identification map.

Consider partition $Y = \{f^{-1}(x) = P_x \subseteq S^1 \times [0,1] \mid x \in D^2\}$

$\forall x \neq (0,0) \quad P_x$ singleton.

$x = (0,0) \Rightarrow P_{(0,0)} = S^1 \times \{1\} \Rightarrow Y \simeq CS^1 \Rightarrow CS^1 \simeq D^2.$

Bonus) (20 pts) Prove or give a counterexample for the following statement:

Let (X, τ) be a topological space, and $A \subset X$.

If A is compact in X, then $\overline{A}$ is compact in X.

let $X = [0,1]$

$\tau = \{ O \subseteq X \mid 1 \in O \}$

$A = \{1\}$ is compact.

$\bar{A} = [0,1]$ not compact.

since $O_x = \{1, x\}$ is open

let $U = \{ O_x \mid x \in [0,1) \}$ is an open cover of $[0,1]$

with no finite subcover.