Math 402/571 Topology

Midterm 2

December 16, 2013

1a) (5 pts) Define the fundamental group, $\pi_1(X, p)$.
1b) (5 pts) Define deformation retraction.
1c) (5 pts) Define group action on a space. Define orbit space of a group action.
1d) (5 pts) Define the surface with boundary.

2) (5 pts each) For each of (a)-(d) below: If the proposition is true, write TRUE. If the proposition is false, write FALSE. No explanations are required for this problem.

2a) Let X be a topological space and $p, q \in X$.
Then $\pi_1(X, p) \simeq \pi_1(X, q)$.

2b) Let A be a subspace of the topological space X and $p \in A$. If $f : X \to A$ is a retraction, then $f_* : \pi(X, p) \to \pi_1(A, p)$ is an isomorphism.

2c) If X and Y are contractible spaces, then $X \times Y$ is also contractible.

2d) Let X be a contractible space. If $f : X \to X$ is a continuous map, then f has a fixed point, i.e. $\exists x \in X$ s.t. $f(x) = x$.

3) Prove or disprove the following statements:

3a) (10 pts) $S^2 \vee S^2$ is contractible.

3b) (10 pts) $\mathbf{R}^3 \vee \mathbf{R}^3$ is contractible.

4a) (10 pts) Let $p, q, r \in S^2$. Let $X = S^2 - \{p, q, r\}$. Compute $\pi_1(X)$.
4b) (10 pts) Let $p, q \in T^2$. Let $Y = T^2/p \sim q$. Compute $\chi(Y)$.

5) (20 pts) Let $Y = \mathbf{R}^3 - S^1 \vee S^1$. Compute $\pi_1(Y)$ and $\chi(Y)$.

6) (20 pts) Prove or give a counterexample to the following statement:
Let X and Y be compact surfaces with boundary. If X is homotopy equivalent to Y, then X is homeomorphic to Y, i.e. $X \sim Y \Rightarrow X \simeq Y$.

2) (5 pts each) For each of (a)-(d) below: If the proposition is true, write TRUE. If the proposition is false, write FALSE. No explanations are required for this problem.

2a) Let X be a topological space and $p, q \in X$.
Then $\pi_1(X, p) \simeq \pi_1(X, q)$.

FALSE. (X may not be path connected) X

2b) Let A be a subspace of the topological space X and $p \in A$. If $f : X \to A$ is a retraction, then $f_* : \pi(X, p) \to \pi_1(A, p)$ is an isomorphism.

FALSE: (f_* onto, but may not be one-to-one.)

ex: $f : S^1 \to \{p\} \in S^1$

2c) If X and Y are contractible spaces, then $X \times Y$ is also contractible.

TRUE.

$X \underset{F}{\sim} \cdot$

$Y \underset{G}{\sim} \cdot$ $\Rightarrow$ $X \times Y \underset{G}{\sim} X \times \cdot \underset{F}{\sim} \cdot \times \cdot$

2d) Let X be a contractible space. If $f : X \to X$ is a continuous map, then f has a fixed point, i.e. $\exists x \in X$ s.t. $f(x) = x$.

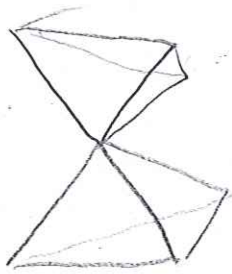
FALSE. $X = \mathbb{R}^2$

$f : \mathbb{R}^2 \to \mathbb{R}^2$

$f(x, y) = (x+1, y)$

3) Prove or disprove the following statements:

3a) (10 pts) $S^2 \vee S^2$ is contractible.

No: $\chi(S^2 \vee S^2) =$  $= 7 - 12 + 8 = 3$

$\chi(\cdot) = 1$

$3 \neq 1 \Rightarrow S^2 \vee S^2 \not\simeq \cdot$

D

3b) (10 pts) $\mathrm{R}^3 \vee \mathrm{R}^3$ is contractible.

TRUE: $\mathbb{R}^3 \vee \mathbb{R}^3 = \mathbb{R}^3_1 \cup \mathbb{R}^3_2 / p \sim q$

$p = (0,0,0) \in \mathbb{R}^3_1$

$q = (0,0,0) \in \mathbb{R}^3_2$

$F : \mathbb{R}^3 \times I \to \mathbb{R}^3$

$F(x,y,z,t) = t(x,y,z)$ $\quad F(\cdot,1) = id$

$F(\cdot,0) = (0,0,0)$

deformation retract.

$X = \mathbb{R}^3 \vee \mathbb{R}^3$.

$\Rightarrow X$ def. retract to $\mathbb{R}^3$ as follows.

$\hat{F} : X \times I \to X$

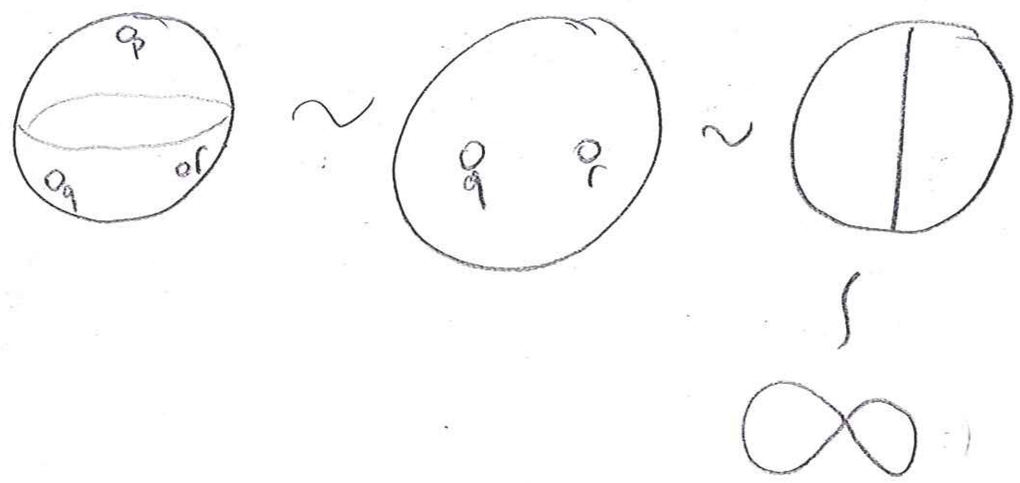
$x \in \mathbb{R}^3_1 \Rightarrow \hat{F}(x,t) = x$

$x \in \mathbb{R}^3_2 \Rightarrow \hat{F}(x,t) = F(x,t) = tx$ ✓

$X \sim \mathbb{R}^3 \sim \cdot$

D

4a) (10 pts) Let $p, q, r \in S^2$. Let $X = S^2 - \{p, q, r\}$. Compute $\pi_1(X)$.



$X \sim S^1 \vee S^1 \Rightarrow \pi_1(X) = \mathbb{Z} * \mathbb{Z}$

4b) (10 pts) Let $p, q \in T^2$. Let $Y = T^2/p \sim q$. Compute $\chi(Y)$.

$Y \sim$  $= T^2 \vee S^1$

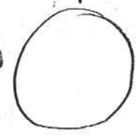
$\chi(T^2 \vee S^1) = \chi(T^2) + \chi(S^1) - \chi(p)$

$= 0 + 0 - 1 = -1$

5) (20 pts) Let $Y = \mathrm{R}^3 - S^1 \vee S^1$. Compute $\pi_1(Y)$ and $\chi(Y)$.

By Van Kampen $\quad \pi_1(Y) = \pi_1(S^3 - S^1 \vee S^1)$

open ball - $S^1 \vee S^1$ $\qquad$ open ball

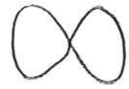
 $\cap$  $= S^3 - S^1 \vee S^1$

$\simeq Y$

$S^3 - S^1 \vee S^1 \simeq \mathbb{R}^3 - \{2 \text{ disjoint lines}\} \simeq \mathbb{R}^2 - \{2\text{ pts}\} \times \mathbb{R}$

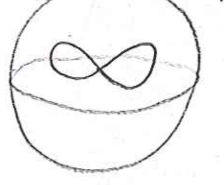
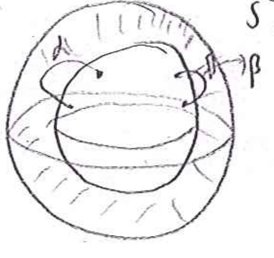
take wedge pt as pt at ∞

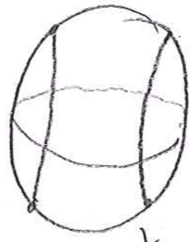
$\sim \mathbb{R}^2 - \{2\text{ pts}\}$

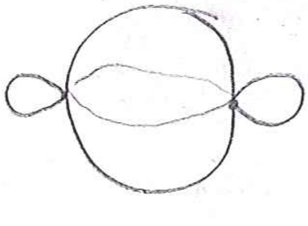
$\simeq$  $\sim$ 

open disk - 2 pts

$\Rightarrow \quad \pi_1(Y) = \mathbb{Z} * \mathbb{Z}$

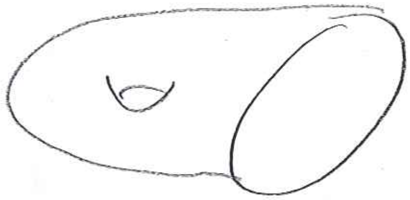
$Y \simeq$  open ball - $S^1 \vee S^1$ $\simeq$ 

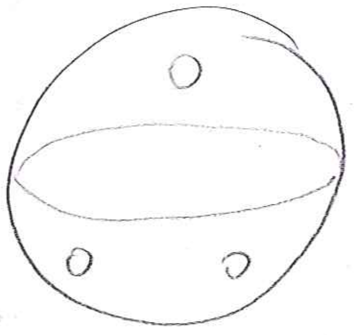
$S^2 \times I - (\alpha \cup \beta)$ $\simeq$  $S^2 \cup 2$ arcs

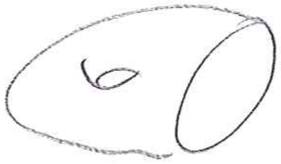
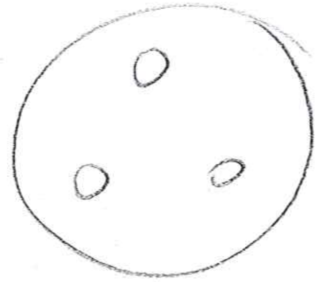
$\simeq$  $= \chi(S^2) - 2 = 0$

Bonus) (20 pts) Prove or give a counterexample to the following statement:

Let X and Y be compact surfaces with boundary. If X is homotopy equivalent to Y, then X is homeomorphic to Y, i.e. $X \sim Y \Rightarrow X \simeq Y$.

T^2 - open disk =  $\sim$ 

S^2 - (3 open disks) =  $\sim$ 

But  $\not\simeq$ 

1 bdry component

3 bdry compnt.

$(X \simeq Y \Rightarrow \partial X \simeq \partial Y)$