Math 402/571 Topology

Final Exam

January 2, 2014

1a) (4 pts) Define connectedness.
1b) (4 pts) Define compactness.
1c) (4 pts) State Jordan Curve Theorem.
1d) (4 pts) State the Classification Theorem for Compact Nonorientable Surfaces.

2) (4 pts each) For each of (a)-(d) below: If the proposition is true, write TRUE. If the proposition is false, write FALSE. No explanations are required for this problem.
2a) Let X be a topological space. Then, any compact subset of X is closed.
2b) Let S be a compact surface. If $\pi_1(S)$ is given, then $\chi(S)$ can be computed.
2c) For any X, $\mathcal{C}X$ (cone on X) is a contractible space.
2d) Fundamental group detects orientability for closed surfaces. i.e.
Let S_1 and S_2 be closed (orientable or nonorientable) surfaces.
If $\pi_1(S_1) \simeq \pi_1(S_2)$, then $S_1 \simeq S_2$.

3) Prove or give a counterexample for the following statements:

3a) (8 pts) Let X be a topological space, and $A \subset X$. If A is connected, then $\overline{A}$ is connected, too.
3b) (8 pts) Let X be a topological space. If X is Hausdorff, then X is a metric space (metrizable).

4) (20 pts) Let S be a compact, connected surface with $\chi(S) = -2$. Write down all possibilities for S.

5) (12 pts) Determine the surface S according to the Classification Theorem.
Compute $\pi_1(S)$.

6) (15 pts) Compute $\pi_1(S^3)$ and $\chi(S^3)$. (S^3 is the 3-dimensional sphere)

7) (15 pts) Show that $\chi(Y \times S^1) = 0$ for any manifold Y.

2) (4 pts each) For each of (a)-(d) below: If the proposition is true, write TRUE. If the proposition is false, write FALSE. No explanations are required for this problem.

2a) Let X be a topological space. Then, any compact subset of X is closed.

FALSE.

$X = I$ (indiscrete topology)

$A = \{\frac{1}{2}\}$

2b) Let S be a compact surface. If $\pi_1(S)$ is given, then $\chi(S)$ can be computed.

TRUE:

If S closed $\Rightarrow$ $\pi_1(S)$ classification ✓

If S has bdry $\Rightarrow$ $\pi_1(S) = *\mathbb{Z}^n \Rightarrow \chi(S) = 1-n$

$S \sim \vee(S^1)^n$

2c) For any X, CX (cone on X) is a contractible space.

TRUE.

contractible to cone pt.

2d) Fundamental group detects orientability for closed surfaces. i.e.
Let S_1 and S_2 be closed (orientable or nonorientable) surfaces.
If $\pi_1(S_1) \simeq \pi_1(S_2)$, then $S_1 \simeq S_2$.

TRUE:

orientable $\Rightarrow$ $\pi_1(S) = \langle a_1, a_2 \dots b_1 \dots b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} \dots \rangle$

nonorientable $\Rightarrow$ $\pi_1(S) = \langle a_1, a_1 \dots b_1 \dots b_g \mid a_1^2 b_1^2 \dots \rangle$

3) Prove or give a counterexample for the following statements:

3a) (8 pts) Let X be a topological space, and $A \subset X$. If A is connected, then $\overline{A}$ is connected, too.

BOOK + LECTURE NOTES.

3b) (8 pts) Let X be a topological space. If X is Hausdorff, then X is a metric space (metrizable).

FALSE. COUNTEREXAMPLE:

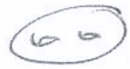
$X = \mathbb{R}$ $\quad \mathcal{T} = \langle [a,b) \mid a < b \in \mathbb{R} \}$

$(X, \mathcal{T})$ Hausdorff.

However, not metrizable.

- $(X, \mathcal{T})$ is separable ($\overline{\mathbb{Q}} = \mathbb{R}$)
- Any separable metric space is second-countable (countable basis)
- However, $(X, \mathcal{T})$ is not second countable.
- WIKIPEDIA - LOWER LIMIT TOPOLOGY.

4) (20 pts) Let S be a compact, connected surface with $\chi(S) = -2$. Write down all possibilities for S.

S closed orientable $\Rightarrow$ $\chi(S) = 2-2g = -2$ $\Rightarrow$ $S \simeq \Sigma_2$  1

closed nonorientable $\Rightarrow$ $\chi(S) = 2-n = -2$ $\Rightarrow$ $S \simeq N_4$ 2

$n = 4$

S compact. $\#\partial S = 1$ $\Rightarrow$ $\chi(S) = 2-n-1 = -2$ $\Rightarrow$ $S \simeq N_3 - (1\text{ disk})$ 3

$\#\partial S = 2$ $\Rightarrow$ $\chi(S) = 2-g-2 = -2$ $\Rightarrow$ $S \simeq T^2 - (2\text{ disks})$ 4

$S \simeq K^2 - (2\text{ disks})$ 5

$\#\partial S = 3$ $\Rightarrow$ $\chi(S) = 2-n-3 = -2$ $\Rightarrow$ $S \simeq P^2 - (3\text{ disks})$ 6

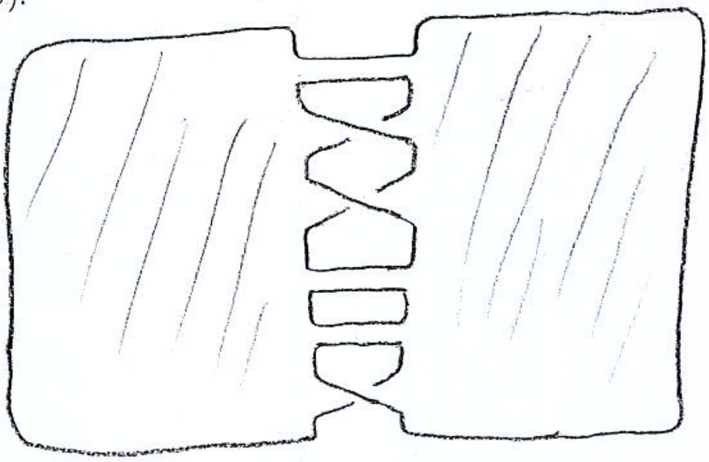
$\#\partial S = 4$ $\Rightarrow$ $\chi(S) = 2-4 = -2$ $\Rightarrow$ $S^2 - (4\text{ disks})$ 7

Orientable: $\underbrace{S^2 - (4\text{ disks})}_{}^{=\Sigma_0}$, $T^2 - (2\text{ disks})^{=\Sigma_1}$, Σ_2

nonorientable: $P^2 - (3\text{ disks})$, $K^2 - (2\text{ disks})$, $N_3 - (1\text{ disk})$, N_4

$P^2 = N_1$, $K^2 = N_2$

5) (12 pts) Determine the surface S according to the Classification Theorem.
Compute $\pi_1(S)$.



S non orientable.

$S \simeq$ 

$\#(\partial S) = 3$

$\chi(S) = 2 - 6 = -4$

$\chi(S) = 2 - n - 3 = -4 \Rightarrow n = 3$

$\Rightarrow S \simeq N_3 - (3 \text{ disks})$

$\pi_1(S) = \underbrace{\mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z} * \mathbb{Z}}_{5}$

6) (15 pts) Compute $\pi_1(S^3)$ and $\chi(S^3)$. (S^3 is the 3-dimensional sphere)

$\pi_1(S^3) = ?$ $\quad S^3 = U \cup V \quad U \simeq S^3 - \{N\}$ north pole

$V \simeq S^3 \setminus \{S\}$ south pole

Van Kampen $\Rightarrow \quad \pi_1(S^3) \simeq \pi_1(U) * \pi_1(V) / \pi_1(U \cap V) \simeq \{0\}$

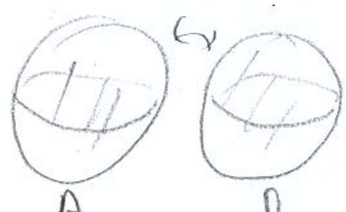
since

$U \simeq \mathbb{R}^3$

$V \simeq \mathbb{R}^3$

$\pi_1(U) = \pi_1(V) \simeq \{0\}$

$\chi(S^3) = ?$ $\quad S^3 \simeq$

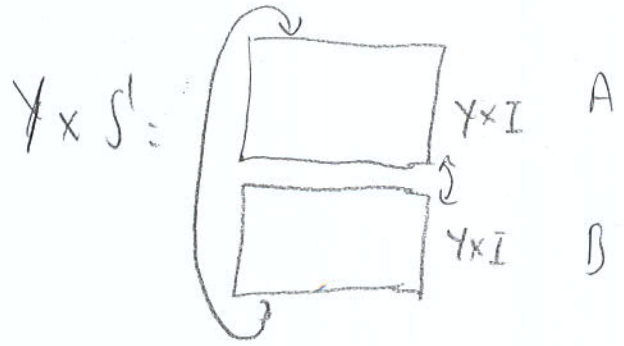


A D

$\chi(S^3) = \chi(A) + \chi(B) - \chi(A \cap B)$

$= 1 + 1 - 2 = 0$

7) (15 pts) Show that $\chi(Y \times S^1) = 0$ for any manifold Y.



$$\chi(A) = \chi(Y \times I) = \chi(Y)$$
$$\chi(B) = \chi(Y \times I) = \chi(Y)$$

$$\chi(A \cap B) = \chi(Y) + \chi(Y) = 2\chi(Y)$$

$$\chi(Y \times S^1) = \chi(A) + \chi(B) - \chi(A \cap B) = \chi(Y) + \chi(Y) - 2\chi(Y)$$
$$= 0$$