Math 402/571 Topology

Midterm 1

October 26, 2015

1a) (5 pts) Define compactness.
1b) (5 pts) Define connectedness.
1c) (5 pts) Define limit point.
1d) (5 pts) Define the topology of a metric space.

2) (5 pts each) TRUE - FALSE:
2a) Let (X, τ) be a topological space. If A is compact in X, then it is closed.
2b) Let Y be a closed set in X. If Z is closed in Y, then Z is closed in X.
2c) Any indiscrete space is path connected.
2d) Let (X, τ) be a topological space. Then, any path component of X is closed.

3) Prove the following statements:
3a) (7 pts) Every metric space is Hausdorff.
3b) (13 pts) Let (X, d) be a metric space, and $A \subset X$. Then,
$d(x, A) = 0$ if and only if $x \in \overline{A}$.

4) Prove or give a counterexample for the following statements:
4a) (10 pts) If $X \times Y \simeq X \times Z$ then $Y \simeq Z$.
4b) (10 pts) If every function $f : X \to \mathbf{R}$ is continuous, then X has discrete topology.

5) Prove or give a counterexample for the following statements:
5a) (10 pts) A component in a topological space X is both open and closed subset of X.
5b) (10 pts) Let (X, d) be a metric space, and $A \subset X$. A is compact if and only if A is closed and bounded.

Bonus) (20 pts) Prove or give a counterexample for the following statement:
Let (X, τ) be a topological space, and $A \subset X$.
If A is compact in X, then $\overline{A}$ is compact in X.

2) (5 pts each) For each of (a)-(d) below: If the proposition is true, write TRUE. If the proposition is false, write FALSE. No explanations are required for this problem.

2a) Let (X, τ) be a topological space. If A is compact in X, then it is closed.

FALSE: $X = [0,1]$ τ: indiscrete

$A = \{\frac{1}{2}\} \Rightarrow A$ compact but not closed

2b) Let Y be a closed set in X. If Z is closed in Y, then Z is closed in X.

TRUE: $Z = \underbrace{A}_{\text{closed in } X} \cap \underbrace{Y}_{\text{closed in } X} \Rightarrow Z$ closed in X

2c) Any indiscrete space is path connected.

TRUE. $(X, \tau_{\text{indiscrete}})$ $\forall x, y \in X$

let $f: [0,1] \to X$ s.t. $f(a) = x,\ a \in [0, \frac{1}{2})$; $f(b) = y,\ b \in [\frac{1}{2}, 1]$

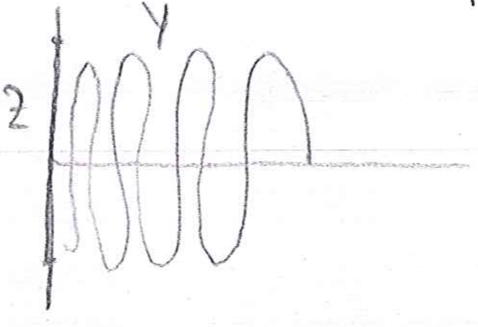
f cts $\Rightarrow X$ path connected.

2d) Let (X, τ) be a topological space. Then, any path component of X is closed.

FALSE: TOPOLOGIST SINE CURVE

$X = Y \cup Z$

Y path component but not closed



3) Prove or give a counterexample for the following statements:

3a) (7 pts) Every metric space is Hausdorff.

(X, d). $x, y \in X$ $x \neq y$ $\Rightarrow a = d(x, y) > 0$

$B_{\frac{a}{2}}(x) \cap B_{\frac{a}{2}}(y) = \emptyset$

3b) (13 pts) Let (X, d) be a metric space, and $A \subset X$. Then,

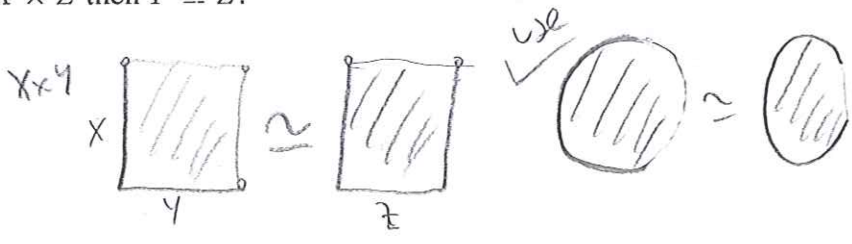
$$d(x, A) = 0 \text{ if and only if } x \in \overline{A}.$$

see HW solutions.

4) Prove or give a counterexample for the following statements:

4a) (10 pts) If $X \times Y \simeq X \times Z$ then $Y \simeq Z$.

$X = [0,1)$
$Y = [0,1)$
$Z = [0,1]$



but $[0,1) \not\simeq [0,1]$

4b) (10 pts) If every function $f : X \to \mathbf{R}$ is continuous, then X has discrete topology.

$\forall a \in X$ $\quad f_a = \begin{cases} 1 & x = a \\ 0 & x \neq a \end{cases}$ $\Rightarrow$ $\forall a \in X$ $\{a\} = f_a^{-1}\left(\frac{1}{2}, \frac{3}{2}\right)$ open

cts $\qquad \Rightarrow (X, \tau)$ discrete topology

5) Prove or give a counterexample for the following statements:

5a) (10 pts) A component in a topological space X is both open and closed subset of X.

FALSE. $X = \mathbb{Q}$ with induced toplgy from $\mathbb{R}$,

$\{q\} \in \mathbb{Q}$ compnet since largest connected set containing $\{q\}$

however, $\{q\}$ is not open since open sets in $\mathbb{Q}$ are $(a,b) \cap \mathbb{Q}$!

open interval in $\mathbb{R}$

5b) (10 pts) Let (X, d) be a metric space, and $A \subset X$. A is compact if and only if A is closed and bounded.

FALSE. let $X = [0,1]$ $T =$ discrete

$A = X$ closed & bounded

but A is not compact!

open covering $\mathcal{O} = \{\{x\} \mid x \in [0,1]\} \Rightarrow \nexists$ no finite cover.

singletons

Bonus) (20 pts) Prove or give a counterexample for the following statement:

Let (X, τ) be a topological space, and $A \subset X$.

If A is compact in X, then $\overline{A}$ is compact in X.

Let $X = [0,1]$

$\tau = \{ O \subseteq X \mid 1 \in O \}$

$A = \{1\}$ is compact.

$\bar{A} = [0,1]$ not compact.

since $O_x = (1, x)$ is open

let $\mathcal{U} = \{ O_x \mid x \in [0,1) \}$ is an open cover of $[0,1]$

with no finite subcover.