Math 402/571 Topology

Final

June 2, 2009

1a) (5 pts) Define topological equivalence of two topological spaces.
1b) (5 pts) Define continuity of a map between two topological spaces. Define continuity of a map between two metric spaces.
1c) (5 pts) Define identification space. Define identification topology.
1d) (5 pts) Define Euler Characteristics of a surface, $\chi(S)$.

2) (5 pts each) For each of (a)-(d) below: If the proposition is true, write TRUE. If the proposition is false, write FALSE. No explanations are required for this problem.
2a) Preimage of a compact set under a continuous function may not be compact.
2b) $D^2 \times D^2$ is a contractible space.
2c) Let X and Y be metric spaces. Then X is deformation retract of $X \times Y$.
2d) Let S_1 and S_2 be closed orientable surfaces. If $\chi(S_1) = \chi(S_2)$ then S_1 is homeomorphic to S_2.

3) Prove or give a counterexample for the following statements:
3a) (10 pts) Let X be a metric space and A be a subspace of X. If A is compact, then A is closed and bounded.
3b) (10 pts) Let X be a metric space and A be a subspace of X. If A is closed and bounded, then A is compact.

4) Let X and Y be the surfaces with boundary such that X is obtained by removing a small open disk from T^2 , and Y is obtained by removing 3 disjoint small open disks from S^2.
4a) (10 pts) Show that X is homotopy equivalent to Y.
4b) (10 pts) Decide whether X is homeomorphic to Y or not.

5) (20 pts) Give an example of a simply connected space which is not contractible. Show your work.

6) (20 pts) Let $X = S^3 - S^1$. Compute $\pi_1(X)$.

2) (5 pts each) For each of (a)-(d) below: If the proposition is true, write TRUE. If the proposition is false, write FALSE. No explanations are required for this problem.

2a) Preimage of a compact set under a continuous function may not be compact.

TRUE: $f: \mathbb{R} \to \cdot$

2b) $D^2 \times D^2$ is a contractible space.

TRUE: $D^2 \times D^2 \sim D^2 \times \cdot \sim \cdot$

2c) Let X and Y be metric spaces. Then X is deformation retract of $X \times Y$.

FALSE: $X = \cdot$, $Y = S^1$ $\Rightarrow$ $X \times Y \simeq S^1$

2d) Let S_1 and S_2 be closed orientable surfaces. If $\chi(S_1) = \chi(S_2)$ then S_1 is homeomorphic to S_2.

TRUE: Classification of surfaces.

3) Prove or give a counterexample for the following statements:

3a) (10 pts) Let X be a metric space and A be a subspace of X. If A is compact, then A is closed and bounded.

Proof: A compact $\Rightarrow$ A bounded.

Let $x_0 \in X$. Consider the covering $\{B_n(x_0)\}$ s.t. $A \subseteq X = \bigcup_{n=1}^{\infty} B_n(x_0)$

Since A is compact, $\exists$ finite cover $\Rightarrow$ $A \subseteq B_{N_0}(x_0)$ for some $N_0 > 0$.

A compact $\Rightarrow$ A closed.

Claim: A^c is open. Let $y \in A^c$. let $\varepsilon_x = d(x,y)$ $\forall x \in A$.

Then $\{B_{\frac{\varepsilon_x}{2}}(x)\}_{x \in A}$ is an open cover for A. Since A is compact,

$\exists$ finite cover, $A = \bigcup_{i=1}^{n} B_{\frac{\varepsilon_{x_i}}{2}}(x)$. Let $O = \bigcap_{i=1}^{n} B_{\frac{\varepsilon_{x_i}}{2}}(y)$. Then O is open and $O \subseteq A^c$.

Hence A^c is open. [Note that we use the metric essentially]

3b) (10 pts) Let X be a metric space and A be a subspace of X. If A is closed and bounded, then A is compact.

$X = [0,1]$ $\quad$ d = discrete metric $\quad d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$

$A = X \Rightarrow$ A closed and bounded $(A \subseteq B_2(0))$

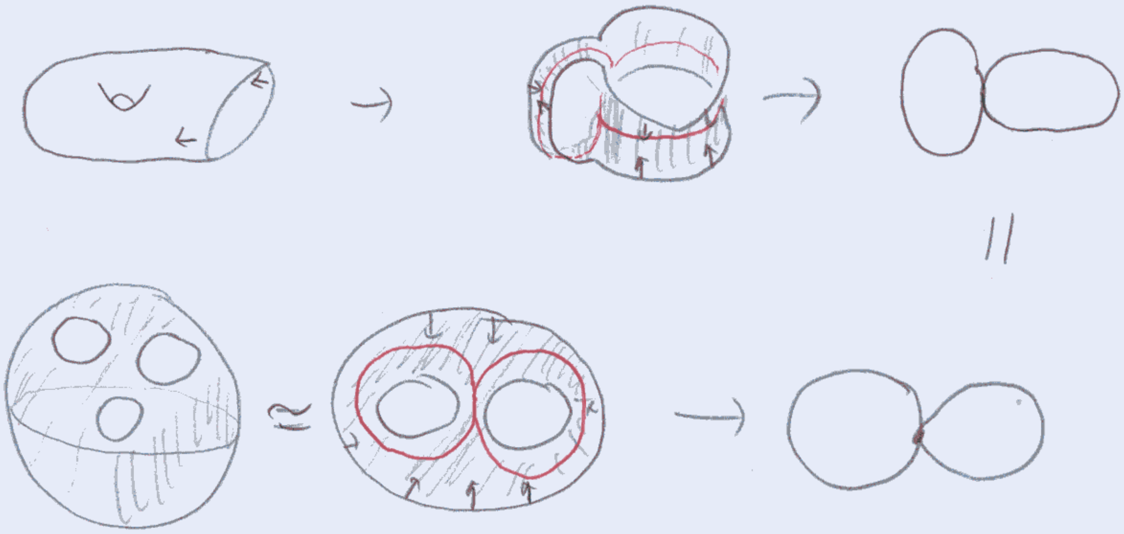
However, A is not compact. $\{B_{\frac{1}{2}}\{x\}\}_{x \in [0,1]}$ is an open cover with no finite subcover.

4) Let X and Y be the surfaces with boundary such that X is obtained by removing a small open disk from T^2 , and Y is obtained by removing 3 disjoint small open disks from S^2.

4a) (10 pts) Show that X is homotopy equivalent to Y.

$X =$

$Y =$



$\Rightarrow X \sim S^1 \vee S^1 \qquad Y \sim S^1 \vee S^1 \Rightarrow X \sim Y$

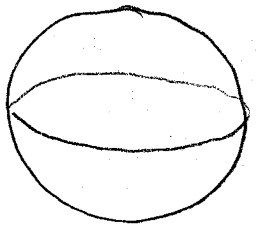
4b) (10 pts) Decide whether X is homeomorphic to Y or not.

X and Y are not homemorphic

since $\partial X = S^1 \quad \rightarrow \quad \partial X \not\cong \partial Y \Rightarrow X \not\cong Y$

$\partial Y = S^1 \sqcup S^1 \sqcup S^1$

5) (20 pts) Give an example of a simply connected space which is not contractible. Show your work.



S^2

$\chi(S^2) = 2$

If S^2 was contractible, then $S^2 \sim \cdot$

We know that Euler characteristic χ is homotopy invariant. But $\chi(\cdot) = 1$

$\chi(S^2) = 2 \neq 1 \Rightarrow S^2 \not\sim \cdot$

So S^2 is simply connected but not contractible!

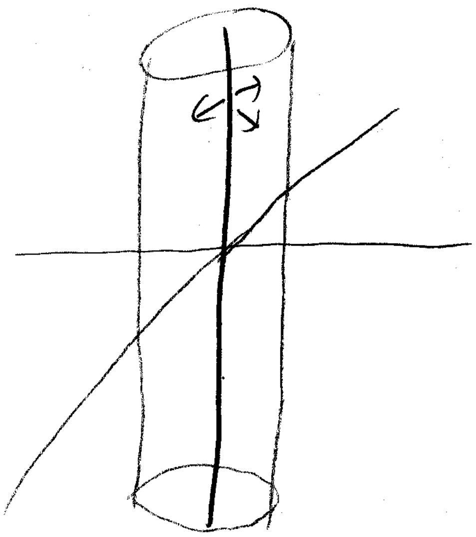
6) (20 pts) Let $X = S^3 - S^1$. Compute $\pi_1(X)$.

$S^3 = \mathbb{R}^3 \cup \{\infty\}$

$S^1 = z\text{-axis} \cup \{\infty\}$

$\Rightarrow \quad S^3 - S^1 = \mathbb{R}^3 - z\text{-axis}$

$\mathbb{R}^3 - z\text{-axis} \sim S^1 \times \mathbb{R}$ where S^1 is unit circle in xy plane

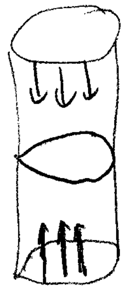


$$F(x,y,z,t) = \left(f_t(|x|)\frac{x}{|x|},\ f_t(|y|)\frac{y}{|y|},\ z\right)$$

where $f_t(r) = t + (1-t)r \quad \left(\begin{array}{l} f_0(r) = r \\ f_1(r) = 1 \end{array}\right)$

$S^1 \times \mathbb{R} \sim S^1 \quad \Rightarrow \quad S^3 - S^1 \sim S^1 \quad \Rightarrow \pi_1(S^3 - S^1) = \pi_1(S^1)$

$= \mathbb{Z}$



$F(x,y,z,t) = (x, y, tz)$