Math 402/571 Topology

Midterm 1

March 16, 2009

1a) (5 pts) Define continuity.

1b) (5 pts) Define compactness.

1c) (5 pts) Define connectedness.

1d) (5 pts) Define path connectedness.

2a) (10 pts) Give an example of a topological space (X, τ) which cannot be written as a metric space. Explain your answer.

2b) (10 pts) Show that metrizability (having a metric) is a topological property.

3) (15 pts) Prove or give a counterexample for the following statement:
If $X \times Y \simeq X \times Z$ then $Y \simeq Z$.

4a) (10 pts) Show that a component is closed.
4b) (10 pts) Prove or give a counterexample for the following statement:
A component in a topological space X is both open and closed subset of X.

5) (15 pts) Prove or give a counterexample for the following statement:
Let (X, d) be a metric space, and $A \subset X$. A is compact if and only if A is closed and bounded.

6) (20 pts) Show that if X is connected and locally path connected, then X is path connected.

MATH 402/571 Midterm I Solutions - Spring 09

1. Check book

2. a.) $X = [0,1]$ $\tau = \mathcal{T}_{indiscrete}$

Assume d is a metric on X inducing $\mathcal{T}_{ind}$. Then $d(0,1) = c > 0$

$\Rightarrow$ $B_{\frac{c}{2}}(0) \cap B_{\frac{c}{2}}(1) = \emptyset$ and $B_{\frac{c}{2}}(0), B_{\frac{c}{2}}(1)$ are open in induced topology by the metric.

$\mathcal{T}_{ind} = \{\emptyset, X\}$ $\Rightarrow$ $\nexists$ no open set in $(X, \mathcal{T}_{ind})$ containing 0 but not containing 1. So, $B_{\frac{c}{2}}(0) \notin \mathcal{T}_{ind}$. This is a contradiction. $\square$

(Alternatively, $(X, \mathcal{T}_{ind})$ is not Hausdorff. However, all metric spaces are Hausdorff.) $\times$

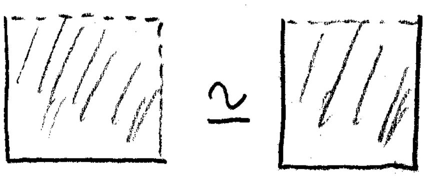
2. b) $X \simeq Y$. if X has a metric $\Rightarrow$ Y has a metric.

(X, d_X) given. $\varphi: X \to Y$ homeomorphism.

Define $d_Y(y_1, y_2) = d_X(\varphi^{-1}(y_1), \varphi^{-1}(y_2))$. Check the properties of metric. Show d_Y is a metric.

Then $\varphi(B_\varepsilon(x)) = B_\varepsilon(\varphi(y))$ d_Y induces the topology on Y. $\square$

3. Counterexample: $X = [0,1)$ $Y = [0,1)$ $Z = [0,1]$



$X \times Y$ $\qquad$ $X \times Z$

but $\underset{Y}{[0,1)} \not\simeq \underset{Z}{[0,1]}$

4. a.) Claim: C connected $\Rightarrow$ $\bar{C}$ connected.

Proof: Otherwise, $\exists$ nonempty proper subset $A \subseteq \bar{C}$, where A is both open and closed.

$\hat{A} = A \cap C$ is nonempty (C is dense in $\bar{C}$), and proper (A^c is open in C.

$\hat{A}$ is both open and closed $\Rightarrow$ C is not connected $\times$.

C component $+$ $C \subseteq \bar{C}$ $+$ $\bar{C}$ connected $\Rightarrow$ $C = \bar{C}$ $\Rightarrow$ C closed.

4. b. Counterexample: $\mathbb{Q} \subseteq \mathbb{R}$ subspace topology.

Claim: $\forall q \in \mathbb{Q}$ $\{q\}$ is a component.

Proof: Let A be the component containing q, and $A \neq \{q\}$. Then let $q' \in A$

$\Rightarrow \exists r \in (q, q')$ which is irrational. Let $A_1 = A \cap (-\infty, r)$, $A_2 = A \cap (r, \infty)$.

$A = A_1 \cup A_2$ $\quad \bar{A}_1 \cap A_2 = \emptyset$ and $A_1 \cap \bar{A}_2 = \emptyset$ $\Rightarrow$ A is not connected! $\times$

Hence $\{q\}$ is a component but $\{q\}$ is not open as $\forall \varepsilon$ $B_\varepsilon(q) \cap \mathbb{Q} \neq \{q\}$. $\square$

5. Counterexample:

$X = [0,1]$. $d = d_{discrete}$ ($d(x,y) = 1$ if $x \neq y$, $d(x,y) = 0$ if $x = y$)

X is closed. X is bounded as $X \subseteq B_2(0)$.

However, X is not compact as $F = \{\{x\} \mid x \in X\}$ is an open cover with no finite subcover.

6. Assume X is not path connected. Then P is a path component in X but $P \neq X$.

Since X is connected either $P \cap \bar{P^c} \neq \emptyset$ or $\bar{P} \cap P^c \neq \emptyset$

If $P \cap \bar{P^c} \neq \emptyset$, let $x \in P \cap \bar{P^c}$. $\exists_{x \in} O_x \overset{open}{\subseteq} X$ which is path connected (X is loc. path. con.)

But then $P \cup O_x$ is path connected as $x \in P \cap O_x$. Also as $x \in \bar{P^c}$

$\exists y \in P^c \cap O_x \Rightarrow P \subsetneq P \cup O_x$, but P is maximal path connected set $\times$.

If $\bar{P} \cap P^c \neq \emptyset$, let $x \in \bar{P} \cap P^c$. Then $P \subsetneq \bar{P}$, as $x \in P^c$.

$\exists_{x \in} O_x \overset{open}{\subseteq} X$ O_x path connected. $O_x \cap P \neq \emptyset$ as $x \in \bar{P}$. Let $y \in O_x \cap P$.

Then $P \cup O_x$ path connected. But $P \subsetneq P \cup \{x\} \subseteq P \cup O_x$.

This contradicts with P being maximal. $\square$