2) (5 pts each) For each of (a)-(d) below: If the proposition is true, write TRUE. If the proposition is false, write FALSE. No explanations are required for this problem.

2a) Let X be a topological space and $p, q \in X$. Then $\pi_1(X,p) \simeq \pi_1(X,q)$.

FALSE.

(If X is not path connected)

2b) Let A be a subspace of the topological space X. If $f : X \to A$ is a retraction, then $f_* : \pi(X,p) \to \pi_1(A,p)$ is isomorphism where $p \in A$.

FALSE.

(f_* is just onto. Example: $f : S^1 \to \{p\}$)

2c) All contractible spaces are homotopy equivalent to each other.

TRUE.

($X \sim \cdot$ and $\sim$ is equivalence relation)

2d) If X, Y path connected and $\pi_1(X) \simeq \pi_1(Y)$, then X is homotopy equivalent to Y.

FALSE.

($S^2 \not\sim \cdot$. $\pi_1(S^2) = \pi_1(\cdot) = \{0\}$)

3) (15 pts) Prove or give a counterexample for the following statement:

Every contractible space has fixed point property.
(*X has the fixed point property if for any continuous map $f : X \to X$, there exists a point $p \in X$ with $f(p) = p$*)

NO:

$X = \mathbb{R}$ $\qquad$ $\mathbb{R}$ contractible $\qquad$ $F(x,t) : \mathbb{R} \times I \to \mathbb{R}$
$(x,t) \mapsto tx$

$\Rightarrow \mathbb{R} \sim \{0\}$

$f : \mathbb{R} \to \mathbb{R}$ $\qquad$ no fixed pt.
$x \mapsto x+1$

4) (15 pts) Show that $\mathbf{R}^2$ is not homeomorphic to $\mathbf{R}^3$.

Assume $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ homeo.

$\Rightarrow \quad \varphi|_{\mathbb{R}^2-\{0\}} : \mathbb{R}^2-\{0\} \to \mathbb{R}^3 - \varphi(0)$ homeo.

(assume $\varphi(0)=0$ by composing φ with translation)

$\Rightarrow \quad \mathbb{R}^2-\{0\} \simeq \mathbb{R}^3-\{0\}$

$\mathbb{R}^n-\{0\} \sim S^{n-1}$ with $F : \mathbb{R}^n-\{0\} \times I \to \mathbb{R}^n-\{0\}$

$(x,t) \quad f_t(|x|)\dfrac{x}{|x|}$

where $f_t : (0,\infty) \to (0,\infty)$

$f_0 = 1 \quad f_1 = id$

(ex: $f_t(a) = a^t \quad a>0$)

$\Rightarrow \quad \mathbb{R}^2-\{0\} \sim S^1$

$\mathbb{R}^3-\{0\} \sim S^2$

$\pi_1(S^1) = \mathbb{Z}$

$\pi_1(S^2) = \{0\}$

$\Rightarrow \quad \pi_1(\mathbb{R}^2-\{0\}) = \mathbb{Z} \neq \{0\} = \pi_1(\mathbb{R}^3-\{0\})$

$\Rightarrow \quad \mathbb{R}^2-\{0\} \not\simeq \mathbb{R}^3-\{0\}$

X

5) (15 pts) Let $p, q \in S^n$ and $X_n = S^n - \{p, q\}$. Compute $\pi_1(X_n)$ for any $n \geq 2$.

$$\underbrace{S^n - \{p\} \cong \mathbb{R}^n}_{\varphi} \quad \text{(stereographic projection)}$$

$$\Rightarrow \quad S^n - \{p, q\} \simeq \mathbb{R}^n - \varphi(q) \simeq \mathbb{R}^n - \{0\}$$

From previous question, $\mathbb{R}^n - \{0\} \sim S^{n-1}$

$$\Rightarrow \quad n = 2 \quad \pi_1(X_n) = \pi_1(S^1) = \mathbb{Z}$$

$$n \geq 3 \quad \pi_1(X_n) = \pi_1(S^{n-1}) = \{0\}$$

6a) (15 pts) Compute $\pi_1(P^2)$. Show your work.

By Definition, $P^2 = S^2/_{x \sim -x} = S^2/\mathbb{Z}_2$ where $\mathbb{Z}_2 \curvearrowright S^2$, $\varphi(x) = -x$

$\mathbb{Z}_2 \curvearrowright S^2$: $\cdot\ \pi_1(S^2) = \{0\}$

$\cdot\ \forall x\ \exists U_x\quad \varphi(U_x) \cap U_x = \emptyset$

$\Rightarrow$ by Theorem 5.13 $\quad \pi_1(P^2) = \pi_1(S^2/\mathbb{Z}_2) = \mathbb{Z}_2$

6b) (10 pts) Let p be a point in P^2 and let $X = P^2 - \{p\}$. Compute $\pi_1(X)$.

$X = P^2 - \{p\} \sim P^2 - \text{disk} = \text{Möbius band} \sim S^1 \Rightarrow \pi_1(X) = \pi_1(S^1) = \mathbb{Z}$