

2) (5 pts each) For each of (a)-(d) below: If the proposition is true, write TRUE. If the proposition is false, write FALSE. No explanations are required for this problem.

2a) Let X be a topological space and $p, q \in X$. Then $\pi_1(X, p) \simeq \pi_1(X, q)$.

FALSE.

(If X is not path connected)

2b) Let A be a subspace of the topological space X . If $f : X \rightarrow A$ is a retraction, then $f_* : \pi_1(X, p) \rightarrow \pi_1(A, p)$ is isomorphism where $p \in A$.

FALSE.

(f_* is just onto. Example: $f : S^1 \rightarrow \{p\}$)

2c) All contractible spaces are homotopy equivalent to each other.

TRUE.

($X \sim$ and \sim is equivalence relation)

2d) If X, Y path connected and $\pi_1(X) \simeq \pi_1(Y)$, then X is homotopy equivalent to Y .

FALSE.

($S^2 \not\sim$. $\pi_1(S^2) = \pi_1(\cdot) = \{0\}$)

3) (15 pts) Prove or give a counterexample for the following statement:

Every contractible space has fixed point property.
(X has the fixed point property if for any continuous map $f : X \rightarrow X$, there exists a point $p \in X$ with $f(p) = p$)

NO:

$$X = \mathbb{R} \quad \mathbb{R} \text{ contractible} \quad f(x, t) : \mathbb{R} \times I \rightarrow \mathbb{R}$$
$$(x, t) + x$$
$$\Rightarrow \mathbb{R} \sim \{0\}$$

$$f : \mathbb{R} \rightarrow \mathbb{R} \quad \text{no fixed pt.}$$
$$x \mapsto x+1$$

4) (15 pts) Show that \mathbb{R}^2 is not homeomorphic to \mathbb{R}^3 .

Assume $\varphi: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ homeo.

$\Rightarrow \varphi|_{\mathbb{R}^2 - \{0\}}: \mathbb{R}^2 - \{0\} \rightarrow \mathbb{R}^3 - \{\varphi(0)\}$ homeo.

$\mathbb{R}^2 - \{0\}$

(assume $\varphi(0)=0$ by composing
 φ with translation)

$\Rightarrow \mathbb{R}^2 - \{0\} \cong \mathbb{R}^3 - \{0\}$

$\mathbb{R}^2 - \{0\} \cong S^{n-1}$ with $F: \mathbb{R}^3 - \{0\} \times I \rightarrow \mathbb{R}^3 - \{0\}$

$$(x, t) \mapsto \frac{\varphi(x)}{|x|}$$

where $f_t: (0, \infty) \rightarrow (0, \infty)$

$$f_0 = 1 \quad f_1 = id$$

(ex: $f_t(a) = a^t \quad a > 0$)

$\Rightarrow \mathbb{R}^2 - \{0\} \cong S^1$

$\mathbb{R}^2 - \{0\} \cong S^1$

$$\pi_1(S^1) = \mathbb{Z}$$

$$\pi_1(S^1) = \{0\}$$

$$\Rightarrow \pi_1(\mathbb{R}^2 - \{0\}) = \mathbb{Z} \neq \{0\} = \pi_1(\mathbb{R}^3 - \{0\})$$

$$\Rightarrow \mathbb{R}^2 - \{0\} \neq \mathbb{R}^3 - \{0\}$$

X

5) (15 pts) Let $p, q \in S^n$ and $X_n = S^n - \{p, q\}$. Compute $\pi_1(X_n)$ for any $n \geq 2$.

$$S^n - \{p\} \xrightarrow{\varphi} \mathbb{R}^n \text{ (stereographic projection)}$$

$$\Rightarrow S^n - \{p, q\} \cong \mathbb{R}^n - \varphi(q) \cong \mathbb{R}^n - \{0\}$$

From previous question, $\mathbb{R}^n - \{0\} \cong S^{n-1}$

$$\Rightarrow n=2 \quad \pi_1(X_2) = \pi_1(S^1) = \mathbb{Z}$$

$$n > 3 \quad \pi_1(X_n) = \pi_1(S^{n-1}) = \{0\}$$

6a) (15 pts) Compute $\pi_1(P^2)$. Show your work.

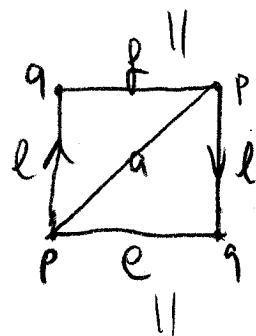
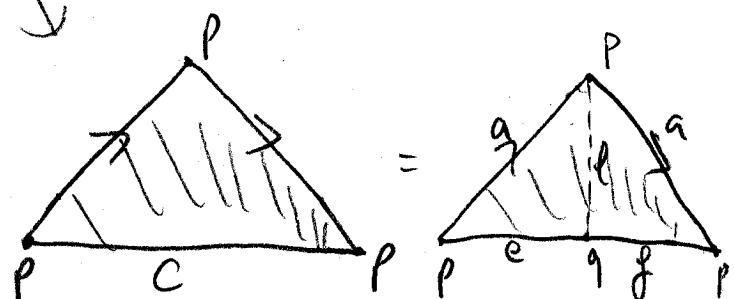
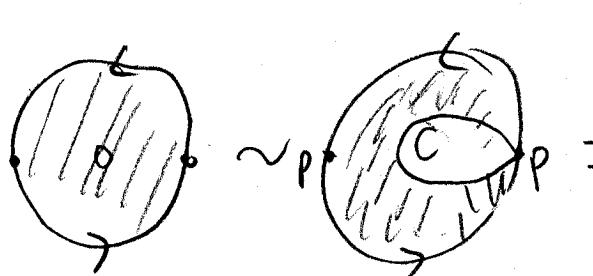
By definition, $P^2 = S^2 / \{x \sim -x\} = S^2 / \mathbb{Z}_2$ where $\mathbb{Z}_2 \curvearrowright S^1$
 $\varphi(x) = -x$

$\mathbb{Z}_2 \curvearrowright S^1$:
• $\pi_1(S^1) = \{0\}$
• $\forall x \exists U_x \quad \ell(U_x) \cap U_{-x} = \emptyset$

\Rightarrow by Theorem 5.1) $\pi_1(P^2) = \pi_1(S^1 / \mathbb{Z}_2) = \mathbb{Z}_2$

6b) (10 pts) Let p be a point in P^2 and let $X = P^2 - \{p\}$. Compute $\pi_1(X)$.

$X = P^2 - \{p\} \sim P^2 - \text{disk} = \text{Möbius band} \sim S^1 \quad (\Rightarrow) \quad \pi_1(X) = \pi_1(S^1) = \mathbb{Z}$



Möbius band