HW #3 - SOLUTIONS

① For $\mathbb{E}^1$ $(-n,n)$, $n \in \mathbb{N}$ is an open cover which does not have any finite subcover. Assume for a contradiction that it has a finite subcover. Then $\mathbb{E}^1 \subseteq \bigcup_{n=1}^{N} (-n,n) = (-N,N)$ for some $N \in \mathbb{N}$ which is clearly not true.

- For $[0,1)$ $[0, 1-\frac{1}{n})$, $n \in \mathbb{N}$ is an open cover in topology of $[0,1)$

Proof = let $x \in [0,1)$. Even if x is very close to 1 by Archimedean property there is sufficiently big $N_x \in \mathbb{N}$ such that $x < 1-\frac{1}{N_x}$ so $x \in [0, 1-\frac{1}{N_x})$. This is a cover for $[0,1)$.

Now assume for a contradiction that it has a finite subcover. Then for some $N \in \mathbb{N}$ $[0,1) \subseteq [0, 1-\frac{1}{N})$

But for instance $1-\frac{1}{N+1} \in [0,1)$ but $\notin [0, 1-\frac{1}{N})$

- For $(0,1)$ $(\frac{1}{n}, 1)$ $n>1$, $n \in \mathbb{N}$ is an open cover.

Proof = let $x \in (0,1)$. By the archimedean property there exists $N_x \in \mathbb{N}$ st. $x > \frac{1}{N_x}$ so $x \in (\frac{1}{N_x}, 1)$.

Assume for a contradiction that it has a finite subcover. Then it must be true that $(0,1) \subseteq (\frac{1}{N}, 1)$ for some $N \in \mathbb{N}$. But $\frac{1}{N+1} \in (0,1)$ which is not in $(\frac{1}{N}, 1)$. so it has no finite subcover.

(15) 1) Any compact space K:

Let $x \in K$ be a point. K is a neighborhood of x which is compact
so K is locally compact.

2) $\mathbb{E}^n$: Let $x \in \mathbb{E}^n$. Consider the open ball $B_1(x)$. Then $\overline{B_1(x)}$
is a closed neighborhood of x. Which is also bounded. By Heine
Borel theorem a closed and bounded subset of $\mathbb{E}^n$ is compact.

3) Any discrete space: Let D be a discrete space. For $x \in D$
consider $\{x\}$ which is also a compact neighborhood of x.

4) Any closed subset C of a locally compact space K:

Let $x \in C$ be a point $\Rightarrow x \in K$. Since K is a locally compact
space there exist a compact neighborhood N of x in K.
$N \cap C$ is also a neighborhood of x which is closed in the subspace
topology of N inherited from K because C is a closed set
in K $\Rightarrow N \cap C$ is closed in N. So closed subset of a
compact set is compact $\Rightarrow N \cap C$ is compact $\Rightarrow C$ locally compact.

5) Assume for a contradiction that $\mathbb{Q}$ is locally compact. Then for
$0 \in \mathbb{Q}$ there exist a neighborhood $N \ni 0$ which is compact.
Consider $(-\varepsilon, \varepsilon) \cap \mathbb{Q} \subseteq N$ Then for $r \in \mathbb{R} \setminus \mathbb{Q}$ there is a sequence
of rationals converging to that point. But this sequence constitutes
an infinite subset of N which has no limit point in N. So
by Bolzano Weier. property N cannot be compact. Result follows.

6) Let K_1 be a locally compact space we need to show that
any homeomorphic image $f(K_1)$ is locally compact. Let $y \in f(K_1)$
$\Rightarrow y = f(x)$ for some $x \in K_1$. $\exists$ a compact neighborhood N of x.
$\Rightarrow f(N)$ is compact since f is continuous and also it is a
neighborhood of $y = f(x)$ / $\exists O$ open $x \in O \subseteq N \Rightarrow f(O)$ open $y \in f(O) \subseteq f(N)$

⑰ $X \cup \{\infty\} = X'$

open sets are the open sets of X and $(X-K) \cup \{\infty\}$

(i) $\phi \subseteq X$ is an open set of X so it is an open set of X'

$X \cup \{\infty\} = (X - \phi) \cup \{\infty\}$ ϕ is a compact set, so $X \cup \{\infty\}$ is open in X'

(ii) When we take arbitrary union of open sets of X'

$$\bigcup_{\alpha \in I}(X - K_\alpha) \cup \{\infty\} \bigcup_{\beta \in I} O_\beta = \bigcup_{\alpha \in I}(X - K_\alpha) \cup O \cup \{\infty\}$$

(since X is a topological space arbitrary union of open sets of X is also open. $\bigcup_{\beta \in I} O_\beta = O$)

$$= X - \bigcap_{\alpha \in I}(K_\alpha \cap O^c) \cup \{\infty\}$$

$K_\alpha \cap O^c$ is compact since O^c is closed, being closed subset of a compact set

and X is Hausdorf $\Rightarrow$ K_α is are closed and so compact.

$\Rightarrow \bigcap K_\alpha \cap O^c \subseteq K_\alpha$ is also closed and so compact.

$\therefore$ Arbitrary union of open sets is open.

iii) Let $(X - K_1) \cup \{\infty\}$ and $(X - K_2) \cup \{\infty\}$ be two open sets $\Rightarrow [(X - K_1) \cup \{\infty\}] \cap [(X - K_2) \cup \{\alpha\}]$

$= (X - K_1 \cup K_2) \cup \{\infty\}$

finite union of compact sets is compact so this set is open.

also $[(X - K) \cup \{\infty\}] \cap O = (\overbrace{X - K}^{\text{open in } X}) \cap O$ - open in X where O is open in X

also open. And finite intersection of O_1, O_2 which is open in X so result follows by inductio

- Let $x, y \in X \cup \{\infty\}$. If $x, y \in X$ then we can find O_x, O_y open in X such that $x \in O_x$ and $y \in O_y$ and $O_x \cap O_y = \phi$ since X is Housdorff.

if $x = \{\infty\}$ and $y \in X$, X locally compact means we can find some compact neighborhood K of y so that $y \in K$ and $x \in (X - K) \cup \{\infty\}$ open. And K is neighborhood means we can find $O \subseteq K$ open and $y \in O$.

$\Rightarrow$ $O \cap (X - K) \cup \{\infty\} = \phi$ $\Rightarrow$ $X \cup \{\infty\}$ is Housdorff.

- Claim = X is dense in $X \cup \{\infty\}$.

Let $x \in X \cup \{\infty\}$ and $x \in O \subseteq X \cup \{\infty\}$ open.

If $y \in X$ then clearly $O \cap X \neq \phi$

If $x = \{\infty\}$ and $O = (X - K) \cup \{\infty\}$ be an open neighborhood of $\{\infty\}$. $\Rightarrow$ $(X - K) \cup \{\infty\} \cap X \neq \phi$ since X is not compact.

Result follows.

- Claim : $X \cup \{\infty\}$ is compact :

Let $\bigcup\limits_{\alpha \in I} O_\alpha$ be an open cover of $X \cup \{\infty\}$.

Then $\{\infty\} \in O_{\alpha_0}$ for some O_{α_0} open in $X \cup \{\infty\}$. Then O_α is in the form of $(X - K) \cup \{\infty\}$

$$\Rightarrow \quad X \cup \{\infty\} \subseteq \underbrace{(X - K) \cup \{\infty\}} \cup K \qquad \overset{\text{compact}}{K} \subseteq \bigcup_{n=1}^{N} O_n$$

$$X \cup \{\infty\} \subseteq O_{\alpha_0} \cup \bigcup_{n=1}^{N} O_n$$

so $X \cup \{\infty\}$ is covered finitely many open sets which means it is compact

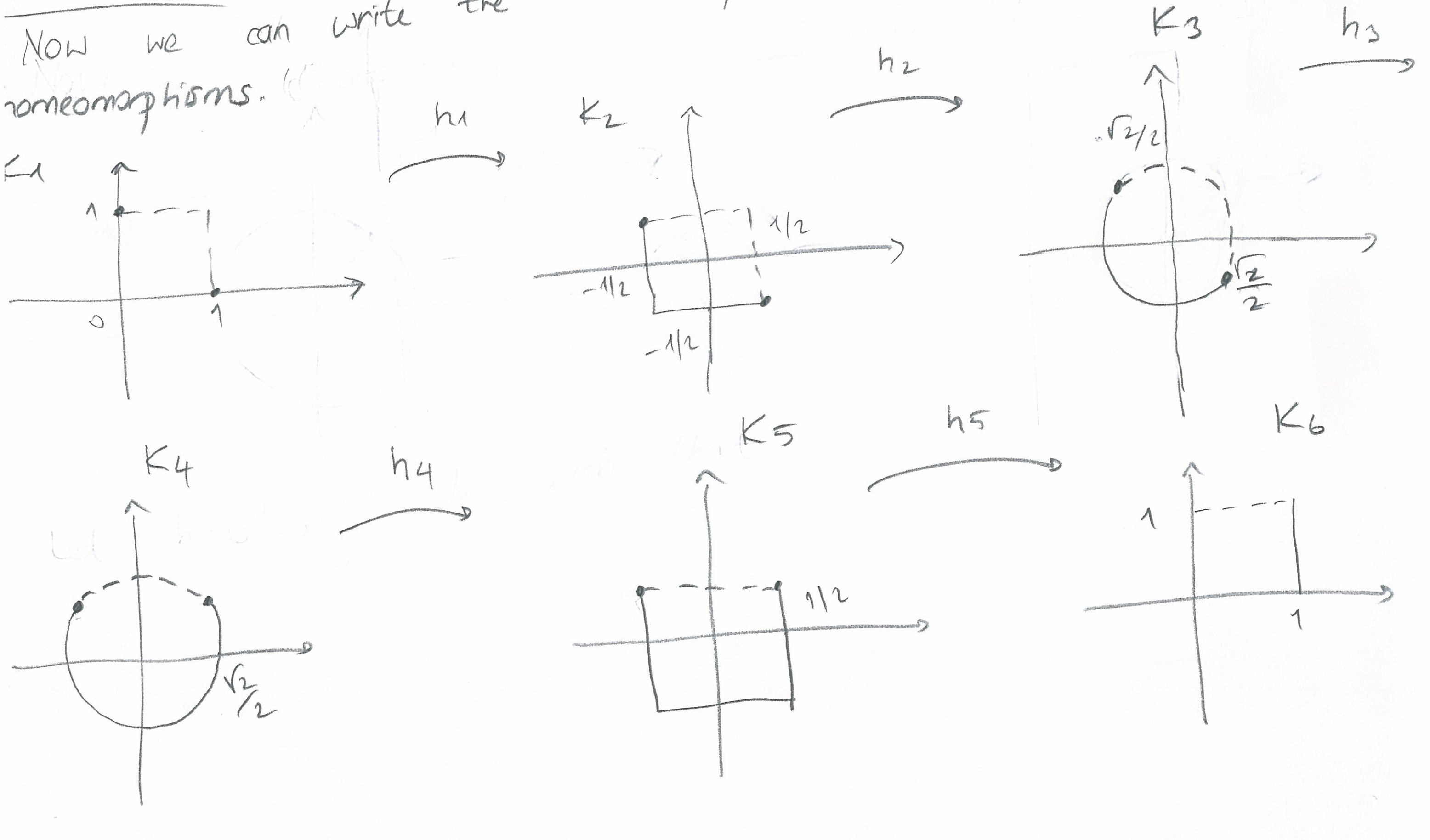
(23) Prove that $[0,1) \times [0,1)$ is homeomorphic to $[0,1] \times [0,1)$

For the solution of this problem we will use the glueing lemma

The Glueing Lemma: Let $X = A \cup B$ where A & B are closed in X. Let $f: A \to Y$ and $g: B \to Y$ be continuous. If $f(x) = g(x)$ $\forall x \in A \cap B$ then f and g combine to give a continuous function $h: X \to Y$ defined by setting $h(x) = f(x)$ if $x \in A$ and $h(x) = g(x)$ if $x \in B$.

Proof: Let C be a closed subset of Y. Then $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$. Since f is continuous $f^{-1}(C)$ is closed in A, so it is closed in X, similarly $g^{-1}(C)$ is closed in B so it is closed in X so their union also closed in X

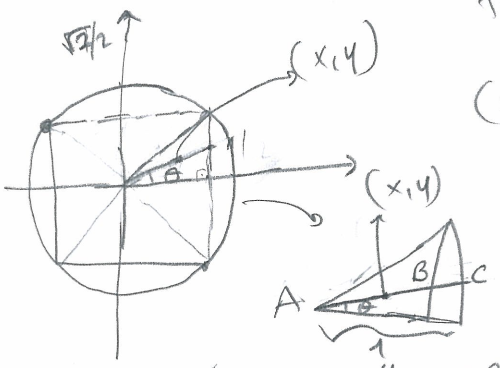
Now we can write the homeomorphism as the composition of five homeomorphisms.



for $h_1 : K_1 \to K_2$

$h_1(x,y) = \left(x - \frac{1}{2}, y - \frac{1}{2}\right)$ is clearly a homeomorphism.

For $h_2 : K_2 \to K_3$



for $0 < y < x$

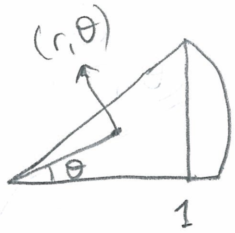
$$(x,y) \longmapsto \left(\frac{\sqrt{x^2+y^2} \cdot \frac{\sqrt{2}}{2}}{\frac{1}{\cos\left(\tan^{-1}\left(\frac{y}{x}\right)\right)}} , \tan^{-1}\left(\frac{y}{x}\right) \right)$$

$$= \left(\frac{\sqrt{x^2+y^2}\ \sqrt{2}\cos\left(\tan^{-1}\left(\frac{y}{x}\right)\right)}{2} , \tan^{-1}\right.$$

(we will map AB to AC) $\left(|AB| = \frac{1}{\cos\theta}\right)$

similarly we can map all 8 pieces of the square to the 8 pieses of circle. By the gluing lemma it is continous and clearly 1-1 and onto.

For the inverse $h_2^{-1} = K_3 \to K_2$



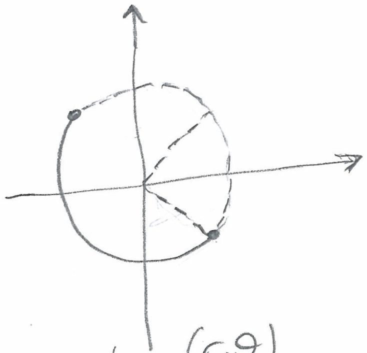
$$(r,\theta) \longmapsto \left(\frac{r}{\frac{\sqrt{2}}{2}} \cdot \cos\theta , \theta \right)$$

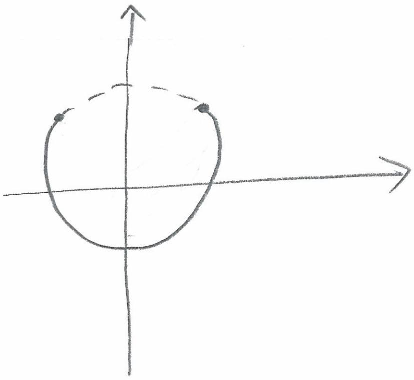
In cartesian coordinates $x = r\cos\theta$, $y = r\sin\theta$

$$h_2^{-1} = (r,\theta) \longleftrightarrow \left(\sqrt{2}\, r\cos\theta \cdot \cos\theta , \sqrt{2}\, r\cos\theta \sin\theta \right)$$

similarly all 8 pieces will be sent. Again by gluing lemma it is continous.

$h_3 : K_3 \to K_4$





① send (r,θ) $-\frac{\pi}{2} \le \theta \le -\frac{\pi}{4}$ to (r,θ) $-\frac{\pi}{2} \le \theta \le \frac{\pi}{4}$ and

② send (r,θ) $-\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$ to (r,θ) $\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$

send other points with identity map. Then again this will be continuous by gluing lemma.

① $(r,\theta) \longmapsto (r, 3\theta + \pi)$

② $(r,\theta) \longmapsto (r, \frac{\theta}{3} + \frac{\pi}{3})$

similarly inverse map can be found.

h_4, h_5 are also can be found similarly.

㉗ For $U_i \subseteq X_i$ we should have $p_i^{-1}(U_i) \in \tau$ where τ is the smallest topology for which all projection maps is continuos.

We know that countably many union and finitely many intersections should also be in the topology.

so $\bigcup_{j=1}^{\infty} p_i^{-1}(U_i) = \prod X_i \in \tau$ and $\bigcap p_i^{-1}(U_i)$ is also in the topology. $\bigcap p_i^{-1}(U_i)$ for finitely many n gives us sets of the form $Y_1 \times Y_2 \times \_\_$ where $Y_j = X_j$ for all but finitely many j and for finitely many $Y_j = U_j$

Claim: $\mathcal{B} = \{ Y_1 \times Y_2 \times Y_3 \_ , Y_j = U_j$ where $U_j \subseteq X_j$ is an open set, for finitely many j and $Y_j = X_j$ for infinitely many $j \}$? for this product topology $\tau \}$ is a base

– This claim follows directly because by construction in τ there exists countable union of sets of the form $\bigcap p_i^{-1}(U_i)$ where intersection is taken over finitely many numbers.

Claim 2: This is the smallest topology satisfying these condition for which all projections are continuos.

Let τ' be another topology

Proof: Let $O \in \tau$ be an open set. Then O is countable union of finite intersect or O is the finite intersections of the sets $p_i^{-1}(U_i)$. since U_i open $p_i^{-1}(U_i) \in \tau'$ $\forall i$. and finite intersections also in τ' and Also any countable union of finite intersections is also in τ' since it is a topology.