HW # 5 - SOLUTIONS

(1) (a) $\cong$ (b)

Let $B = \{$ lines through the origin that does not contain the origin$\}$

Define the map $f : S^n \to B$ which is continuous
$x \mapsto tx : t \in \mathbb{R} \setminus \{0\}$

and onto. S^n is compact and B is Hausdorff since any two lines can be contained in two disjoint open sets because 0 is not contained in these lines. So by Corollary 4.4 f is an identification map.

Now consider $\{f^{-1}(y) : y \in B\}$. This gives a partition of S^n which is exactly defined in (a). So by thm 4.2 (a) result follows.

(a) $\cong$ (c)

Let S^{n+} denote the upper half and S^{n-} denote the lower half of S^n. Since $S^{n+} \cong B^n$ there is a map $f^+ : S^{n+} \to B^n$ which is a homeomorphism. Now define the map $h : S^{n-} \to S^{n+}$
$x \mapsto -x$

which sends every point to its antipodal.

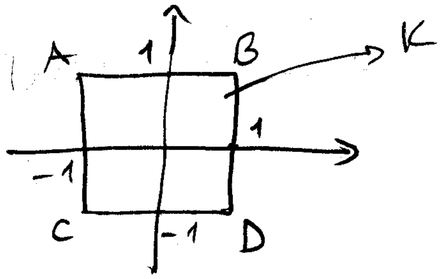
And also define $f^- : S^{n-} \to B^n$ $\quad f^- = f^+ \circ h$

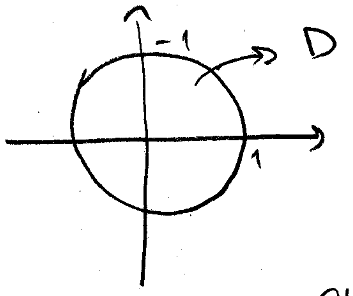
Now consider the map $\pi : B^n \to B^n / x \sim -x \quad x \in \partial B^n$

Now πf^+ and πf^- are continuous and they have the same value on $S^{n-} \cap S^{n+}$. So by gluing lemma $f = \pi f^+ \cup \pi f^-$ is continuous and it is onto clearly.

$\{f^{-1}(y) : y \in B^n / x \sim -x,\ x \in \partial B^n\}$ gives the partition which is defined in a. So (a) $\cong$ (c)

(2)





Define the map $f : K \to D$. Clearly f is onto and continuous.
$(x,y) \mapsto (x\sqrt{1-y^2}, y)$

Also define the projection map $\pi : D \to D/\{x \sim -x : x \in \partial D\}$

We know that $D/\{x \sim -x : x \in \partial D\}$ is homeomorphic to projective plane. Write P instead of it. Now $f = \pi \circ f : K \to P$ is continuous and onto. Also K is compact and P is Hausdorff. So by Corollary 4.4 f is an identification map. f sends $[AB]$ to $(0,1)$ and $[CD]$ to $(0,-1)$.

Consider $\{f^{-1}(y) : y \in P\} = \{\{(x,y)\} : x \in (-1,1), y \in (-1,1)\} \cup$
$\{\{(x,1), (x,-1)\}, x \in [-1,1]\} \cup \{(1,y), (-1, 1-y)\} : y \in [-1,1]\}$

This partition gives us exactly the one which is defined in question. So by Thm 4.2 a) result follows. ie. $M/\sim \cong P$.

(5) X is compact being closed and bounded subset of $\mathbb{R}^2$.
Y is not compact. $\pi : \mathbb{R} \to Y$ sends each point to the set it belongs
Let $O_n = \left(n+\frac{1}{3}, n+\frac{2}{3}\right)$ $\pi^{-1}(O_n) = \left(n+\frac{1}{3}, n+\frac{2}{3}\right)$ which is open in $\mathbb{R}$ so O_n is open in Y.

Now consider $G_n = \left(n-\frac{5}{12}, n+\frac{5}{12}\right) \setminus \{n\}$ and $A = \{n : n \in \mathbb{N}\}$

A is a point in Y.

Now $\pi^{-1}\{\bigcup G_n \cup A\} = \bigcup_{n \in \mathbb{N}} \left(n-\frac{5}{12}, n+\frac{5}{12}\right)$ is open in $\mathbb{R} \Rightarrow$

$\bigcup_{n \in \mathbb{N}} G_n \cup A$ is also open in Y. So $\bigcup_{n \in \mathbb{N}} O_n \cup \bigcup_{n \in \mathbb{N}} G_n \cup A$

is an open cover for Y which has no subcover $\Rightarrow$ no finite subcover

(Because if $O_{n_0} \notin$ subcover $\Rightarrow n_0 + \frac{1}{2} \notin$ union)

(16) $f : SO(n) \times \mathbb{Z}_2 \longmapsto O(n)$ $\left(\mathbb{Z}_2 \cong \left\{ \begin{bmatrix} 1 & & \\ & 1 & \\ & & \ddots \\ & & & 1 \end{bmatrix}, \begin{bmatrix} -1 & & \\ & 1 & \\ & & \ddots \\ & & & 1 \end{bmatrix} \right\}\right)$

$$\left(A, \begin{bmatrix} \pm 1 & & \\ & 1 & \\ & & \ddots \\ & & & 1 \end{bmatrix} \right) \longmapsto A \cdot \begin{bmatrix} \pm 1 & & \\ & 1 & \\ & & \ddots \\ & & & 1 \end{bmatrix}$$

this is onto and continuous also 1-1.
$SO(n) \times \mathbb{Z}_2$ is compact $O(n)$ is Hausdorff so f is a homeomorphism.

Assume that they are isomorphic as topological groups so there is an isomorphism $\varphi : SO(n) \times \mathbb{Z}_2 \mapsto O(n)$ $\varphi((I \times 1)(I \times 1)) = \varphi(I \times 1)\varphi(I \times 1) = \varphi(I \times 1)$

and $\varphi((I \times 0)(I \times 0)) = \varphi(I \times 0)\varphi(I \times 0) = \varphi(I \times 0)$

so there are two elements in $O(n)$ satisfying $A^2 = A$

but only I satisfies this. $\times$