HW - 7 / SOLUTIONS

(11) Let X be a path connected space and $p, q \in X$. Also suppose that γ_1 and γ_2 be two paths from p to q.

$$\gamma_{1*} = \pi_1(X,p) \to \pi_1(X,q)$$
$$\langle \alpha \rangle \mapsto \langle \gamma_1^{-1}.\alpha.\gamma_1 \rangle$$

$$\gamma_2 = \pi_1(X,p) \to \pi_2(X,q)$$
$$\langle \alpha \rangle \mapsto \langle \gamma_2^{-1}.\alpha.\gamma_2 \rangle$$

We want $\langle \gamma_1^{-1}.\alpha.\gamma_1 \rangle = \langle \gamma_2^{-1}.\alpha.\gamma_2 \rangle$

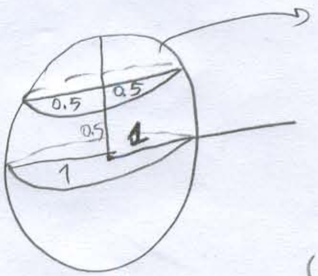
$$\Rightarrow \langle \gamma_1^{-1} \rangle \langle \alpha \rangle \langle \gamma_1 \rangle = \langle \gamma_2^{-1} \rangle \langle \alpha \rangle \langle \gamma_2 \rangle$$
$$\Rightarrow \langle \gamma_1 \gamma_1^{-1} \rangle \langle \alpha \rangle \langle \gamma_1 \rangle = \langle \gamma_1 \rangle \langle \gamma_2^{-1} \rangle \langle \alpha \rangle \langle \gamma_2 \rangle$$
$$\Rightarrow \langle \alpha.\gamma_1 \rangle = \langle \gamma_1 . \gamma_2^{-1}.\alpha.\gamma_2. \rangle$$
$$\Rightarrow \langle \alpha \rangle = \langle \gamma_1 \gamma_2^{-1}.\alpha.\gamma_2.\gamma_1^{-1} \rangle \quad \text{if } \pi_1(X,p) \text{ is abelian}$$

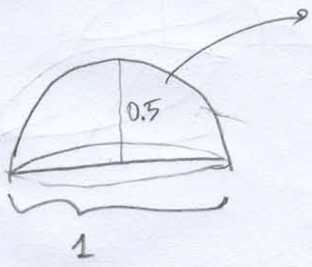
$\gamma_1\gamma_2^{-1}$ is a loop in $\pi_1(X,p)$

then $\langle \alpha \rangle = \langle \alpha \rangle \langle \gamma_1 \gamma_2^{-1} \gamma_2 \gamma_1^{-1} \rangle = \langle \alpha \rangle \langle e \rangle = \langle \alpha \rangle$ ✓

(16) We can cover S^n with sets such that any two points in this set has distance less than 1.

For instance in S^2





Any two points in this set has distance < 1.
We can cover S^2 with such sets.
Generalize this idea to S^n.

by Lebesgue lemma we can find points $0 = t_0 < t_1 \ldots < t_n = 1$ in I such that $\alpha([t_{k-1}, t_k])$ is contained in one of the sets in cover, where α is a loop on S^n. Let β_k be the path joining $\alpha(t_{k-1})$ to $\alpha(t_k)$ by a straight line inside the ball. Now consider $\beta = \beta_1\beta_2 \ldots \beta_k$ which is a loop in $\mathbb{R}^n$ by construction. Also $\|\beta(s) - \alpha(s)\| \leq 1$.

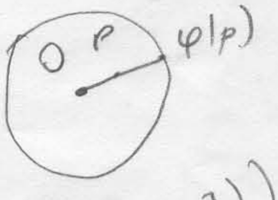
Now write the homotopy between β and α

$F(x,t) = t\beta(x) + (1-t)\alpha(x)$ this straight line homotopy does not pass through the origin since $\|\beta(s) - \alpha(s)\| \leq 1$. $\forall s$.

Now consider α in $\mathbb{R}^n$.

Also take a point p inside S^n such that any straight line from that point to $\beta(s)$ $\forall s$ does not pass through the origin. Then we can also write the straight line homotopy H to that point such that $H(x,0) = \beta(s)$ $H(x,1) = p$

Now consider the projection map φ which sends any point to the intersection point of S^n and the straight line starting at origin and passing through that point. ie



Consider $\varphi \circ H : \mathbb{R}^n \times [0,1] \to S^n$

$$\varphi \circ H(x,0) = \varphi(\beta(s))$$
$$\varphi \circ H(x,1) = \varphi(p)$$

(φ is defined since $H(\mathbb{R}^n \times [0,1])$ does not contain origin.)

$\Rightarrow \varphi(\beta(s)) \sim$ constant $= \varphi(p) \in S^n$

we also know that $\alpha(s) \sim \beta(s) \Rightarrow \varphi \circ \alpha \sim \varphi \circ \beta$

but $\varphi \circ \alpha = \alpha$ since $\alpha(s) \in S^n$ $\forall s$.

$\Rightarrow \alpha \sim \varphi \circ \beta \sim$ constant $\Rightarrow \pi_1(S^n)$ is trivial

since it is path connected it is simply connected.

For S^2 writing H is not possible because it must pass through the origin.

(21) (a) Consider the generator α of $\pi_1(S^1, 1)$, which is the loop homeomorphic to S^1. If α moves around S^1 clockwise

$f_* : \pi_1(S^1, 1) \to \pi_1(S^1, f(1))$

$\alpha \longmapsto f \circ \alpha$

$f \circ \alpha$ moves around S^1 anticlockwise one time. So this map sends the generator to the generator which is an isomorphism.

(b) f sends generator α to the loop moving around S^1 n times.

(c) 

f sends the generator α to the trivial element of $\pi_1(S^1, f(1))$ so it is the trivial map.

$f_*(\pi_1(S^1, 1)) \cong \{0\}$