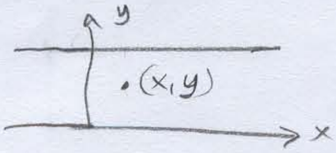
HW # 8 – SOLUTIONS

(15) $\varphi_n(x,t) = \left(x+n, \frac{1}{2} + (-\frac{1}{2})^n + (-1)^n y\right)$

For $n \in \mathbb{Z}$ define the map φ_n as above.
The orbit space is Möbius band. $\mathbb{R} \times I / \mathbb{Z} \cong M$

Let $p = (x,y) \in \mathbb{R} \times I$



Choose $\varepsilon = \min\{y, 1-y\}$

$\Rightarrow \varphi_n(B_\varepsilon(p)) \cap B_\varepsilon(p) = \phi$

since for $(x,y) \in B_\varepsilon(p)$ $\left(x+n, \frac{1}{2} + (-\frac{1}{2})^n + (-1)^n y\right) \in \varphi_n(B_\varepsilon(p))$

$\varepsilon < 1$ so intersection is clearly empty.

Also $\mathbb{R} \times I$ is simply connected $\Rightarrow$ by Thm 5.13

$\pi_1(M) \cong \mathbb{Z}$.

Define $\varphi_n(x,t) = (x+n, y)$ for the cylinder.

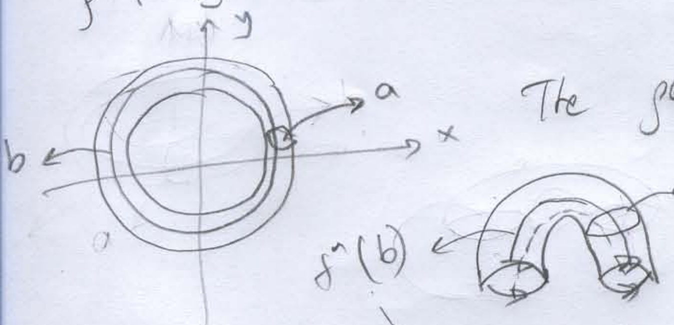
The orbit space is $\mathbb{R} \times I / \mathbb{Z} \cong S^1 \times I$

result follows similarly.

(22) 1) $g : \mathbb{Z}_2 \to T$ $\qquad f : T \to T/\mathbb{Z}_2$ identification map

$g(x,y,z) = (x,-y,-z)$ $\qquad f^* : \pi_1(T) \to \pi_1(T/\mathbb{Z}_2)$

The generators of $\pi_1(T)$ are a and b.



$f^*(a)$ which is homotopic to a point.

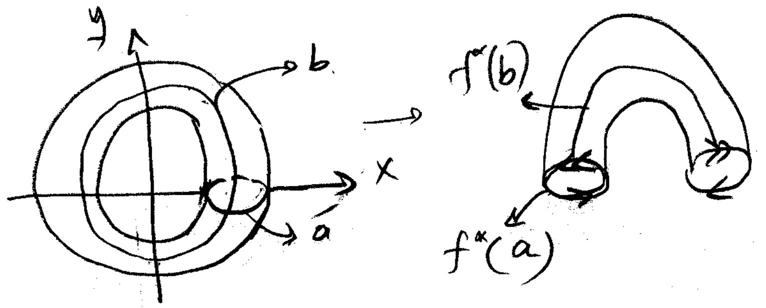
$f^*(b)$ which is also homotopic to a point

$\Rightarrow f^*$ is trivial map.

2) $\rho(x,y,z) = (-x,-y,z)$

$f: T \to T/\mathbb{Z}_2$

$f^*: \pi_1(T) \to \pi_1(T)$



$f^*(a)$ and $f^*(b)$ represents the same loops as T.

so f^* is an isomorphism

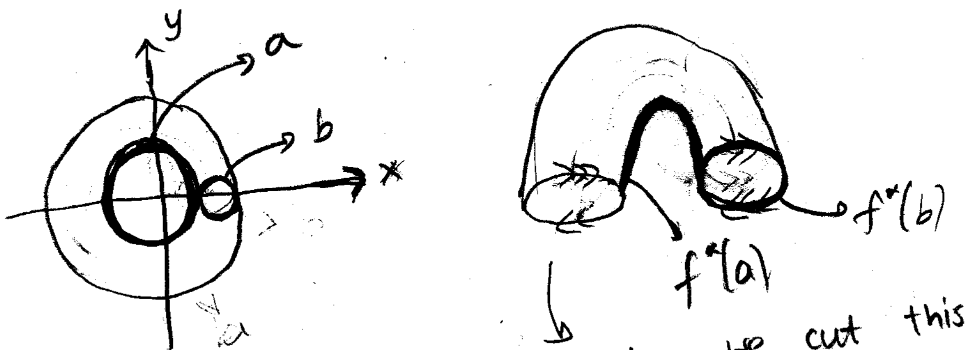
3) $\rho(x,y,z) = (-x,-y,-z)$

$f: T \to T/\mathbb{Z}_2$

$f^*: \pi_1(T) \to \pi_1(K^2)$

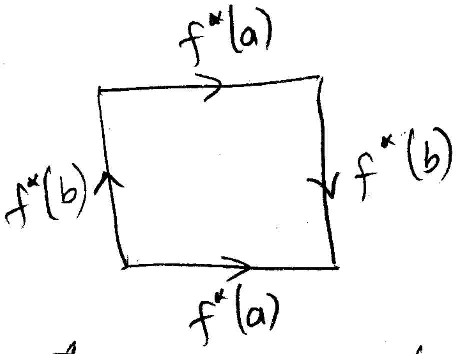
$a \mapsto f^*(a)$

$b \mapsto f^*(b)$



a and b are generators of $\pi_1(T^2)$

when we cut this from the line $f^*(a)$ we get



$\Rightarrow \quad f^*(b) f^*(a) f^*(b) (f^*(a))^{-1} = id$ (trivial loop).

let $f^*(a) = c$ $\qquad \pi(K^2) = \{ c, d \mid dcdc^{-1} = 1 \}$

$f^*(b) = d$

and $\quad f^*: \pi_1(T^2) \to \pi^1(K^2)$

$a \mapsto c$

$b \mapsto d$

(25) A punctured disk deformation retracts onto its boundary.

$H(z,t) = (1-t)\, z + t \dfrac{z}{|z|}$ $\quad H : (D \setminus \{0\}) \times I \to D \setminus \{0\}$

(We consider the disk that does not contain 0)

A punctioned disk is also homeomorphic to punctured $I \times I$.

Homeomorphism $f : D \setminus \{0\} \to I \times I \setminus \{p\}$

$f \circ H$ is the deformation retraction of $I \times I \setminus \{p\}$ onto the boundary of $I \times I$. Let $f \circ H = \tilde{H} : I \times I \setminus \{p\} \times I \to I \times I \setminus \{p\}$

Now consider the usual projection map $\pi : I \times I \to T_*$ which sends every point to the set which is belongs in the partition T_*

$g : I \times I \to T$

$(s,t) \mapsto ((R + r\cos 2\pi s) \cos 2\pi t, (2 + r\cos 2\pi s) \sin 2\pi t, r \sin 2\pi t)$

$\{g^{-1}(t)\}$ for $t \in T$ gives the partition of T^*.

so T is homeomorphic to T^* since g is an identification map.

We can use T instead of T^*.

We need to find a deformation retraction of T onto the one point union of two circles. Let $\pi(p) = q$

Define the relation $\pi^{-1} : T \to I \times I$ as

$\pi^{-1}(\langle x \rangle) = x$ for $x \in \langle x \rangle$ which is a representative of $\langle x \rangle$.

Now $\tilde{H} \circ (\pi^{-1} \times id) = T \setminus \{q\} \times I \to I \times I \setminus \{p\}$

is a function since $\tilde{H}(\pi^{-1}(\hat{x}), t) = \tilde{H}(x,t)$

and $\pi \circ \tilde{H} \circ (\pi^{-1} \times id) = T \setminus \{q\} \times I \to T \setminus \{q\}$

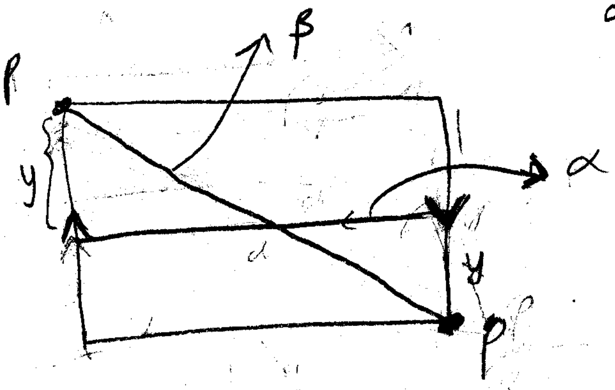
$\pi \circ \tilde{H} \circ (\pi^{-1} \times id)(\hat{x}, 0) = \pi \circ \tilde{H}(x, 0) = \pi(x) = x$ for $\hat{x} \in T \setminus \{q\} \times I$

$\pi \circ \tilde{H} \circ (\pi^{-1} \times id)(\hat{x}, 1) = \pi(\tilde{H}(x,1)) \in$ one point union of two circles

since $\tilde{H}(x,1) \in$ boundary $I \times I$

(26) a) $i : C \to S$ $\qquad$ S = Möbius strip

$i^* : \pi_1(C) \to \pi_1(S)$

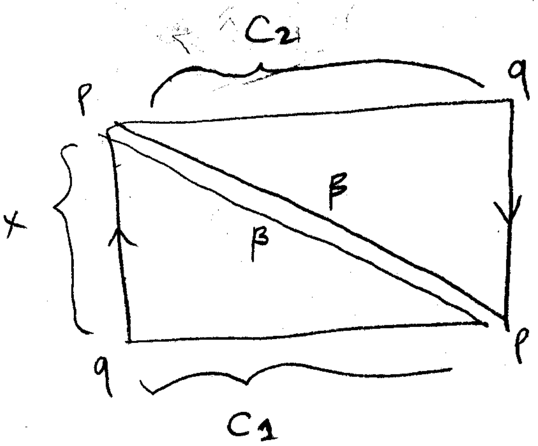


α is the generator of $\pi_1(S)$

$$\langle y^{-1} \alpha y \beta^{-1} \rangle = 1$$

$$\Rightarrow \langle y^{-1} \alpha \rangle = \langle y^{-1} \beta \rangle$$

$$\Rightarrow \langle \alpha \rangle = \langle \beta \rangle$$



$$C = C_1 \cdot C_2$$

$$\langle x \beta^{-1} C_1 \rangle = 1$$

$$\langle C_2 \beta^{-1} x^{-1} \rangle = 1$$

$$\langle C_2 \beta^{-1} \cancel{x^{-1}}\, x \beta^{-1} C_1 \rangle = 1$$

$$\Rightarrow \langle C_2 \beta^{-1} \beta^{-1} C_1 \rangle = 1$$

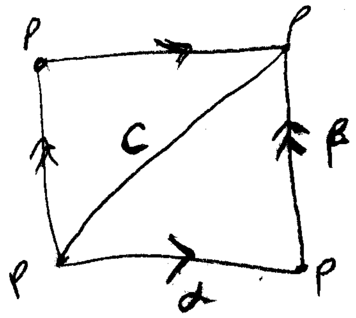
$$\Rightarrow \langle C_2 C_1 \rangle = \langle \beta \cdot \beta \rangle$$

$$\Rightarrow \langle C \rangle = \langle \beta \beta \rangle = \langle \alpha \cdot \alpha \rangle$$

so $\pi_1(C) \to \pi_1(S)$

$n \mapsto 2n$

b) S = torus , C = diagonal circle



$\langle c \rangle = \langle \alpha \rangle \langle \beta \rangle$

$i : C \to S$

$i_* : \pi_1(C) \to \pi_1(S)$

$\langle c \rangle \mapsto \langle \alpha \rangle \langle \beta \rangle$

$n \mapsto \langle n, n \rangle$

$\alpha = (1,0)$

$\beta = (0,1)$

α and β are generators of $\mathbb{Z} \times \mathbb{Z}$

c) S: cylinder C: one of the boundary circles

Since C is deformation retract of S they have isomorphic fundamental groups also inclusion map sends a generator of S to a generator of C so

$i_* : \pi_1(C) \to \pi_1(S)$

$\langle c \rangle \mapsto \langle \alpha \rangle$

$\Rightarrow$ i_* is an isomorphism.