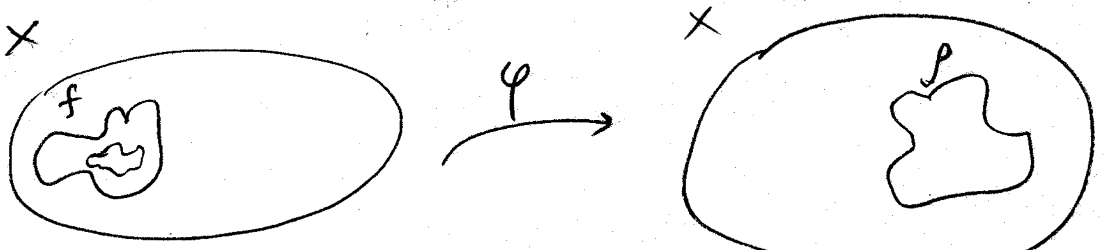
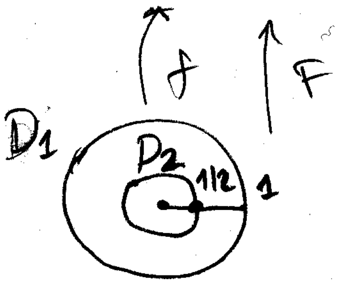
HW8 – SOLUTIONS

(27) Let $f, g : S^1 \to X$ are homotopic maps.
Then there exists a homotopy $H : S^1 \times I \to X$
such that $H(\theta, 0) = f(\theta)$ and $H(\theta, 1) = g(\theta)$





consider the extensions F and G of f and g respectively to D_1 and D_3..
$F(\theta, 1) = f(\theta)$ and $G(\theta, 1) = g(\theta)$

$F : D_1 \to F(D_1)$
$(\theta, r) \mapsto F(\theta, r)$

$F|_{\partial D_1} = f$

Let $D_2 = \{ x \in D_1 : \|x\| < 1/2 \}$ Then consider the homeomorphism $\phi : D_2 \to D_1$
$(\theta, r) \mapsto (\theta, 2r)$

Then define
$\varphi : X \cup_g D_1 \to X \cup_g D_2$

$$\varphi(x) = \begin{cases} x & \text{if } x \in X \\ G(\phi(F^{-1}(x))) & \text{if } x \in F(D_2) \\ K(x) & \text{if } x \in F(D_1 \setminus D_2) \ (K \text{ to be defined}) \end{cases}$$

For $x \in F(D_1 \setminus D_2)$ consider $F^{-1}(x) \in D_1 \setminus D_2$

if $F^{-1}(x) = (r, \theta) \Rightarrow \frac{1}{2} \le r \le 1$

Consider the functions $T_1 : D_1 \setminus D_2 \to S_1$ & $T_2 : D_1 \setminus D_2 \to I$
$(r, \theta) \mapsto \theta$ $\quad$ $(r, \theta) \mapsto -2r + 2$

$K : F(D_1 \setminus D_2) \to X$

$K = H(T_1 \circ F^{-1}, T_2 \circ F^{-1})$

for $x \in f(S^1)$ $\varphi(x) = x$ since $x \in X$ at the same time.
and $x \in f(S^1) \Rightarrow x \in F(D_1 \setminus D_2) \Rightarrow G(x) = H(T_1 \circ F^{-1}(x), T_2 \circ F^{-1}(x)) =$
$= H(\theta, 0) = f(\theta) = x$

$x = f(\theta)$

For $x \in f(\partial D_2)$ if $f(\theta, \frac{1}{2}) = x$

$\varphi(x) = G(\phi(f^{-1}(x))) = G(\phi(\theta, r)) = G(\theta, 1) = \rho(\theta)$

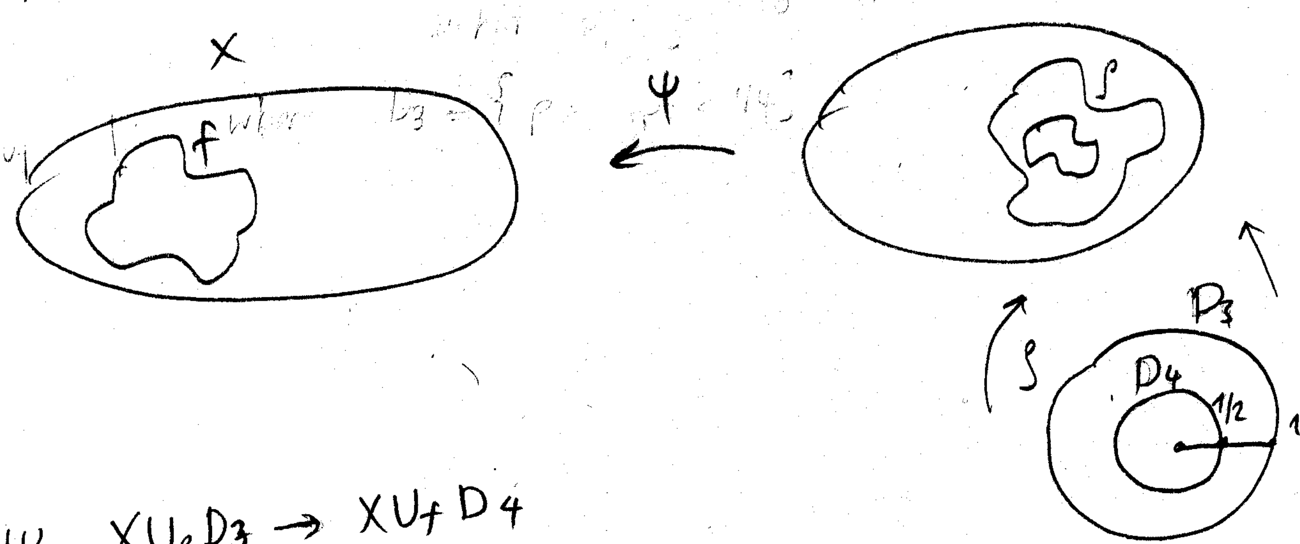
$\varphi(x) = K(x) = H(T_1(F^{-1}(x)), T_2(F^{-1}(x))) = H(T_1(\theta, \frac{1}{2}), T_2(\theta, \frac{1}{2}))$

$= H(\theta, 1) = \rho(\theta)$

so φ is continuous by gluing lemma since x and $G(\phi(F^{-1}(x)))$ have the same values on $f(S^1) = f(\partial D^1)$ and $G(\phi(F^{-1}(x)))$ and $K(x)$ have the same values on $f(\partial D^2)$.

Now define the function $\psi = X \cup_\rho D \to X \cup_f D$ in the same way.

X



$G|_{\partial D^3} = \rho$

$\psi = X \cup_\rho D_3 \to X \cup_f D_4$

$$\psi(x) = \begin{cases} x & \text{if } x \in X \\ F(\phi(G^{-1}(x))) & \text{if } x \in G(D_4) \\ L(x) & \text{if } x \in G(D_4 \setminus D_3) \end{cases}$$

$L = H^{-1}(T_1 \circ G^{-1}, T_2 \circ G^{-1})$ where H^{-1} is the homotopy $H(x, 1-t)$.

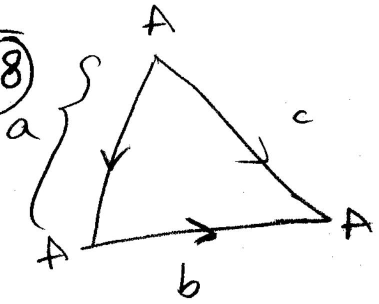
The final picture is $\psi \circ \varphi$ sends $X \cup_f D \to X \cup_f D$.

the disk radius $1/4$ is sent to the whole disk.

the annulus $r > 1/2$ is sent to the homotopy H and for $1/4 < r < 1/2$ is sent to the $H^{-1} = H(x, 1-t)$. sends $x \in X$ to itself

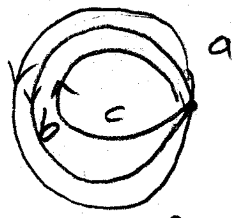
$\psi \circ \varphi$ is homotopic to the identity.

(28)



a is homotopic to a loop. b is also homotopic to a loop in the same direction c is homotopic to a loop in the opposite direction.

We can consider



$\alpha = abc$

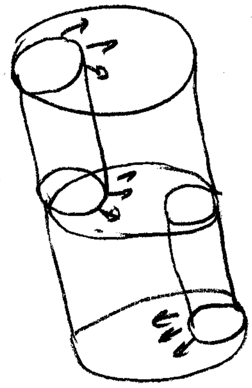
$$\alpha(s) = \begin{cases} \exp 4\pi i s & 0 \le s \le 1/2 \\ \exp 4\pi i (2s-1) & 1/2 \le s \le 3/4 \\ \exp 8\pi i (1-s) & 3/4 \le s \le 1 \end{cases}$$

$\beta(s) = \exp 2\pi i s \quad 0 \le s \le 1$

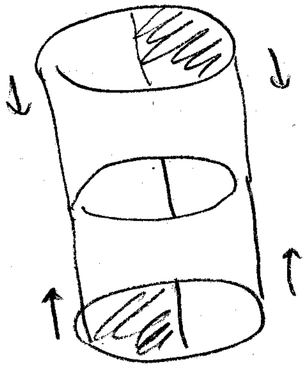
α and β are homotopic. (Given in page 89)

Dunce hat is a space formed from S_1 by attaching a disk using α and disk is a space formed from S_1 by attaching a disk using the identity map. Then by qn 27 disk and dunce hat have the same homotopy type $\Rightarrow$ dunce hat is also contractible.

(29)



$\simeq$



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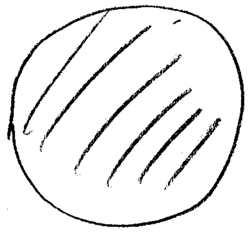

$\simeq \bullet$

(32) Assume that $p(z)$ is never zero then we define a map $f_t : S^1 \to S^1$ by $f_t(z) = p(tz)/|p(tz)|$ $\forall\ t \in \mathbb{R}$.

For $t_1, t_2 \in \mathbb{R}$
Define the homotopy $F : S^1 \times I \to S^1$ by

$$F(z,s) = \frac{p(t_1 z s + t_2 z(1-s))}{|p(t_1 z s + t_2 z(1-s))|}$$

$$F(z,0) = \frac{p(t_2 z)}{|p(t_2 z)|} = f_{t_2}(z) \qquad F(z,1) = \frac{p(t_1 z)}{|p(t_1 z)|} = f_{t_1}(z)$$

$\Rightarrow$ for any $t_1, t_2 \in \mathbb{R}$ f_{t_1} and f_{t_2} is homotopic to each other.

Now for t large enough consider the homotopy

$H : S^1 \times I \to S^1$

$$H(z,s) = \frac{z^n + s\left(\frac{a_{n-1} z^{n-1}}{t^1} + \frac{a_{n-2} z^{n-2}}{t^2} + \dots + \frac{a_0}{t^n}\right)}{\left|z^n + s\left(\frac{a_{n-1} z^{n-1}}{t} + \dots + \frac{a_0}{t^n}\right)\right|}$$

$$H(z,0) = \frac{z^n}{|z^n|} = z^n \quad \text{since } z \in S^1, \quad H(z,1) = f_t(z)$$

For t large enough we can say $z^n > \left|s\left(\frac{a_{n-1} z^{n-1}}{t} + \dots + \frac{a_0}{t^n}\right)\right|$

so $g : S^1 \to S^1$, $z \mapsto z^n$ is homotopic to f_t for t large enough. But we know that any two maps f_{t_1}, f_t are homotopic to each other.

homotopic, so f_0 and z^n are homotopic

But this is not possible since g is a loop turning around S^1 n times but f_0 is a constant map.

Contradiction