2) (5 pts each) For each of (a)-(d) below: If the proposition is true, write TRUE. If the proposition is false, write FALSE. No explanations are required for this problem.

2a) Image of a compact set under a continuous map may not be compact.

FALSE.

2b) $S^2 \times I$ is a contractible space.

FALSE. $S^2 \times I \sim S^2 \not\sim \cdot$

2c) Let X and Y be two manifolds. Then X is deformation retract of $X \vee Y$.

FALSE. $\cancel{Y} = S^1$ $\quad X \vee Y = S^1 \vee S^1$

$X = S^1$ $\quad S^1 \vee S^1 \not\sim S^1$

2d) Let S_1 and S_2 be compact orientable surfaces. If $\chi(S_1) = \chi(S_2)$ then S_1 is homeomorphic to S_2.

FALSE. $X = T^2 - \text{disk}$

$Y = S^2 - 3 \text{ disks}$

3) (15 pts) Prove or give a counterexample for the following statement. If you are giving a counterexample, verify your answer.

Let X be compact, and Y be Hausdorff. Then if $f : X \to Y$ is a 1-1, onto and continuous, then f is a homeomorphism.

Need to show $f^{-1}: Y \to X$ cts.

Let $A \subseteq X$ closed. Since f bijection, $(f^{-1})^{-1}(A) = f(A)$.

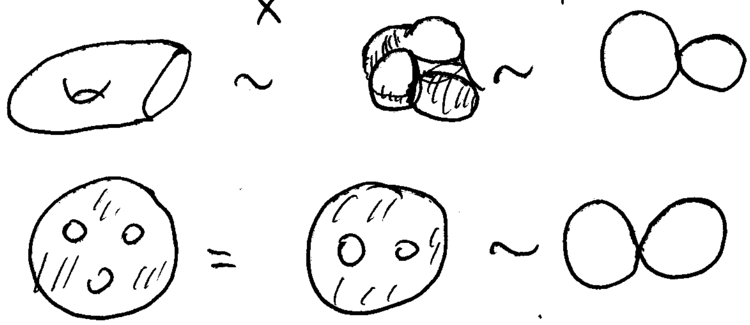
If we show $(f^{-1})^{-1}(A) = f(A)$ closed in Y, we are done.

$$\underbrace{A \subseteq X \text{ closed} \Rightarrow A \text{ compact in } X}_{X \text{ compact}} \quad \underbrace{\Rightarrow f(A) \text{ compact in } Y}_{f \text{ cts}} \quad \underbrace{\Rightarrow f(A) \text{ closed in } Y}_{Y \text{ Hausdorff}}$$

4) (15 pts) Prove or give a counterexample for the following statement. If you are giving a counterexample, verify your answer.

Let X and Y be two surfaces with boundary. If X is homotopy equivalent to Y, then X is homeomorphic to Y.

Ex 1: $\underset{X}{T^2 - \text{disk}} \sim \underset{Y}{S^2 - 3 \text{ disks}}$



but $\partial X = S^1$ $\quad \partial Y = S^1 \sqcup S^1 \sqcup S^1$

$X \simeq Y \Rightarrow \partial X \simeq \partial Y$ but $\partial X \not\simeq \partial Y$

Ex 2. $X = S^1 \times I$

$Y =$ Möbius band

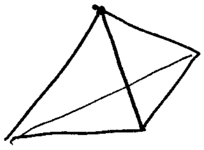
5) (15 pts) Prove or give a counterexample for the following statement. If you are giving a counterexample, verify your answer.

Every simply connected space is contractible.

Ex: S^2 simply connected.

However, $S^2 \not\simeq \cdot$

Since $\chi(S^2) = 2$

$S^2 \cong$  $\Rightarrow \chi(S^2) = 4-6+4 = 2$

$\chi(\cdot) = 1$

$X \sim Y \Rightarrow \chi(X) = \chi(Y)$

but $\chi(S^2) \neq \chi(\cdot) \Rightarrow S^2 \not\sim \cdot$

15 pts

6) ([redacted]) Determine the following surfaces described below.

(like $S_1 \simeq \Sigma_g - nD^2$ i.e. S_1 is the surface obtanined by removing n small disks from Σ_g)

a) S is a compact, orientable surface with 3 boundary components, and $\chi(S) = -7$.

$S^* = S + 3$ disk   closed orientable

$\chi(S^*) = -7 + 3 = -4 \Rightarrow S^* = \Sigma_3 \quad (2 - 2g = -4 \Rightarrow g = 3)$

$S = \Sigma_3 - 3$ disks

b) T is a compact, non-orientable surface with 2 boundary components, and $\chi(T) = -2$.

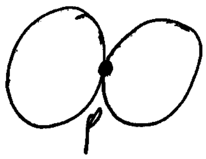
$T^* = T + 2$ disks   closed, non-orientable

$\chi(T^*) = -2 + 2 = 0 \Rightarrow T^* = \underset{\text{Klein bottle}}{M(2)} \quad (2 - g = 0 \Rightarrow g = 2)$

$S^2 - 2$ disks $+ 2$ möbius bands

$T = M(2) - 2$ disks $=$ Klein bottle $- 2$ disks

Bonus) (20 pts) Let $X = S^3 - (S^1 \vee S^1)$. Compute $\pi_1(X)$.

$S^3 = \mathbb{R}^3 \cup \{\infty\}$ $\qquad$ $S^1 \vee S^1 =$ 

Let p be ∞ point in S^3.

Then $X = S^3 - (S^1 \vee S^1) = \mathbb{R}^3 - 2$ disjoint lines $=$
(Think 2 parallel lines to z-axis)
$\{p\} \times \mathbb{R}$ and $\{q\} \times \mathbb{R}$
$p, q \in$ xy plane.

$\Rightarrow \quad X \sim \mathbb{R}^2 - (\{p\} \cup \{q\}) \sim S^1 \vee S^1$

$\Rightarrow \quad \cancel{\pi_1} \; \pi_1(X) = \pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z}$